TOKYO J. MATH.
VOL. 12, NO. 2, 1989

On a Global Realization of a Discrete Series for $SU(n, 1)$ as Applications of Szegö Operator and Limits of Discrete Series

Takeshi KAWAZOE

Keio University

§1. Introduction.

In [KW] Knapp and Wallach gave an explicit imbedding of the discrete series of a connected semisimple Lie group G with finite center as a subrepresentation in the nonunitary principal series. However, it was in an infinitesimally equivalent fashion. Recently, when real rank of G is 1, Blank [B] gave an explicit projection operator that transfers a reducible unitary principal series onto a limit of discrete series in a global level. In this paper, applying Zuckerman's technique (see [Z]), we shall shift Blank's result and construct a representation of G which is infinitesimally equivalent to a discrete series. Then the unitarity of the representation corresponds to the square-integrability on G of the image of the Szegö operator, which was conjectured in [KW]. When $G=SU(n, 1)$, we shall obtain the square-integrability by applying the complex structure of the hermitian symmetric space G/K, and then we get a global construction of the discrete series.

This method is completely different from ordinary one, for it starts with a limit of discrete series. This implies that the representations constructed by our method must be attached to a limit of discrete series, and thus they are unfortunately a part of the discrete series of G (see §6). Square-integrability of the image of the Szegö operator is still an unsettled problem except for $G=SU(n, 1)$, however, all others obtained in this paper are valid for all real rank 1 semisimple Lie groups.

Let G be a connected semisimple Lie group with finite center and fix a maximal compact subgroup K of G. We assume that $\mathrm{rank}\, G=\mathrm{rank}\, K$, that is, G has a compact Cartan subgroup $T\subset K$. Then by Harish-Chandra [HC] this condition is equivalent with that G has a discrete series. Let $\mathfrak{t}$ be the Lie algebra of T and W_K the Weyl group of K. Then the set

Received December 24, 1988

of discrete series is in bijective correspondence with the set of W_K-orbits of non-singular integral forms on $\mathfrak{t}$. We denote by π_Λ the discrete series corresponding to a nonsingular integral form Λ on $\mathfrak{t}$.

Let $G=ANK$ be an Iwasawa decomposition of G and M the centralizer of A in K; let $\mathfrak{t}_c^*$ and $\mathfrak{a}_c^*$ be the dual spaces of the complexifications of $\mathfrak{t}$ and the Lie algebra $\mathfrak{a}$ of A respectively. Let $(\tau_\lambda, V_\lambda)$ $(\lambda \in \mathfrak{t}_c^*)$ be the lowest K-type of π_Λ and $(\sigma_\lambda, H_\lambda)$ the representation of M given by restricting $\tau_\lambda(M)$ to the M-cyclic subspace H_λ generated by the highest weight vector of V_λ. Let

$$\begin{aligned} C^\infty(K, \sigma_\lambda) &= \{f \in C^\infty(K, H_\lambda)\ ;\ f(mk)=\sigma_\lambda(m)f(k),\ m \in M,\ k \in K\}\ , \\ C^\infty(G, \tau_\lambda) &= \{f \in C^\infty(G, V_\lambda)\ ;\ f(kx)=\tau_\lambda(k)f(x),\ k \in K,\ x \in G\}\ . \end{aligned} \tag{1.1}$$

