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Examples on an Extension Problem of Holomorphic Maps and a Holomorphic 1-Dimensional Foliation

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§0. Introduction.

Let C^2 be the two dimensional complex vector space with a standard system of coordinates $z=(z_1, z_2)$. Put

$$B=\{z\in C^2: |z|<1\}\ ,$$
$$\partial B(\varepsilon)=\{z\in C^2: 1-\varepsilon<|z|<1\}\ ,$$
$$\Sigma_1=\{z\in C^2: |z|=1\}\ ,\quad \text{and}$$
$$\Sigma_2=\{z\in C^2: |z|=1-\varepsilon\}\ ,$$

where ε is a constant such that $0<\varepsilon<1$, and

$$|z|^2=|z_1|^2+|z_2|^2\ .$$

In this note, first *we shall construct compact complex* 3*-folds M which admit a holomorphic map*

$$f\ :\ \partial B(\varepsilon)\longrightarrow M$$

such that the inner boundary Σ_2 of $\partial B(\varepsilon)$ is a natural boundary of f. That is, for any point $x\in\Sigma_2$, we *cannot* find any neighborhood W of x in C^2 such that f can be extended to a holomorphic map of $W\cup\partial B(\varepsilon)$ into M. Secondly, we study a 1-dimensional holomorphic foliation on the associated projective bundle $P(TM)$ of the tangent bundle TM. We shall show that *in $P(TM)$ there are a subdomain W, $P(TM)-[W]\neq\emptyset$, and a thin subset S of $P(TM)-[W]$ such that every leaf in W is biholomorphic to P^1 and all compact leaves outside $[W]$ are contained in* S, where $[W]$ indicates the closure of W in $P(TM)$.

In §1, we shall construct our compact complex 3-fold M. In §2, we shall prove the non-extendibility of a certain holomorphic map into M (see also [2]). In §3, we study the holomorphic foliation on $P(TM)$.

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The idea of the construction of M can be found in Atiyah-Hitchin-Singer [1, p. 439, Example 4].

§ 1. Construction of the 3-fold.

Let U be an open subdomain in the complex 3-dimensional projective space $\boldsymbol{P}^3$ defined by

$$U=\{[z_0 : z_1 : z_2 : z_3] \in \boldsymbol{P}^3 \ : \ |z_0|^2+|z_1|^2<|z_2|^2+|z_3|^2\} ,$$

where $[z_0 : z_1 : z_2 : z_3]$ is a system of homogeneous coordinates on $\boldsymbol{P}^3$. Consider the Lie group $Sp(1, 1)$, which is defined by

$$\{g \in M_4(\boldsymbol{C}) \ : \ {}^t\bar{g}\cdot H\cdot g=H, \ J\cdot g=\bar{g}\cdot J\} \tag{1.1}$$

where

$$H=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The condition ${}^t\bar{g}\cdot H\cdot g=H$ implies $g(U)=U$. Put

$$\boldsymbol{H}=\left\{M \in M_2(\boldsymbol{C}) \ : \ M=\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \ \alpha, \beta \in \boldsymbol{C}\right\} .$$

It is easy to see that

$$g=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4(\boldsymbol{C}) , \qquad A, B, C, D \in M_2(\boldsymbol{C}) ,$$

is in $Sp(1, 1)$ if and only if

$$\begin{cases} A, B, C, D \in \boldsymbol{H} , \\ A^*A-C^*C=D^*D-B^*B=I , \\ A^*B=C^*D , \end{cases} \tag{1.2}$$

where $M^*={}^t\bar{M}$.

LEMMA 1.1. *$Sp(1, 1)$ acts transitively on U as a holomorphic automorphism group.*

PROOF. By (1.2), it is easy to see that every element of $Sp(1, 1)$ defines a holomorphic automorphism of U as an element of $PGL(4, \boldsymbol{C})$.

It is enough to prove that the action is transitive. Take any point $z=[z_0 : z_1 : z_2 : z_3] \in U$. Put $\lambda=|z_0|^2+|z_1|^2$ and $\mu=|z_2|^2+|z_3|^2$. If $\lambda\neq 0$, then we put

$$A=\lambda^{-1/2}(\mu-\lambda)^{-1/2}\begin{pmatrix} z_0\bar{z}_2+\bar{z}_1z_3 & z_0\bar{z}_3-\bar{z}_1z_2 \\ -\bar{z}_0z_3+z_1\bar{z}_2 & \bar{z}_0z_2+z_1\bar{z}_3 \end{pmatrix},$$

$$B=(\mu-\lambda)^{-1/2}\begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{pmatrix},$$

$$C=\lambda^{1/2}(\mu-\lambda)^{-1/2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$D=(\mu-\lambda)^{-1/2}\begin{pmatrix} z_2 & -\bar{z}_3 \\ z_3 & \bar{z}_2 \end{pmatrix}.$$

