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Some Results on Additive Number Theory V

Minoru TANAKA

Gakushuin University

§1. The main theorem.

Let $\omega(n)$ denote the number of distinct prime factors of a positive integer n.

Let k and l be positive integers;
let $a_1, \cdots, a_k$ be distinct *non-zero* integers;
let $a_{k+1}, \cdots, a_{k+l}$ be distinct integers.

We put, for $\alpha_i<\beta_i$, $i=1, \cdots, k+l$,

$$\Phi(\alpha_i, \beta_i)=\frac{1}{\sqrt{2\pi}}\int_{\alpha_i}^{\beta_i}\exp\left(-\frac{x^2}{2}\right)dx\ .$$

Let N be a positive integer, which will be assumed to be sufficiently large as occasion demands.

THEOREM. *Let $A(N)=A(N; a_1, \cdots, a_{k+l}; \alpha_1, \beta_1, \cdots, \alpha_{k+l}, \beta_{k+l})$ denote the number of representations of N as the sum of the form $N=p+n$, where p is prime, and n is a positive integer such that*

$$\log\log N+\alpha_i\sqrt{\log\log N}<\omega(p+a_i)<\log\log N+\beta_i\sqrt{\log\log N}$$

for $i=1, \cdots, k$, and

$$\log\log N+\alpha_i\sqrt{\log\log N}<\omega(n+a_i)<\log\log N+\beta_i\sqrt{\log\log N}$$

for $i=k+1, \cdots, k+l$ simultaneously. Then, as $N\to\infty$, we have

$$A(N)\sim\frac{N}{\log N}\cdot\prod_{i=1}^{k+l}\Phi(\alpha_i, \beta_i)\ .$$

The paper will be read without making any references to author's previous papers, except for the proof of Lemma 4.

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§ 2. Proof of the theorem.

For any non-negative integers b and c, we put

$$E(b, c)=\sum_{i=0}^{2c}(-1)^i\binom{b}{i}, \qquad F(b, c)=\sum_{i=0}^{2c+1}(-1)^i\binom{b}{i},$$

then, as simple properties of binomial coefficients, we have

$$E(b, c)\begin{cases}=1 \text{ when } b=0, \\ \geqq 0 \text{ when } b>0,\end{cases} \qquad F(b, c)\begin{cases}=1 \text{ when } b=0, \\ \leqq 0 \text{ when } b>0.\end{cases}$$

LEMMA 1. *Let $b_1, \cdots, b_t$ and $c_1, \cdots, c_t$ be non-negative integers, then*

$$\sum_{j=1}^{t}\Big\{\prod_{\substack{i=1\\ i\neq j}}^{t} E(b_i, c_i)\cdot F(b_j, c_j)\Big\}-(t-1)\prod_{i=1}^{t} E(b_i, c_i)$$
$$\begin{cases}=1, \textit{ when } b_1=\cdots=b_t=0, \\ \leqq 0, \textit{ when at least one of the } b\textit{'s is positive}.\end{cases}$$

We omit the proof.

We put

$$\begin{aligned} D(N)&=D(N; a_1, \cdots, a_{k+l}) \\ &=\prod_{i=1}^{k} a_i\cdot\prod_{j=k+1}^{k+l}(N+a_j)\cdot\prod_{i=1}^{k}\prod_{j=k+1}^{k+l}(N+a_i+a_j) \\ &\quad\cdot\prod_{1\leq i<j\leq k}(a_i-a_j)\cdot\prod_{k+1\leq i<j\leq k+l}(a_i-a_j), \end{aligned}$$

and define the set S consisting of primes p^* as

$$S=\{p^* : p^*\nmid D(N),\ e^{(\log\log N)^2}<p^*<N^{(\log\log N)^{-2}}\}.$$

We put

$$y(N)=\sum_{p^*\in S}\frac{1}{p^*}.$$

LEMMA 2. $y(N)=\log\log N+O(\log\log\log N)$.

PROOF. As is well-known,

$$(1) \qquad \sum_{p^*\leqq x}\frac{1}{p^*}=\log\log x+O(1).$$

Since

$$2^{\omega\{D(N)\}} \leqq D(N) , \quad \omega\{D(N)\} \leqq \log D(N)/\log 2 ,$$

replacing x in (1) by $\log D(N)/\log 2$, we have

$$\sum_{p^*|D(N)} \frac{1}{p^*} \leqq \sum_{p^* \leqq D(N)} \frac{1}{p^*} = O(\log\log\log N) . \tag{2}$$

Hence if we replace x in (1) by the upper and lower limits of p^* in the definition of the set S, easy calculations will give the lemma.

By this lemma, $y(N) < 2\log\log N$ for large N. This inequality will frequently be used in the sequel.

We denote by $\omega^*(N; n)$ the number of distinct prime factors of a positive integer n, which belong to the set S:

$$\omega^*(N; n) = \sum_{p^*|n,\, p^* \in S} 1 .$$

We shall in the sequel mainly concern with $\omega^*(N; n)$, and a result obtained for $\omega^*(N; n)$ will in the last step be transformed to our theorem for $\omega(n)$.

We define, for any positive integer t, the set $M(t)$ consisting of positive integers as

$$\begin{aligned} M(t) = \{m;\ & m \text{ is squarefree,} \\ & m \text{ is composed only of primes} \in S, \\ & m \text{ has } t \text{ prime factors}\}; \end{aligned}$$

the 2nd and 3rd conditions may be rewritten as $\omega(n) = \omega^*(N; n) = t$. We put for convenience $M(0) = \{1\}$.

For any positive integers $t_1, \cdots, t_{k+l}$, we denote by $F(N; t_1, \cdots, t_{k+l})$ the number of representations of N as the sum of the form $N = p + n$, where p is prime, and n is a positive integer such that

$$\begin{aligned} \omega^*(N; p+a_i) &= t_i \qquad \text{for} \quad i = 1, \cdots, k , \\ \omega^*(N; n+a_i) &= t_i \qquad \text{for} \quad i = k+1, \cdots, k+l \end{aligned}$$

simultaneously.