Then the discrete series π_Λ is realized on the L^2 kernel of the Schmid operator D on $C^\infty(G, \tau_\lambda)$ (see [Sc] and § 2 in [KW]) and the (non) unitary principal series $\pi_{\sigma_\lambda,\nu}$ $(\nu \in \mathfrak{a}_c^*)$ is realized on the space $C^\infty(K, \sigma_\lambda)$ as the compact picture (see § 2.1). According to the induced picture of $\pi_{\sigma_\lambda,\nu}$, each function $f \in C^\infty(K, \sigma_\lambda)$ can be extended to the function f on G by defining

$$f(ank)=e^{\nu(\log(a))}f(k) \qquad (a \in A,\ n \in N \text{ and } k \in K) \tag{1.2}$$

and this extension belongs to $C^\infty(G, \sigma_\lambda \times e^\nu)$. Then the Szegö map

$$S: C^\infty(K, \sigma_\lambda) \longrightarrow C^\infty(G, \tau_\lambda) \tag{1.3}$$

is defined by

$$S(f)(x)=\int_K \tau_\lambda(k)^{-1}f(kx)dk\ . \tag{1.4}$$

Knapp and Wallach in [KW] notice that the Szegö map S gives a relation between π_Λ and $\pi_{\sigma_\lambda,\nu}$; actually, for $\nu=\nu_\lambda \in \mathfrak{a}_c^*$ defined by λ (see (2.15 a) and (2.15 b)) S carries $C^\infty(K, \sigma_\lambda)$ into the kernel of the Schmid operator D on $C^\infty(G, \tau_\lambda)$ and moreover, $\pi_{\sigma_\lambda,\nu_\lambda}$ onto π_Λ in an infinitesimally equivariant fashion. Here "infinitesimally" means that the correspondence holds between K-finite vectors of the domain and the range of the mapping. Therefore, as conjectured in § 11 in [KW], it is worth realizing the discrete series π_Λ on the image of S without the K-finiteness assumption.

Now we assume that G has a simply connected complexification G_c and that G has real rank one. Then the above result can be extended to a singular integral form Λ such that $\langle \Lambda, \alpha_0 \rangle = 0$ for a noncompact simple root α_0 and $\langle \Lambda, \beta \rangle \neq 0$ for all other positive roots β. In this case we have two choices of the system of positive roots, we say Δ^+ and $\Delta^{+\prime} = \Delta^+ - \{\alpha_0\} \cup \{-\alpha_0\}$. Then we can define Szegö maps S and S' corresponding to Δ^+ and $\Delta^{+\prime}$ respectively (see [KW], § 12). Since ν_λ equals ρ, half the sum of the positive restricted roots with multiplicities, π_Λ corresponds to a limit of discrete series and $\pi_{\sigma_\lambda, \rho}$ to a reducible unitary principal series. In particular, $\pi_{\sigma_\lambda, \rho}$ is infinitesimally equivalent with the direct sum of the K-finite images of S and S', which give two irreducible constituents of the reducible principal series (see [KW], Theorem 12.6).

The boundary value map

$$L : \text{the image of } S \longrightarrow C^\infty(K, \sigma_\lambda) \tag{1.5}$$

is defined as follows (see § 2.2):

$$L(S(f))(k) = \lim_{a \to \infty} E(e^{\rho(\log(a))}(\pi_{\sigma_\lambda, \rho}(w^{-1}k)f)(a)) \qquad (k \in K) , \tag{1.6}$$

where E denotes the orthogonal projection from V_λ onto H_λ and w a representative of the nontrivial coset of the Weyl group W of A, which has order 2. Then in [B] Blank shows that in a G-equivariant fashion the composition map

$$L \circ S : C^\infty(K, \sigma_\lambda) \longrightarrow C^\infty(K, \sigma_\lambda) \tag{1.7}$$

is a projection operator and, as shown in [KS], it consists of a linear combination of the identity operator and a principal value operator (see [B] and § 2.3). In his method the K-finiteness assumption does not required. This means that, in a global fashion, the limit of discrete series π_Λ ($\nu_\lambda = \rho$) is realized on the image of $L \circ S$ equipped with the L^2-norm on K.