If $\lambda=0$, then we put $A=I$, $B=C=0$, and

$$D=\mu^{-1/2}\begin{pmatrix} z_2 & -\bar{z}_3 \\ z_3 & \bar{z}_2 \end{pmatrix}.$$

Then, in both cases, $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an element of $Sp(1, 1)$. Moreover $g(e)=z$, where $e=[0:0:1:0]\in U$. Hence $Sp(1, 1)$ acts transitively on U. □

LEMMA 1.2. *The isotropy subgroup K of $Sp(1, 1)$ with respect to the action on U is a compact group isomorphic to $Sp(1)\times SO(2)$.*

PROOF. If $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(1, 1)$ fixes $e=[0:0:1:0]$, then it follows easily from (1.2) that

$$B=0\ , \quad C=0\ , \quad A^*A=I\ , \quad \text{and} \quad D^*D=I\ .$$

Since

$$D\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\delta\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \delta\in C^*\ ,$$

D is of the form

$$D=\begin{pmatrix} \delta & 0 \\ 0 & \bar{\delta} \end{pmatrix}, \qquad |\delta|=1\ ,$$

which is identified naturally with an element of $SO(2)$. Hence $g=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in Sp(1)\times SO(2)$. Conversely, every element of this form fixes e. Hence K is isomorphic to $Sp(1)\times SO(2)$. □

By Lemmas 1.1 and 1.2, we have the following

LEMMA 1.3. $U\cong Sp(1, 1)/Sp(1)\times SO(2)$.

There is a well-known exact sequence of Lie groups:

$$1 \longrightarrow \{\pm 1\} \longrightarrow Sp(1, 1) \xrightarrow{\rho} SO^0(4, 1) \longrightarrow 1 , \tag{1.3}$$

where $SO^0(4, 1)$ is the connected component of $SO(4, 1)$ containing the unit. By Vinberg [4] (or by a more general result of A. Borel), we know that there are many finitely generated cocompact discrete subgroups in $SO^0(4, 1)$. Let $\bar{\Gamma}$ be one of them and put $\Gamma'=\rho^{-1}(\bar{\Gamma})$. Since ρ is a double covering, Γ' is also a finitely generated cocompact discrete subgroup of $Sp(1, 1)$. By a well-known theorem of Selberg, there is a subgroup Γ of Γ' such that the index $[\Gamma' : \Gamma]$ is finite and such that Γ contains no elements of finite order. If $\gamma(x)=x$ for some $\gamma\in\Gamma$ and $x\in U$, it follows readily that $\gamma=1$. Since the isotropy group K of $Sp(1, 1)$ with respect to the action on U is compact by Lemma 1.2, we see that the action of Γ on U is properly discontinuous. Therefore we have the following.

THEOREM 1. *There are discrete subgroups $\Gamma\subset Sp(1, 1)$ such that the quotient space $\Gamma\backslash U$ are non-singular compact complex 3-folds.*

§2. An example of non-extendible holomorphic maps.

Let ε be any real number satisfying $0<\varepsilon<1$. Define a holomorphic injective map

$$j : \partial B(\varepsilon) \longrightarrow U$$

by

$$j(w_1, w_2)=[\alpha_0 : \alpha_1 : w_1 : w_2] ,$$

where α_0, α_1 are any complex numbers satisfying

$$|\alpha_0|^2+|\alpha_1|^2=(1-\varepsilon)^2 .$$

Let M be the manifold in Theorem 1. Let

$$\pi : U \longrightarrow M=\Gamma\backslash U$$

be the canonical projection. Define a holomorphic map

$$f : \partial B(\varepsilon) \longrightarrow M$$

by

$$f = \pi \circ j .$$

Then we can show the following.