For any positive integers $m_1, \cdots, m_{k+l}$ such that $m_1 \in M(t_1), \cdots, m_{k+l} \in M(t_{k+l})$ with some positive integers $t_1, \cdots, t_{k+l}$, we denote by $G(N; m_1, \cdots, m_{k+l})$ the number of representations of N as the sum of the form $N = p + n$, where p is prime, and n is a positive integer such that

$$(3) \qquad \prod_{\substack{p^* | (p+a_i) \\ p^* \in S}} p^* = m_i \quad \text{for} \quad i=1, \cdots, k\,,$$

$$(4) \qquad \prod_{\substack{p^* | (n+a_i) \\ p^* \in S}} p^* = m_i \quad \text{for} \quad i=k+1, \cdots, k+l$$

simultaneously.

We have

$$F(N; t_1, \cdots, t_{k+l}) = \sum_{m_1 \in M(t_1)} \cdots \sum_{m_{k+l} \in M(t_{k+l})} G(N; m_1, \cdots, m_{k+l})\,.$$

We shall give some more functions. For any positive integers $t_1, \cdots, t_{k+l}, T_1, \cdots, T_{k+l}$, we put

$$\mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) = \sum_{m_1 \in M(t_1)} \cdots \sum_{m_{k+l} \in M(t_{k+l})} \mathscr{K}^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots T_{k+l})\,,$$

$$\mathscr{K}^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l}) = \sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_{k+l}=0}^{2T_{k+l}} (-1)^{\tau_1+\cdots+\tau_{k+l}} \mathscr{L}(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})\,,$$

$$\mathscr{L}(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l}) = \sum_{\substack{\mu_1 \in M(\tau_1) \\ (\mu_1, m_1)=1}} \cdots \sum_{\substack{\mu_{k+l} \in M(\tau_{k+l}) \\ (\mu_{k+l}, m_{k+l})=1}} \sum_{\substack{p+n=N \\ m_i\mu_i | (p+a_i) \ (i=1,\cdots,k) \\ m_i\mu_i | (n+a_i) \ (i=k+1,\cdots,k+l)}} 1\,,$$

where the summation-variables p and n in the innermost sum run through primes and positive integers respectively such that $p+n=N$ and the conditions

$$\mu_i m_i | (p+a_i) \quad \text{for} \quad i=1, \cdots, k\,,$$
$$\mu_i m_i | (n+a_i) \quad \text{for} \quad i=k+1, \cdots, k+l$$

hold simultaneously.

Similarly we put, for $i=1, \cdots, k+l$,

$$\mathscr{H}^{(i)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) = \sum_{m_1 \in M(t_1)} \cdots \sum_{m_{k+l} \in M(t_{k+l})} \mathscr{K}^{(i)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})\,,$$

$$\mathscr{K}^{(i)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l}) = \sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_i=0}^{2T_i+1} \cdots \sum_{\tau_{k+l}=0}^{2T_{k+l}} (-1)^{\tau_1+\cdots+\tau_{k+l}} \mathscr{L}(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})\,,$$

where the summation-variable τ_i runs through the integers $0, \cdots, 2T_i+1$, and other τ_j's run through the integers $0, \cdots, 2T_j$ respectively.

LEMMA 3.

$$\sum_{i=1}^{k+l}\mathscr{H}^{(i)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})$$
$$-(k+l-1)\mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})$$
$$\leqq F(N; t_1; \cdots, t_{k+l}) \leqq \mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) .$$

PROOF. Changing the order of summations, we have

$$\mathscr{L}(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})$$
$$= \sum_{\substack{p+n=N \\ m_i \mid (p+a_i)\ (i=1,\cdots,k) \\ m_i \mid (n+a_i)\ (i=k+1,\cdots,k+l)}} \prod_{i=1}^{k} \sum_{\substack{\mu_i \in M(\tau_i) \\ (\mu_i, m_i)=1 \\ \mu_i \mid (p+a_i)}} 1 \cdot \prod_{i=k+1}^{k+l} \sum_{\substack{\mu_i \in M(\tau_i) \\ (\mu_i, m_i)=1 \\ \mu_i \mid (n+a_i)}} 1$$
$$= \sum_{\substack{p+n=N \\ m_i \mid (p+a_i)\ (i=1,\cdots,k) \\ m_i \mid (n+a_i)\ (i=k+1,\cdots,k+l)}} \prod_{i=1}^{k} \binom{\omega^*(N; p+a_i)-t_i}{\tau_i} \prod_{i=k+1}^{k+l} \binom{\omega^*(N; n+a_i)-t_i}{\tau_i} .$$

Hence if we put

$$\mathscr{K}^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l}) = \sum_{\substack{p+n=N \\ m_i \mid (p+a_i)\ (i=1,\cdots,k) \\ m_i \mid (n+a_i)\ (i=k+1,\cdots,k+l)}} \delta(p, n) ,$$

$$\sum_{i=1}^{k+1}\mathscr{K}^{(i)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})$$
$$-(k+l-1)\mathscr{K}^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})$$
$$= \sum_{\substack{p+n=N \\ m_i \mid (p+a_i)\ (i=1,\cdots,k) \\ m_i \mid (n+a_i)\ (i=k+1,\cdots,k+l)}} \delta'(p, n) ,$$

then, using the notations in Lemma 1, we have

$$\delta(p, n) = \prod_{i=1}^{k} E\{\omega^*(N; p+a_i)-t_i; T_i\} \cdot \prod_{i=k+1}^{k+l} E\{\omega^*(N; n+a_i)-t_i, T_i\} ,$$