We retain all the assumptions on G. Our aim of this paper is to give a global, not infinitesimal, realization of a discrete series. As mentioned above, when π_Λ is a limit of discrete series ($\nu_\lambda = \rho$), the Szegö map $S: C^\infty(K, \sigma_\lambda) \to C^\infty(G, \tau_\lambda)$ gives a global realization of π_Λ by taking the boundary value. Therefore, if we can shift the realization of the limit of discrete series π_Λ to a discrete series, we can construct the discrete series in a global fashion; therefore, the discrete series we shall treat below must be attached to a limit of discrete series. In order to shift the realization of π_Λ, we shall apply Zuckerman's technique introduced in [Z], roughly speaking, we shall form a suitable projection of tensor products of π_Λ and a finite dimensional representation of G.

Let μ be a dominant integral form on $\mathfrak{t}$ and let (π, U) be a finite dimensional representation of G with lowest weight $-\mu$. Suppose that π satisfies some conditions related with the order of weights (see §3 and Theorem 4.6). Then a discrete series $\pi_{\Lambda+\mu}$ is realized as a subrepresentation in the nonunitary principal series:

$$\pi_{\Lambda+\mu}\subset(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}, C^{\infty}(K, \sigma_{\lambda-\mu})) . \tag{1.8}$$

Actually, first we take the tensor product of $\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}$ and π, and then define a map

$$C^{\infty}(K, \sigma_{\lambda-\mu})\longrightarrow C^{\infty}(G, \sigma_{\lambda}\times e^{\rho}\times\beta) , \tag{1.9}$$

where β is the restriction of π to MAN (see (3.1)); next we extract a component of $C^{\infty}(G, \sigma_{\lambda}\times e^{\rho}\times\beta)$, which is contained in $C^{\infty}(G, \sigma_{\lambda}\times e^{\rho})\cong C^{\infty}(K, \sigma_{\lambda})$, and we apply the Szegö maps S and S' on the component (see (3.3)). Combining these proceedings, we can define the G-equivariant operators

$$S_{\mu} \text{ and } S_{\mu}' : C^{\infty}(K, \sigma_{\lambda-\mu})\longrightarrow C^{\infty}(G, V_{\lambda}) \tag{1.10}$$

(see Proposition 3.2). Let $\Omega_{\lambda,\mu}$ be the kernel of S_{μ}' on $C^{\infty}(K, \sigma_{\lambda-\mu})$. Then $\Omega_{\lambda,\mu}$ is nontrivial, G-invariant and moreover, S_{μ} is injective on $\Omega_{\lambda,\mu}$ (see Lemmas 4.5 and 5.4). In their proofs we use the fact that the limit of discrete series π_{Λ} is realized in a global fashion. When G/K is hermitian; $G=SU(n, 1)$, we see that $S_{\mu}(\Omega_{\lambda,\mu})$ is contained in $L^{2}(G, V_{\lambda})$ (see Theorem 4.6). Therefore, inducing the L^{2} norm of $\Omega_{\lambda,\mu}$ from the one of the image $S_{\mu}(\Omega_{\lambda,\mu})$, we can obtain a unitary representation $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$. Finally, in Theorem 5.6 we show that the representation is irreducible and matrix coefficients are square-integrable on G, so $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$ is a discrete series of $G=SU(n, 1)$. This completes a global realization of a discrete series started with a limit of discrete series.

§2. Notation and preliminaries.

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Throughout this paper we assume that $\operatorname{rank} G=\operatorname{rank} K$, that G has a simply connected complexification G_{c}, and that real rank $G=1$.