THEOREM 2. *For any point $x \in \Sigma_2$, there is no neighborhood W of x in C^2 such that f extends to a holomorphic map $\hat{f}$ of $W \cup \partial M(\varepsilon)$ into M.*

PROOF. Suppose that there were such an open neighborhood W of x such that $W \cap \partial B(\varepsilon)$ is connected. Put $y = \hat{f}(x) \in M$. Since $\pi : U \rightarrow M$ is a Galois covering, we can choose a small relatively compact subdomain Δ around y in M and a relatively compact subdomain $\tilde{\Delta}$ in U such that $\pi^{-1}(\Delta) = \cup_{\gamma \in \Gamma} \gamma(\tilde{\Delta})$. Moreover we can assume that each connected component of $\pi^{-1}(\Delta)$ is relatively compact in U. Since $\hat{f} | W : W \rightarrow M$ is continuous, we can assume that $\hat{f}(W) \subset \Delta$. Hence $f(W \cap \partial B(\varepsilon)) = \hat{f}(W \cap \partial B(\varepsilon)) \subset \Delta$. Therefore, since $W \cap \partial B(\varepsilon)$ is connected, $j(W \cap \partial B(\varepsilon))$ is contained in a connected component of $\pi^{-1}(\Delta)$. Since each connected component of $\pi^{-1}(\Delta)$ is relatively compact in U, we see that the closure $[j(W \cap \partial B(\varepsilon))]$ is compact in U. Hence, for any sequence $\{x_\lambda\}$, $\lambda = 1, 2, \cdots$ of points in $W \cap \partial B(\varepsilon)$ which converges to $x \in W \cap \Sigma_2$, we can choose a subsequence of $\{j(x_\lambda)\}$ which converges to an interior point of U. But this contradicts the definition of the map j. □

REMARK 2.1. The above f does not extend even as a continuous mapping across Σ_2. This is clear from the above argument.

REMARK 2.2. The manifold M is the twistor space over a conformally flat real hyperbolic differentiable 4-manifold.

§ 3. An example of holomorphic foliations.

For a complex manifold X, we let TX denote the tangent bundle and $P(TX)$ the associated projective bundle. Let M be the manifold in Theorem 1 and put $Z = P(TM)$. In this section, we shall construct a holomorphic foliation of dimension 1 on Z and study its leaves.

On $P(TP^3)$, we can consider two fibre bundle structures. One is the natural projection

$$p_1 : P(TP^3) \longrightarrow P^3$$

and the other is the projection

$$q_1 : P(TP^3) \longrightarrow Gr(4, 2)$$

to the Grassmannian manifold of all lines in P^3. The fibre of q_1 passing through a point $v \in P(TP^3)$ corresponds to the line in P^3 passing through $p_1(v)$ with direction v. By the natural inclusion $U \subset P^3$, we regard $P(TU)$ as a subdomain in $P(TP^3)$. Then q_1 defines a holomorphic mapping

$$q_2 : P(TU) \longrightarrow Gr(4, 2) .$$

Obviously, every element of $PGL(4, C)$ induces a holomorphic automorphism of $P(TP^3)$ and $Gr(4, 2)$. Note also that every element of Γ induces a holomorphic automorphism of $P(TU)$. Thus we have the commutative diagram

$$\begin{array}{ccc} P(TU) & \xrightarrow{q_2} & Gr(4, 2) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ P(TU) & \xrightarrow[q_2]{} & Gr(4, 2) , \end{array}$$

for $\gamma \in \Gamma$. The action of Γ on $P(TU)$ is properly discontinuous and we have

$$Z = P(TM) = \Gamma \backslash P(TU) .$$

Hence the mapping q_2 defines a holomorphic foliation F on Z whose leaves are images of the fibres of q_2 in $\Gamma \backslash P(TU)$. Now we shall study the leaves of F. Let

$$\pi_1 : P(TU) \longrightarrow Z$$

be the projection, which is an unramified Galois covering. Put

$$\begin{aligned} &\tilde{W} = \{w \in P(TU) : q_2^{-1}(q_2(w)) \text{ is compact}\} , \\ &W = \pi_1(\tilde{W}) , \quad \text{and} \\ &\tilde{D} = q_2(\tilde{W}) . \end{aligned}$$

For $w \in \tilde{W}$, $q_2^{-1}(q_2(w))$ is biholomorphic to P^1, and is projected by p_1 onto a projective line in U. There are many projective lines in P^3 which are not contained in $[U]$. Hence $P(TU) - [\tilde{W}]$ is not empty.

LEMMA 3.1. *$\tilde{W}$ is a Γ-invariant subdomain.*

PROOF. Take any $w \in \tilde{W}$ and $\gamma \in \Gamma$. Put $\tilde{L} = q_2^{-1}(q_2(w))$. Since $p_1(\tilde{L})$ is a projective line contained in U, so is $\gamma(p_1(\tilde{L}))$. Hence $\gamma(\tilde{L})=q_2^{-1}(q_2(\gamma(w)))$ is biholomorphic to $\boldsymbol{P}^1$. Therefore $\gamma(w)\in\tilde{W}$. Thus $\tilde{W}$ is Γ-invariant. That $\tilde{W}$ is connected follows from the fact that any projective line in U can be displaced continuously in U to the line $z_0=z_1=0$. It is clear that $\tilde{W}$ is open. □