$$\delta'(p, n) = \sum_{j=1}^{k}\Bigg[\prod_{\substack{i=1 \\ i\neq j}}^{k} E\{\omega^*(N; p+a_i)-t_i, T_i\} \cdot F\{\omega^*(N; p+a_j)-t_j, T_j\}$$
$$\cdot \prod_{i=k+1}^{k+l} E\{\omega^*(N; n+a_i)-t_i, T_i\}\Bigg]$$
$$+\sum_{j=k+1}^{k+l}\Bigg[\prod_{i=1}^{k} E\{w^*(N; p+a_i)-t_i, T_i\}$$
$$\cdot \prod_{\substack{i=k+1 \\ i\neq j}}^{k+l} E\{\omega^*(N; n+a_i)-t_i, T_i\} \cdot F\{\omega^*(N; n+a_j)-t_j, T_j\}\Bigg]$$
$$-(k+l-1)\prod_{i=1}^{k} E\{\omega^*(N; p+a_i)-t_i, T_i\} \cdot \prod_{i=k+1}^{k+l} E\{\omega^*(N; n+a_i)-t_i, T_i\} .$$

Now in the definition of $G(N; m_1, \cdots, m_{k+l})$, (3) and (4) hold simultaneously when and only when

$$(5) \qquad \omega^*(N; p+a_i)=t_i \quad \text{for} \quad i=1, \cdots, k\ ,$$

$$(6) \qquad \omega^*(N; n+a_i)=t_i \quad \text{for} \quad i=k+1, \cdots, k+l$$

hold simultaneously, and by Lemma 1, for such p and n, we have $\delta(p, n)=\delta'(p, n)=1$; when at least one of the equations (3) and (4) does not hold, or at least one of the equations (5) and (6) does not hold, we have $\delta(p, n)\geqq 0$ and $\delta'(p, n)\leqq 0$. It follows that

$$\sum_{i=1}^{k+l} \mathscr{K}^{(i)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})$$
$$-(k+l-1)\mathscr{K}^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})$$
$$\leqq G(N; m_1, \cdots, m_{k+l}) \leqq \mathscr{K}^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})\ .$$

Summing up thus obtained inequalities, we get the lemma.

We shall further introduce some functions. We put

$$H^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})$$
$$=\sum_{m_1\in M(t_1)} \cdots \sum_{m_{k+l}\in M(t_{k+l})} K^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})\ ,$$

$$K^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})$$
$$=\sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_{k+l}=0}^{2T_{k+l}} (-1)^{\tau_1+\cdots+\tau_{k+l}} L(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})\ ,$$

$$L(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})$$
$$=\sum_{\substack{\mu_1\in M(\tau_1)\\ (\mu_1, m_1)=1}} \cdots \sum_{\substack{\mu_{k+l}\in M(\tau_{k+l})\\ (\mu_{k+l}, m_{k+l})=1}}^{*} \frac{1}{\varphi(m_1\mu_1 \cdots m_{k+l}\mu_{k+l})}\ ,$$

where the asterisk attached to the summation symbols means that $m_1\mu_1, \cdots, m_{k+l}\mu_{k+l}$ are relatively prime in pairs, and $\varphi(m_1\mu_1\cdots m_{k+l}\mu_{k+l})$ is Euler's function of $m_1\mu_1\cdots m_{k+l}\mu_{k+l}$.

$$H_1^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})$$
$$=\sum_{m_1\in M(t_1)} \cdots \sum_{m_{k+l}\in M(t_{k+l})} K_1^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})\ ,$$

$$K_1^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})$$
$$=\sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_{k+l}=0}^{2T_{k+l}} (-1)^{\tau_1+\cdots+\tau_{k+l}} L_1(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})\ ,$$

$$L_1(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})$$
$$= \sum_{\substack{\mu_1 \in M(\tau_1) \\ (\mu_1, m_1)=1}} \cdots \sum^{*}_{\substack{\mu_{k+l} \in M(\tau_{k+l}) \\ (\mu_{k+l}, m_{k+l})=1}} \frac{1}{m_1\mu_1 \cdots m_{k+l}\mu_{k+l}},$$

where the asterisk has the same meaning as before, and, in each summand, $\varphi(m_1\mu_1 \cdots m_{k+l}\mu_{k+l})$ is replaced by $m_1\mu_1 \cdots m_{k+l}\mu_{k+l}$;

$$H_2^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})$$
$$= \sum_{m_1 \in M(t_1)} \cdots \sum_{m_{k+1} \in M(t_{k+l})} K_2^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l}),$$

$$K_2^{(0)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l})$$
$$= \sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_{k+l}=0}^{2T_{k+l}} (-1)^{\tau_1+\cdots+\tau_{k+l}} L_2(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l}),$$

$$L_2(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})$$
$$= \sum_{\substack{\mu_1 \in M(\tau_1) \\ (\mu_1, m_1)=1}} \cdots \sum^{**}_{\substack{\mu_{k+l} \in M(\tau_{k+l}) \\ (\mu_{k+l}, m_{k+l})=1}} \frac{1}{m_1\mu_1 \cdots m_{k+l}\mu_{k+l}},$$

where the double asterisks indicate that $m_1\mu_1, \cdots, m_{k+l}\mu_{k+l}$ are not coprime, or not relatively prime in pairs.

LEMMA 4. *Let* $t_i < 2y(N)$ *and* $T_i = [5y(N)]$ *for* $i = 1, \cdots, k+l$. *Then, as* $N \to \infty$,

$$H_1^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) + H_2^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})$$
$$= \frac{\{y(N)\}^{t_1+\cdots+t_{k+l}} e^{-(k+l)y(N)}}{t_1! \cdots t_{k+l}!} \{1 + o(1)\}$$

uniformly in $t_1, \cdots, t_{k+l}$.