Let $\mathfrak{g}$ be the Lie algebra of G. For a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ we denote the complexification and its dual space by $\mathfrak{u}_{c}$ and $\mathfrak{u}_{c}^{*}$ respectively. Let θ denote the Cartan involution of $\mathfrak{g}$ determined by K and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{t}\subset\mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$, Δ the root system of $(\mathfrak{g}_{c}, \mathfrak{t}_{c})$ and Δ_{n} (resp. Δ_{k}) the set of noncompact (resp. compact) roots of Δ. Root vectors E_α $(\alpha\in\Delta)$ can be selected in such a way that $B(E_\alpha, E_{-\alpha})=2\langle\alpha,\alpha\rangle^{-1}$ and $\theta(E_\alpha)^-=-E_{-\alpha}$, where bar denotes conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$ and B is the Killing form on $\mathfrak{g}_c$. Then $\alpha(H_\alpha)=2$ for $H_\alpha=[E_\alpha, E_{-\alpha}]$ (cf. [He], p. 155-156). We fix a noncompact simple root, say α_0, and let Δ^+ be the set of positive roots of Δ so that α_0 is positive. Put $\Delta_n{}^+=\Delta_n\cap\Delta^+$ and $\Delta_k{}^+=\Delta_k\cap\Delta^+$. Then $\mathfrak{a}=\boldsymbol{R}(E_{\alpha_0}+E_{-\alpha_0})$ is a maximal abelian subspace of $\mathfrak{p}$. Let $\mathfrak{h}^-$ denote a Cartan subalgebra of the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$. Then $\mathfrak{t}=\mathfrak{h}^-+i\boldsymbol{R}H_{\alpha_0}$ and $\mathfrak{h}=\mathfrak{h}^-+\mathfrak{a}$ is a noncompact Cartan subalgebra of $\mathfrak{g}$. Let $u=\exp\frac{1}{4}\pi(E_{\alpha_0}-E_{-\alpha_0})$. Then the standard Cayley transform relative to α_0 is given by $\mathrm{Ad}(u)$. It carries $\mathfrak{t}_c$ to $\mathfrak{h}_c$; in fact, Ad(u) acts trivially on $\mathfrak{h}_c{}^-$ and $\mathrm{Ad}(u)H_{\alpha_0}=-(E_{\alpha_0}+E_{-\alpha_0})$.

Let Ψ be the root system of $(\mathfrak{g}_c, \mathfrak{h}_c)$ and $\Psi_m\subset\Psi$ the root system of $(\mathfrak{m}_c, \mathfrak{h}_c{}^-)$. Let Ψ^+ be the set of positive roots of Ψ obtained by requiring that $\mathfrak{a}$ comes before $\mathfrak{h}^-$, and let $\Psi_m{}^+=\Psi_m\cap\Psi^+$. Then $\Psi^+=\{\gamma\circ\mathrm{Ad}(u)^{-1};\ \gamma\in S\subset\Delta\}$, where $S=\Psi_m{}^+\cup\{\gamma\in\Delta;\ \langle\gamma,\alpha_0\rangle<0\}$ (cf. [KW], Lemma 8.5). Let Σ denote the set of restricted roots of $(\mathfrak{g}_c, \mathfrak{a}_c)$ and let Σ^+ be the set of positive restricted roots obtained by requiring that $E_{\alpha_0}+E_{-\alpha_0}$ is contained in the positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$. Then the orderings defined by Δ^+, Ψ^+ and Σ^+ satisfy compatibility. Let δ, δ_n and δ_k be half the sum of the roots in Δ^+, $\Delta_n{}^+$ and $\Delta_k{}^+$ respectively, and let ρ be half the sum of the roots in Σ^+ with multiplicities.

Let A and N be the analytic subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$ respectively, where $\mathfrak{n}$ is the sum of positive restricted root spaces. Then an Iwasawa decomposition of G is given by $G=ANK$. Let M and M' be the centralizer and normalizer of A in K respectively and let $W=M'/M$. W has order 2; let w be a representative of the nontrivial coset. Then $G=MAN\cup MANwMAN$, and if we put $V=\theta(N)$, we see that $V=wNw^{-1}$ and $MAN\cap V=\{1\}$. Let "exp" denote the exponential mapping of $\mathfrak{a}$ onto A and "log" the inverse mapping. Then each element g in G and in the open dense subset $MANV$ of G respectively can be written as:

$$\begin{aligned} g&=\exp H(g)\cdot n(g)\cdot k(g) \qquad (H(g)\in\mathfrak{a},\ n(g)\in N,\ k(g)\in K),\\ &=m(g)\cdot a(g)\cdot n\cdot v(g) \qquad (m(g)\in M,\ a(g)\in A,\ n\in N,\ v(g)\in V)\,. \end{aligned} \tag{2.1}$$