LEMMA 3.2. *Γ acts on $\tilde{D}$ and the action is properly discontinuous.*

PROOF. Since $\tilde{W}$ is Γ-invariant by Lemma 3.1, Γ acts on $\tilde{D}$. Note that $\tilde{W}$ is a fibre bundle over $\tilde{D}$ with compact fibres $\boldsymbol{P}^1$. Therefore, since the action of Γ on $P(TU)$ is properly discontinuous, so is the action on $\tilde{W}$. Consequently, the action on $\tilde{D}$ is properly discontinuous. □

By Lemma 3.2, the quotient space $\Gamma\backslash\tilde{D}$ becomes naturally a normal complex space. Moreover the projection $q_2:\tilde{W}\to\tilde{D}$ defines a fibre bundle structure $\bar{q}:W\to\Gamma\backslash\tilde{D}$ on W, whose reduced fibres are biholomorphic to $\boldsymbol{P}^1$. Since $\tilde{W}$ is Γ-invariant, W is a domain in Z such that $Z-[W]\cong\Gamma\backslash(P(TU)-[\tilde{W}])$ is non-empty.

Let L be a compact leaf of F. Let $\tilde{L}_0$ be a connected component of $\pi_1^{-1}(L)$. Then $\tilde{L}_0$ is a fibre of q_2 and $\pi_1^{-1}(L)=\cup_{\gamma\in\Gamma}\gamma(\tilde{L}_0)$. If $\tilde{L}_0$ is compact, then $\tilde{L}_0\subset\tilde{W}$, and consequently $L\subset W$. Suppose that $\tilde{L}_0$ is not compact. Note that there is a compact curve $\tilde{L}\cong\boldsymbol{P}^1$, which is a fibre of q_1 in $P(T\boldsymbol{P}^3)$, such that $\tilde{L}$ contains $\tilde{L}_0$ as a connected subdomain. Put $l=p_1(\tilde{L})$. Note that $p_1|\tilde{L}:\tilde{L}\to l$ is biholomorphic. It is easy to show that $U\cap l$ is biholomorphic to $\boldsymbol{C}$ or a unit disk. Hence so is $\tilde{L}^0$. Since L is compact, there is a non-trivial subgroup Γ_0 of Γ such that Γ_0 leaves $\tilde{L}_0$ invariant and such that $\Gamma_0\backslash\tilde{L}_0\cong L$. Thus we have, in particular, the following correspondence.

$$\begin{array}{l} C=\{\tilde{L}\subset P(TU)\ :\ \tilde{L}\text{ is a non-empty non-compact component of a fibre of } q_2 \text{ such that } \pi_1(\tilde{L}) \text{ is compact}\} \\ \qquad\qquad\downarrow \Phi \\ S=\{l\in Gr(4,2)\ :\ \text{The isotropy subgroup } \Gamma_l \text{ of } \Gamma \text{ at } l \text{ is an infinite group}\}, \end{array}$$

where $\Phi(\tilde{L})$ corresponds to the projective line in $\boldsymbol{P}^3$ which contains $p_1(\tilde{L})$ as a subdomain. Then the mapping Φ is injective. Put

$$S_\gamma=\{l\in Gr(4,2)\ :\ \gamma(l)=l\}\,.$$

Then S_γ is a proper analytic subset in $Gr(4, 2)$. Therefore we have

THEOREM 3. *For the holomorphic foliation F on Z, there is a non-*

empty subdomain W in Z, $Z-[W]\neq\emptyset$, and a thin set S in $Z-[W]$ with the following properties.

(1) *Every leaf L of F with $L\cap W\neq\emptyset$ is contained in W, and is biholomorphic to $\boldsymbol{P}^1$.*

(2) *All compact leaves in $Z-[W]$ are contained in S.*

Our last example, Theorem 3, shows that a theorem of Nishino [3] on parametrizing compact divisors does not hold in higher codimensional cases.

References

[1] M. ATIYAH, N. HITCHIN and I. M. SINGER, Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A, **362** (1978), 425-461.

[2] MA. KATO, An example of compact complex 3-folds and an extension problem of holomorphic maps, preprint, 1983.

[3] T. NISHINO, L'existence d'une fonction analytique sur une variété analytique complexe a deux dimensions, Publ. RIMS Kyoto Univ., **18** (1982), 387-419.

[4] E. B. VINBERG, Discrete groups generated by reflections in Lobacevski spaces, Math. Sbornik, **72 (111)** (1967), 471-488, Math. USSR-Sbornik, **1** (1967), 429-444.

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