PROOF. We have

$$H_1^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) + H_2^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})$$
$$= \prod_{i=1}^{k+l} \sum_{m_i \in M(t_i)} \frac{1}{m_i} \sum_{\tau_i=0}^{2T_i} (-1)^{\tau_i} \sum_{\substack{\mu_i \in M(\tau_i) \\ (\mu_i, m_i)=1}} \frac{1}{\mu_i}.$$

Hence it will suffice to show that

$$\sum_{m_i \in M(t_i)} \frac{1}{m_i} \sum_{\tau_i=0}^{2T_i} (-1)^{\tau_i} \sum_{\substack{\mu_i \in M(\tau_i) \\ (\mu_i, m_i)=1}} \frac{1}{\mu_i} = \frac{\{y(N)\}^{t_i} e^{-y(N)}}{t_i!} \{1 + o(1)\}$$

uniformly in $t_i < 2y(N)$ for each index i. The proof can be carried out similarly as [3], the proof of Lemma 6.

LEMMA 5. *Let $t_i<2y(N)$ and $T_i=[5y(N)]$ for $i=1,\cdots,k+l$. Then, as $N\to\infty$,*

$$H_2^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})=\frac{\{y(N)\}^{t_1+\cdots+t_{k+l}}e^{-(k+l)y(N)}}{t_1!\cdots t_{k+l}!}\cdot o(1)$$

uniformly in $t_1,\cdots,t_{k+l}$.

PROOF. Recalling the definition of $H_2^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})$, we have

$$(7)\qquad H_2^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})$$
$$\leqq\sum_{m_1\in M(t_1)}\cdots\sum_{m_{k+l}\in M(t_{k+l})}\sum_{\tau_1=0}^{2T_1}\cdots\sum_{\tau_{k+l}=0}^{2T_{k+l}}$$
$$\sum_{\substack{\mu_1\in M(\tau_1)\\(\mu_1,m_1)=1}}\cdots\sum^{**}_{\substack{\mu_{k+l}\in M(\tau_{k+l})\\(\mu_{k+l},m_{k+l})=1}}\frac{1}{m_1\mu_1\cdots m_{k+l}\mu_{k+l}},$$

where the double asterisks indicate that, for each summand, $m_1\mu_1,\cdots,m_{k+l}\mu_{k+l}$ are not relatively prime in pairs, or there exists a couple of indices i_1 and i_2 with $1\leqq i_1<i_2\leqq k+l$ such that $(m_{i_1}\mu_{i_1},m_{i_2}\mu_{i_2})>1$. If we put $d=(m_{i_1}\mu_{i_1},m_{i_2}\mu_{i_2})$, $m_{i_1}\mu_{i_1}=dm'_{i_1}\mu'_{i_1}$, $m'_{i_1}|m_{i_1}$, $\mu'_{i_1}|\mu_{i_1}$, $m_{i_2}\mu_{i_2}=dm'_{i_2}\mu'_{i_2}$, $m'_{i_2}|m_{i_2}$, $\mu'_{i_2}|\mu_{i_2}$, and $m'_i=m_i$, $\mu'_i=\mu_i$ for $i\neq i_1,i_2$, then

$$\frac{1}{m_1\mu_1\cdots m_{k+l}\mu_{k+l}}=\frac{1}{d^2}\cdot\frac{1}{m'_1\mu'_1\cdots m'_{k+l}\mu'_{k+l}}.$$

Hence rewriting the right-hand side of (7), we have

$$H_2^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})$$
$$\leqq\sum_d\frac{1}{d^2}\cdot\sum_{t'_1=0}^{t_1}\cdots\sum_{t'_{k+l}=0}^{t_{k+l}}\sum_{m'_1\in M(t'_1)}\cdots\sum_{m'_{k+l}\in M(t'_{k+l})}$$
$$\sum_{\tau'_1=0}^{2T_1}\cdots\sum_{\tau'_{k+l}=0}^{2T_{k+l}}\sum_{\mu'_1\in M(\tau'_1)}\cdots\sum_{\mu'_{k+l}\in M(\tau'_{k+l})}\frac{1}{m'_1\mu'_1\cdots m'_{k+l}\mu'_{k+l}},$$

which gives

$$(8)\qquad H_2^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})\leqq\sum_d\frac{1}{d^2}\cdot\Big(\sum_{t=0}^{\infty}\sum_{m\in M(t)}\frac{1}{m}\Big)^{2(k+l)},$$

where the summation-variable d runs through integers greater than 1, and each factor of d belongs to S, so that $d>\exp\{(\log\log N)^2\}$, hence

$$\sum_d\frac{1}{d^2}=O(e^{-(\log\log N)^2}).$$

Also

$$\sum_{t=0}^{\infty}\sum_{m\in M(t)}\frac{1}{m}\leqq\sum_{t=0}^{\infty}\frac{\{y(N)\}^t}{t!}=e^{y(N)}=O(e^{2\log\log N})\ .$$

Hence it follows from (8) that

$$H_2^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})=O(e^{2(k+l)\log\log N-(\log\log N)^2})\ .$$

Since

$$\frac{\{y(N)\}^{t_1\cdots+t_{k+l}}}{t!\cdots t_{k+l}!}>\frac{\{y(N)\}^{t_1+\cdots+t_{k+l}}}{\{2y(N)\}^{t_1+\cdots+t_{k+l}}}2^{-(t_1+\cdots+t_{k+l})}>e^{-2(k+l)y(N)}>e^{-4(k+l)\log\log N}\ ,$$

we can write

$$\begin{aligned}&H_2^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})\\&\quad=\frac{\{y(N)\}^{t_1+\cdots+t_{k+l}}}{t_1!\cdots t_{k+l}!}\cdot O(e^{6(k+l)\log\log N-(\log\log N)^2})\ ,\end{aligned}$$

which holds uniformly in $t_1,\cdots,t_{k+l}$.

LEMMA 6. *Let* $t_i<2y(N)$ *and* $T_i=[5y(N)]$ *for* $i=1,\cdots,k+l$. *Then, as* $N\to\infty$,

$$H_1^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})=\frac{\{y(N)\}^{t_1+\cdots+t_{k+l}}e^{-(k+l)y(N)}}{t_1!\cdots t_{k+l}!}\{1+o(1)\}$$

uniformly in $t_1,\cdots,t_{k+l}$.