We shall normalize Haar measures dk on K, dm on M and dv on V so that dk and dm have total mass 1 and dv satisfies $\int_V e^{2\rho H(v)}dv=1$. Let da denote the Haar measure on A that corresponds to a fixed Euclidean structure on $\mathfrak{g}$ under the exponential mapping. Then Haar measures dn on

N and dg on G respectively can be normalized by the integral formulas:

$$\int_N f(n)dn=\int_V f(wvw^{-1})dv$$

and (2.2)

$$\int_G f(g)dg=\int_A\int_N\int_K f(ank)e^{2\rho(\log(a))}dadndk$$

for integrable functions f on N and G respectively. Let $A^+=\exp(\mathfrak{a}^+)$. Then $G=KCL(A^+)K$ and there exists a continuous function $D(a)\geqq 0$ on A^+ such that

$$dg=D(a)dkdadk' , \tag{2.3a}$$

where $g=kak'\in KA^+K$, and

$$e^{2\rho(\log(a))}D(a)\leqq C \qquad \text{for} \quad a\in A^+ \tag{2.3b}$$

(cf. [He], pp. 381-382).

Let

$$\Delta^{+\prime}=s_0(\Delta^+)=(\Delta^+-\{\alpha_0\})\cup\{-\alpha_0\} , \tag{2.4}$$

where s_0 is the reflection with respect to α_0. Then $\Delta^{+\prime}$ is a new positive root system of $(\mathfrak{g}_c, \mathfrak{t}_c)$; since $E_{\alpha_0}+E_{-\alpha_0}=E_{-\alpha_0}+E_{\alpha_0}$, it follows that $\Delta^{+\prime}{}_k=\Delta^+{}_k$ and the corresponding Iwasawa decomposition is the same as before (cf. [KW], p. 198).

2.1. Non unitary principal series and intertwining operators. We shall recall three realizations: induced, compact and noncompact pictures of (non)unitary principal series representations $\pi_{\sigma,\nu}$ of G, where $\nu\in\mathfrak{a}_c{}^*$ and (σ, H) is a finite dimensional irreducible unitary representation of M (cf. [KS]). Then the representation space of $\pi_{\sigma,\nu}$ in each picture is respectively given by

$$C^\infty(G, \sigma\times e^\nu)=\{f\in C^\infty(G, H) \ ; \ f(mang)=\sigma(m)e^{\nu(\log(a))}f(g) , \quad man\in MAN,\ g\in G\} , \tag{2.5}$$
$$C^\infty(K, \sigma)=\{f\in C^\infty(K, H) \ ; \ f(mk)=\sigma(m)f(k),\ m\in M,\ k\in K\}$$

and $C^\infty(V, H)$; the action of $\pi_{\sigma,\nu}(g)$ $(g\in G)$ on each space is given by

$$\begin{aligned}
&\pi_{\sigma,\nu}(g)f(x)=f(xg) \qquad (x\in G) ,\\
&\pi_{\sigma,\nu}(g)f(k)=e^{\nu(H(kg))}f(k(kg)) \qquad (k\in K) ,\\
&\pi_{\sigma,\nu}(g)f(v)=\sigma(vg)e^{\nu(\log(vg))}f(v(vg)) \qquad (v\in V) ,
\end{aligned} \tag{2.6}$$

where σ and log are respectively extended to the operator and the function defined almost everywhere on G by letting

$$\sigma(manv)=\sigma(m) \quad \text{and} \quad \log(manv)=\log(a) \qquad (manv \in MANV) . \tag{2.7}$$