PROOF. From Lemmas 4 and 5, we obtain the lemma.

LEMMA 7. *Let* $t_i<2y(N)$ *and* $T_i=[5y(N)]$ *for* $i=1,\cdots,k+l$. *Then, as* $N\to\infty$,

$$H^{(0)}(N;t_1,\cdots,t_{k+l};T_1,\cdots,T_{k+l})=\frac{\{y(N)\}^{t_1+\cdots+t_{k+l}}e^{-(k+l)y(N)}}{t_1!\cdots t_{k+l}!}\{1+o(1)\}$$

uniformly in $t_1,\cdots,t_{k+l}$.

PROOF. Since $\omega(m_1\mu_1\cdots m_{k+l}\mu_{k+l})<12(k+l)y(N)<24(k+l)\log\log N$, and each factor of $m_1\mu_1\cdots m_{k+l}\mu_{k+l}$ belongs to S,

$$\begin{aligned}\frac{1}{m_1\mu_1\cdots m_{k+l}\mu_{k+l}}&\leqq\frac{1}{\varphi(m_1\mu_1\cdots m_{k+l}\mu_{k+l})}\\&=\frac{1}{m_1\mu_1\cdots m_{k+l}\mu_{k+l}}\prod_{p\mid m_1\mu_1\cdots m_{k+l}\mu_{k+l}}\left(1-\frac{1}{p}\right)^{-1}\end{aligned}$$

$$\leqq \frac{1}{m_1\mu_1\cdots m_{k+l}\mu_{k+l}} \prod_{p|m_1\mu_1\cdots m_{k+l}\mu_{k+l}}\left(1+\frac{2}{p}\right)$$

$$\leqq \frac{1}{m_1\mu_1\cdots m_{k+l}\mu_{k+l}}(1+2e^{-(\log\log N)^2})^{24(k+l)\log\log N},$$

so that

$$\frac{1}{\varphi(m_1\mu_1\cdots m_{k+l}\mu_{k+l})}=\frac{1+o(1)}{m_1\mu_1\cdots m_{k+l}\mu_{k+l}}$$

uniformly in $m_1\mu_1, \cdots, m_{k+l}\mu_{k+l}$. From this and Lemma 6, we obtain the lemma.

Next we put

$$H^{(i)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) = \sum_{m_1\in M(t_1)}\cdots\sum_{m_{k+l}\in M(t_{k+l})} K^{(i)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l}),$$

$$K^{(i)}(N; m_1, \cdots, m_{k+l}; T_1, \cdots, T_{k+l}) = \sum_{\tau_1=0}^{2T_1}\cdots\sum_{\tau_i=0}^{2T_i+1}\cdots\sum_{\tau_{k+l}=0}^{2T_{k+l}}(-1)^{\tau_1+\cdots+\tau_{k+l}}L(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l}),$$

where the summation variable τ_i runs through the integers $0, \cdots, 2T_i+1$, and other τ_j's run through the integers $0, \cdots, 2T_j$ respectively.

LEMMA 8. *Let* $t_i<2y(N)$ *and* $T_i=[5y(N)]$ *for* $i=1, \cdots, k+l$. *Then, as* $N\to\infty$,

$$H^{(i)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})=\frac{\{y(N)\}^{t_1+\cdots+t_{k+l}}e^{-(k+l)y(N)}}{t_1!\cdots t_{k+l}!}\{1+o(1)\}$$

uniformly in $t_1, \cdots, t_{k+l}$.

PROOF. Similarly as Lemmas 4, 5, 6 and 7, we obtain the lemma.

In the proof of next lemma, we shall use Bombieri's theorem (cf. [1], [2]).

LEMMA 9. *Let* $t_i<2y(N)$ *and* $T_i=[5y(N)]$ *for* $i=1, \cdots, k+l$. *Then, as* $N\to\infty$,

$$\mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) - H^{(0)}(N-1; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})\mathrm{li}(N-1)=O\left(\frac{N}{\log^\alpha N}\right)$$

uniformly in $t_1, \cdots, t_{k+l}$, *where* $\mathrm{li}(N-1)$ *is the logarithmic integral of* $N-1$, *and* α *is an arbitrary positive constant.*

Proof. In the definition of $\mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})$, each summand of $\mathscr{L}(N; m_1, \cdots, m_{k+l}; \tau_1, \cdots, \tau_{k+l})$ has the form

$$\sum_{\substack{p+n=N \\ m_i\mu_i|(p+a_i)\ (i=1,\cdots,k) \\ m_i\mu_i|(n+a_i)\ (i=k+1,\cdots,k+l)}} 1 ; \tag{9}$$

the prime factors of $m_1\mu_1, \cdots, m_{k+l}\mu_{k+l}$ belong to the set S, and do not contain the prime factors of $D(N)$. It follows that $m_1\mu_1, \cdots, m_{k+l}\mu_{k+l}$ are relatively prime in pairs; in fact, if a prime p^* exists such that $p^*|(p+a_i)$ and $p^*|(p+a_j)$ with $1\leqq i<j\leqq k$, then $p^*|(a_i-a_j)$, and such p^* is excluded from the set S; if a prime p^* exists such that $p^*|(n+a_i)$ and $p^*|(n+a_j)$ with $k+1\leqq i<j\leqq k+l$, then $p^*|(a_i-a_j)$ and such p^* is excluded from the set S; if a prime p^* exists such that $p^*|(p+a_i)$ and $p^*|(n+a_j)$ with $1\leqq i\leqq k$ and $k+1\leqq j\leqq k+l$, then $p^*|(N+a_i+a_j)$, and such p^* is excluded from the set S.

Now the simultaneous congruences

$$p\equiv -a_i \pmod{m_i\mu_i}$$

for $i=1, \cdots, k$ with relatively prime moduli in pairs are equivalent to a congruence

$$p\equiv r_1 \pmod{m_1\mu_1\cdots m_k\mu_k} \tag{10}$$

with suitable r_1. Since the set S excludes the prime factors of $a_1, \cdots, a_k$, we have $(r_1, m_1\mu_1\cdots m_k\mu_k)=1$.

The simultaneous congruences

$$n\equiv -a_i \pmod{m_i\mu_i}$$

for $i=k+1, \cdots, k+l$ with relatively prime moduli in pairs, since $p+n=N$, may be rewritten as

$$p\equiv N+a_i \pmod{m_i\mu_i} .$$

These congruences are equivalent to a congruence

$$p\equiv r_2 \pmod{m_{k+1}\mu_{k+1}\cdots m_{k+l}\mu_{k+l}} . \tag{11}$$

Since the set S excludes the prime factors of $N+a_{k+1}, \cdots, N+a_{k+l}$, we have $(r_2, m_{k+1}\mu_{k+1}\cdots m_{k+l}\mu_{k+l})=1$.

The two congruences (10) and (11) are again equivalent to a single congruence

$$p\equiv r \pmod{m_1\mu_1\cdots m_{k+l}\mu_{k+l}} ,$$

where $(r, m_1\mu_1 \cdots m_{k+l}\mu_{k+l}) = 1$, and (9) may be written as $\pi(N-1; m_1\mu_1 \cdots m_{k+l}\mu_{k+l}, r)$.

It follows from the considerations on (9) that

$$
\begin{aligned}
(12) \quad & |\mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) \\
& \quad - H^{(0)}(N-1; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})\mathrm{li}(N-1)| \\
& \leqq \sum_{m_1 \in M(t_1)} \cdots \sum_{m_{k+l} \in M(t_{k+l})} \sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_{k+l}=0}^{2T_{k+l}} \sum_{\substack{\mu_1 \in M(\tau_1) \\ (\mu_1, m_1)=1}} \cdots \sideset{}{^*}\sum_{\substack{\mu_{k+l} \in M(\tau_{k+l}) \\ (\mu_{k+l}, m_{k+l})=1}} \\
& \quad \left| \pi(N-1; m_1\mu_1 \cdots m_{k+l}\mu_{k+l}, r) - \frac{\mathrm{li}(N-1)}{\varphi(m_1\mu_1 \cdots m_{k+l}\mu_{k+l})} \right| .
\end{aligned}
$$

In order to avoid the complexity caused by r, we replace each summand by

$$\max_{\rho} \left| \pi(N-1; m_1\mu_1 \cdots m_{k+l}\mu_{k+l}, \rho) - \frac{\mathrm{li}(N-1)}{\varphi(m_1\mu_1 \cdots m_{k+l}\mu_{k+l})} \right| ,$$

where ρ runs through a reduced residue system modulo $m_1\mu_1 \cdots m_{k+l}\mu_{k+l}$.

If we put $\nu = m_1\mu_1 \cdots m_{k+l}\mu_{k+l}$, then

$$\omega(\nu) < 12(k+l)y(N) < 24(k+l)\log\log N \,;$$

each prime factor of ν belongs to S, so that

$$\nu < (N^{(\log\log N)^{-2}})^{24(k+l)\log\log N} = N^{24(k+l)/\log\log N} ,$$

for each ν, the number of summands for which $m_1\mu_1 \cdots m_{k+l}\mu_{k+l} = \nu$ does not exceed $\{\omega(\nu)\}^{2(k+l)} < \{24(k+l)\log\log N\}^{2(k+l)}$.

Thus from (12) we obtain

$$
\begin{aligned}
(13) \quad & |\mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) \\
& \quad - H^{(0)}(N-1; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})\mathrm{li}(N-1)| \\
& \leqq \{24(k+l)\log\log N\}^{2(k+l)} \sum_{\nu} \max_{\rho} \left| \pi(N-1; \nu, \rho) - \frac{\mathrm{li}(N-1)}{\varphi(\nu)} \right| ,
\end{aligned}
$$

where ν runs through positive integers not exceeding $N^{24(k+l)/\log\log N}$. Now by Bombieri's theorem

$$\sum_{\nu} \max_{\rho} \left| \pi(N-1; \nu, \rho) - \frac{\mathrm{li}(N-1)}{\varphi(\nu)} \right| = O\left(\frac{N}{\log^{\alpha} N} \right)$$

with arbitrary positive constant α. Hence by (13) we obtain the lemma.

LEMMA 10. *Let* $t_i < 2y(N)$ *and* $T_i = [5y(N)]$ *for* $i = 1, \cdots, k+l$. *Then, as* $N \to \infty$,

$$\mathscr{H}^{(i)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) \\ -H^{(i)}(N-1; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})\mathrm{li}(N-1) = O\Big(\frac{N}{\log^\alpha N}\Big)$$

uniformly in $t_1, \cdots, t_{k+l}$, *where* α *is an arbitrary positive constant.*

Proof. Similarly as (12) and (13), we have

$$|\mathscr{H}^{(i)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) \\ -H^{(i)}(N-1; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})\mathrm{li}(N-1)| \\ \leqq \sum_{m_1 \in M(t_1)} \cdots \sum_{m_{k+l} \in M(t_{k+l})} \sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_i=0}^{2T_i+1} \cdots \sum_{\tau_{k+l}=0}^{2T_{k+l}} \sum_{\substack{\mu_1 \in M(\tau_1) \\ (\mu_1, m_1)=1}} \cdots \sum^*_{\substack{\mu_{k+l} \in M(\tau_{k+l}) \\ (\mu_{k+l}, m_{k+l})=1}} \\ \left|\pi(N-1; m_1\mu_1 \cdots m_{k+l}\mu_{k+l}, r) - \frac{\mathrm{li}(N-1)}{\varphi(m_1\mu_1 \cdots m_{k+l}\mu_{k+l})}\right| \\ \leqq \{24(k+l)\log\log N\}^{2(k+l)} \sum_\nu \max_\rho \left|\pi(N-1; \nu, \rho) - \frac{\mathrm{li}(N-1)}{\varphi(\nu)}\right| .$$

Applying again Bombieri's theorem, we obtain the lemma.