The intertwining operator between the induced picture and the compact one (resp. the noncompact one) is given by restricting $f \in C^\infty(G, \sigma\times e^\nu)$ to K (resp. to V) and conversely, the G-equivariant extension of $f \in C^\infty(K, \sigma)$ (resp. $f \in C^\infty(V, H)$) to an element in $C^\infty(G, \sigma\times e^\nu)$ is given by letting

$$\begin{aligned} &f(x)=e^{\nu(H(x))}f(k(x)) \\ &(\text{resp. } f(x)=\sigma(x)e^{\nu(\log(x))}f(v(x))) . \end{aligned} \tag{2.8}$$

Therefore, giving attention to the restriction and extension, we use the notation "$\pi_{\sigma,\nu}(g)f$" without distinguishing the three pictures.

Let $\nu=(1+z)\rho$ $(z \in \boldsymbol{C})$. If $z \in i\boldsymbol{R}$, then the L^2 norm with respect to the Haar measure on K is preserved by the action given in (2.6), so it determines a unitary structure of the representation $\pi_{\sigma,\nu}$. We put $w\sigma(m)=\sigma(wmw^{-1})$ $(m \in M)$. Then it follows from [KS], Proposition 20 that $\pi_{\sigma,\nu}$ is reducible if and only if (1) σ is equivalent with $w\sigma$, (2) $z=0$ and (3) the mean value of $\sigma(xw)^{-1}$ $(x \in G)$ equlas 0. Under the assumption on G, $\pi_{\sigma,\rho}$ is a reducible unitary principal series of G (cf. [KS], §16). Let $w\nu(H)=\nu(wH)$ $(H \in \mathfrak{a})$. If $\mathrm{Re}(z)>0$, then an intertwining operator $A(w, \sigma, z)$ between $\pi_{\sigma,\nu}$ and $\pi_{w\sigma,w\nu}$ is given by

$$A(w, \sigma, z)f(k)=\int_K e^{(1-z)\rho\log(k'w)}\sigma^{-1}(k'w)f(k'k)dk' \qquad (k \in K) \tag{2.9}$$

(see [KS], §9) and moreover, if $z=0$, intertwining operators between $\pi_{\sigma,\rho}$ and $\pi_{\sigma,\rho}$ are all of the form: aA_0+bI $(a, b \in \boldsymbol{C})$, where I is the identity operator and A_0 is the principal value operator given by

$$A_0f(k)=\int_K e^{\rho\log(k'w)}\sigma^{-1}(k'w)f(k'k)dk' \qquad (k \in K) \tag{2.10}$$

(see Corollary in [KS], p. 517).

2.2. Szegö map and boundary value map. For an integral Δ_k^+-dominant form $\lambda \in \mathfrak{t}_c^*$ let $(\tau_\lambda, V_\lambda)$ be an irreducible unitary representation of K with highest weight λ. Let ϕ_λ be a nonzero highest weight vector and H_λ the M-cyclic subspace of V_λ generated by ϕ_λ. Let $(\sigma_\lambda, H_\lambda)$ denote the representation of M given by restricting τ_λ to H_λ, and E_λ the orthogonal projection from V_λ onto H_λ. Then for $\eta \in \mathfrak{a}_c^*$ the Szegö map

$$S_{\eta,\lambda}:\ C^\infty(K,\sigma_\lambda)\longrightarrow C^\infty(G,\tau_\lambda) \tag{2.11}$$

is defined by

$$\begin{aligned} S_{\eta,\lambda}f(g)&=\int_K e^{\eta H(kg^{-1})}\tau_\lambda^{-1}(k(kg^{-1}))f(k)dk \\ &=\int_K \tau_\lambda(k^{-1})f(kg)dk\ , \end{aligned} \tag{2.12}$$

where in the second integral we denote by the same letter "f" the G-equivariant extension of $f\in C^\infty(K,\sigma_\lambda)$ to G according to the induced picture of $\pi_{\sigma_\lambda,\nu}$ with $\nu=2\rho-\eta$ (see (2.8) and [KW], Lemma 6.2). Then this map is G-equivariant. If we put emphasis on the dependence of $S_{\eta,\lambda}$ on the choice of the positive root system Δ^+, we use the notation "$S_{\eta,\lambda}(\Delta^+)$".