Lemma 11. *Let* $t_i < 2y(N)$ *and* $T_i = [5y(N)]$ *for* $i = 1, \cdots, k+l$. *Then, as* $N \to \infty$,

$$\mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) = \frac{N\{y(N)\}^{t_1+\cdots+t_{k+l}}e^{-(k+l)y(N)}}{t_1! \cdots t_{k+l}! \log N}\{1+o(1)\} ,$$

and

$$\mathscr{H}^{(i)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) = \frac{N\{y(N)\}^{t_1+\cdots+t_{k+l}}e^{-(k+l)y(N)}}{t_1! \cdots t_{k+l}! \log N}\{1+o(1)\}$$

for $i = 1, \cdots, k+l$ *uniformly in* $t_1, \cdots, t_{k+l}$.

Proof.

$$\frac{\{y(N)\}^{t_i}}{t_i!} > \Big(\frac{t_i}{2}\Big)^{t_i} \cdot \frac{1}{t_i^{t_i}} = 2^{-t_i} > e^{-2y(N)} > (\log N)^{-4}$$

for $i = 1, \cdots, k+l$. Hence by the preceding lemma we have

$$\mathscr{H}^{(0)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) \\ -H^{(0)}(N-1; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l})\mathrm{li}(N-1) \\ = O\left[\frac{N\{y(N)\}^{t_1+\cdots+t_{k+l}}(\log N)^{4(k+l)}}{t_1! \cdots t_{k+l}! \log^\alpha N}\right],$$

and similarly for

$$\mathscr{H}^{(i)}(N; t_1, \cdots, t_{k+l}; T_1, \cdots, T_{k+l}) .$$

Since $e^{y(N)} < e^{2\log\log N} = \log^2 N$ and α is arbitrary, from above formulas, Lemmas 7 and 8, we obtain the lemma.

LEMMA 12. *Let $t_i < 2y(N)$ for $i=1, \cdots, k+l$. Then, as $N \to \infty$,*

$$F(N; t_1, \cdots, t_{k+l}) = \frac{N\{y(N)\}^{t_1+\cdots+t_{k+l}} e^{-(k+l)y(N)}}{t_1! \cdots t_{k+l}! \log N}\{1+o(1)\}$$

uniformly in $t_1, \cdots, t_{k+l}$.

PROOF. From Lemmas 3 and 11, we obtain the lemma.

LEMMA 13. *Let $\alpha_i < \beta_i$, and let t_i be positive integers such that $t_i = y(N) + x_i\sqrt{y(N)}$ with $\alpha_i < x_i < \beta_i$ for $i=1, \cdots, k+l$. Then, as $N \to \infty$,*

$$F(N; t_1, \cdots, t_{k+l}) = \frac{N}{\log N}\{2\pi y(N)\}^{-(k+l)/2} e^{-(x_1^2+\cdots+x_{k+l}^2)/2}\{1+o(1)\}$$

uniformly in $t_1, \cdots, t_{k+l}$.

PROOF. In Stirling's formula

$$t! = \sqrt{2\pi} t^{t+1/2} e^{-t}\left\{1+O\left(\frac{1}{t}\right)\right\} ,$$

we put $t = t_i = y(N) + x_i\sqrt{y(N)}$, then easy calculations give

$$t_i! = \sqrt{2\pi}\{y(N)\}^{y(N)+k_i\sqrt{y(N)}+1/2} e^{-y(N)+x_2^2/2}\left\{1+O\left(\frac{1}{\sqrt{y(N)}}\right)\right\} ,$$

or

$$\frac{\{y(N)\}^{t_i} e^{-y(N)}}{t_i!} = \frac{e^{-x_2^2/2}}{\sqrt{2\pi y(N)}}\left\{1+O\left(\frac{1}{\sqrt{y(N)}}\right)\right\}$$

for $i=1, \cdots, k+l$. Multiplying thus obtained formulas, we have the lemma.

LEMMA 14. *Let $a_1, \cdots a_{k+l}, \alpha_1, \beta_1, \cdots, \alpha_{k+l}, \beta_{k+l}$ be as in the theorem. Let $A^*(N) = A^*(N; a_1, \cdots, a_{k+l}; \alpha_1, \beta_1, \cdots, \alpha_{k+l}, \beta_{k+l})$ denote the number of representations of N as the sum of the form $N = p + n$, where p is prime, and n is a positive integer such that*

$$y(N) + \alpha_i\sqrt{y(N)} < \omega^*(N; p+a_i) < y(N) + \beta_i\sqrt{y(N)}$$

for $i=1, \cdots, k$, and

$$y(N)+\alpha_i\sqrt{y(N)}<\omega^*(N;\ n+a_i)<y(N)+\beta_i\sqrt{y(N)}$$

for $i=k+1, \cdots, k+l$ *simultaneously. Then, as* $N\to\infty$, *we have*

$$A^*(N)\sim\frac{N}{\log N}\cdot\prod_{i=1}^{k+l}\Phi(\alpha_i,\ \beta_i)\ .$$

PROOF. By the definition of $F(N;\ t_1, \cdots, t_{k+l})$, we can write

$$A^*(N)=\sum_{t_1}\cdots\sum_{t_{k+l}}F(N;\ t_1, \cdots, t_{k+l})\ ,$$

the summation extending over the $k+l$ positive integers such that

$$y(N)+\alpha_i\sqrt{y(N)}<t_i<y(N)+\beta_i\sqrt{y(N)}$$

for $i=1, \cdots, k+l$; let these values of t_i be $t_{i\nu}=y(N)+x_{i\nu}\sqrt{y(N)}$ with $\nu=1, \cdots, s_i$; then

$$x_{i,\nu+1}-x_{i\nu}=\{y(N)\}^{-1/2}$$

for $\nu=1, \cdots, s_i-1$; thus by Lemma 13, we can write

$$A^*(N)=\{1+o(1)\}\frac{N}{\log N}\cdot(2\pi)^{-(k+l)/2}\prod_{i=1}^{k+1}\sum_{\nu=1}^{s_i-1}e^{-x_{i\nu}^2/2}(x_{i,\nu+1}-x_{i\nu})\ ,$$

which proves the lemma.

PROOF OF THE THEOREM. By the aid of Titchmarsh's theorem (cf. [1], [2]), we can transform the result for $A^*(N)$ to that for $A(N)$.

Let $S'=\{1, \cdots, 2N\}-S$, then, for $i=1, \cdots, k$, we have

$$\sum_{p+n=N}\{\omega(p+a_i)-\omega^*(N;\ p+a_i)\}$$
$$=\sum_{p+n=N}\ \sum_{\substack{p^*\mid(p+a_i)\\ p^*\in S'}}1=\sum_{p<N}\ \sum_{\substack{p^*\mid(p+a_i)\\ p^*\in S'}}1\leqq\sum_{\substack{p^*\leqq N+a_i\\ p^*\in S'}}\pi(N;\ p^*,\ -a_i)\ .$$

Since any positive number has at most one prime factor greater than the square root of itself, the last sum may be estimated by

$$\sum_{\substack{p^*\leqq\sqrt{2N}\\ p^*\in S'}}\pi(N;\ p^*,\ -a_i)+O\Big(\frac{N}{\log N}\Big)\ .$$

Here we quote Titchmarsh's theorem, then we have

$$\sum_{\substack{p^*\leqq\sqrt{2N}\\ p^*\in S'}}\pi(N;\ p^*,\ -a_i)=O\Big(\frac{N}{\log N}\cdot\sum_{p^*\in S'}\frac{1}{p^*}\Big)\ .$$

Let N be so large that the set $\{1, \cdots, 2N\}$ contains the set S and prime factors of $D(N)$. Then S' contains the prime factors of $D(N)$. But (2) holds. Hence, if we replace x in (1) by $2N$ and then by upper and lower limits of p^* in the definition of the set S, easy calculations will give

$$\sum_{p^* \in S'} \frac{1}{p^*} = O(\log\log\log N) .$$

Thus, for $i=1, \cdots, k$, we have

$$\sum_{p+n=N} \{\omega(p+a_i) - \omega^*(N; p+a_i)\} = o\Big(\frac{N}{\log N}\sqrt{y(N)}\Big) . \tag{14}$$

For $i=k+1, \cdots, k+l$, using again Titchmarsh's theorem, we also have

$$\begin{aligned}\sum_{p+n=N} \{\omega(n+a_i) - \omega^*(N; n+a_i)\} &= \sum_{p+n=N} \sum_{\substack{p^* \mid (n+a_i) \\ p^* \in S'}} 1 = \sum_{p<N} \sum_{\substack{p^* \mid (N-p+a_i) \\ p^* \in S'}} 1 \\ &\leqq \sum_{\substack{p^* \leqq N+a_i \\ p^* \in S'}} \pi(N; p^*, N+a_i) = \sum_{\substack{p^* \leqq \sqrt{2N} \\ p^* \in S'}} \pi(N; p^*, N+a_i) + O\Big(\frac{N}{\log N}\Big) \\ &= O\Big(\frac{N}{\log N} \sum_{p^* \in S'} \frac{1}{p^*}\Big) = O\Big(\frac{N}{\log N} \log\log\log N\Big) ,\end{aligned}$$

so that, for $i=k+1, \cdots, k+l$, we have

$$\sum_{p+n=N} \{\omega(n+a_i) - \omega^*(N; n+a_i)\} = o\Big(\frac{N}{\log N}\sqrt{y(N)}\Big) . \tag{15}$$

It follows from (14), (15) and Lemma 2 that, for any given $\varepsilon > 0$, we can take $N_1 = N_1(\varepsilon)$ so large that, for $N > N_1$,

$$\begin{aligned}&A^*(N; a_1, \cdots, a_{k+l}; \alpha_1+\varepsilon, \beta_1-\varepsilon, \cdots, \alpha_{k+l}+\varepsilon, \beta_{k+l}-\varepsilon) - \frac{\varepsilon N}{\log N} \\ &\quad < A(N; a_1, \cdots, a_{k+l}; \alpha_1, \beta_1, \cdots, \alpha_{k+l}, \beta_{k+l}) \\ &\quad < A^*(N; a_1, \cdots, a_{k+l}; \alpha_1-\varepsilon, \beta_1+\varepsilon, \cdots, \alpha_{k+l}-\varepsilon, \beta_{k+l}+\varepsilon) + \frac{\varepsilon N}{\log N} .\end{aligned}$$

From this and Lemma 14, we obtain

$$\begin{aligned}&\prod_{i=1}^{k+l} \Phi(\alpha_i+\varepsilon, \beta_i-\varepsilon) - \varepsilon \\ &\quad \leqq \liminf_{N\to\infty} \frac{A(N; a_1, \cdots, a_{k+l}; \alpha_1, \beta_1, \cdots, \alpha_{k+l}, \beta_{k+l}) \log N}{N}\end{aligned}$$

$$\leqq \limsup_{N\to\infty} \frac{A(N; a_1, \cdots, a_{k+l}; \alpha_1, \beta_1, \cdots, \alpha_{k+l}, \beta_{k+l})\log N}{N}$$

$$\leqq \prod_{i=1}^{k+l} \Phi(\alpha_i-\varepsilon, \beta_i+\varepsilon)+\varepsilon .$$

Since ε is an arbitrary positive number, this gives the theorem.

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Present Address:
DEPARTMENT OF MATHEMATICS, GAKUSHUIN UNIVERSITY
MEJIRO, TOSHIMA-KU, TOKYO 171, JAPAN