On the image of $S_{\eta,\lambda}$ a boundary value map

$$L_\eta\ :\ \text{image of } S_{\eta,\lambda}\longrightarrow C^\infty(K,\sigma_\lambda) \tag{2.13}$$

is defined by

$$L_\eta(S_{\eta,\lambda}(f))(k)=\lim_{a\to\infty}E_\lambda(e^{\eta(\log(a))}S_{\eta,\lambda}(\pi_{\sigma_\lambda,\nu}(w^{-1}k)f)(a))\ . \tag{2.14}$$

Then following [B] and [GTKS], we see that

THEOREM 2.1. *Let* $\nu=2\rho-\eta=(1+z)\rho$. *If* $\mathrm{Re}(z)>0$, *then*

$$L_\eta\circ S_{\eta,\lambda}=A(w,\sigma_\lambda,z)$$

(see (2.9)) *and* L_η *is* G-*equivariant.*

If $z=0$, $L_\rho\circ S_{\rho,\lambda}$ also can be defined by (2.14). On the other hand, $A(w,\sigma_\lambda,z)$ is not defined for $z=0$, because the integral (2.9) in the definition does not converge. However, as mentioned in 2.1, we know that the limiting case $z=0$ must be of the form aA_0+bI, so $L_\rho\circ S_{\rho,\lambda}$ is of the same form. This fact is directly investigated in [B].

THEOREM 2.2. L_ρ *transfers* $S_{\rho,\lambda}(L^2(K,\sigma_\lambda))$ *into* $L^2(K,\sigma_\lambda)$ *in a* G-*equivariant manner and* $L_\rho\circ S_{\rho,\lambda}$ *is the projection operator of the form* $a_\lambda I+A_0$, *where* A_0 *is given by* (2.10) *and* a_λ *is the constant given by* $E_\lambda\int_V e^{\rho H(v)}\tau_\lambda(k(v)w)^{-1}dv=a_\lambda I$.

2.3. Discrete series and limits of discrete series. Let us suppose that $\Lambda=\lambda-\delta_n+\delta_k$ is Δ^+-dominant, and that Λ is nonsingular or singular with respect to just one pair of roots $\pm\alpha_0$. Then, as shown by [HC] and [KO], if Λ is nonsingular, it corresponds to a discrete series, otherwise, to a limit of discrete series of G. Both of them we denote by π_Λ. Then by [Sc] we know that the lowest K-type of π_Λ is given by τ_λ.

We define η_λ and $\nu_\lambda \in \mathfrak{a}_c^*$ as follows:

$$\eta_\lambda(E_{\alpha_0}+E_{-\alpha_0})=\frac{2\langle\lambda+n_0\alpha_0, \alpha_0\rangle}{\langle\alpha_0, \alpha_0\rangle}, \tag{2.15a}$$

and

$$\nu_\lambda=2\rho-\eta_\lambda, \tag{2.15b}$$

where n_0 is the number of positive noncompact roots γ satisfying that γ is not strongly orthogonal to α_0 and $\gamma+\alpha_0 \in \Delta$ (see [KW], (6.5a), (6.5b)). Let

$$S_\lambda=S_\lambda(\Delta^+)=S_{\eta_\lambda,\lambda}(\Delta^+) \quad \text{and} \quad S_\lambda'=S_{s_0\lambda}(\Delta^{+\prime}). \tag{2.16}$$

Then by Theorems 1.1 and 12.6 in [KW] the Szegö maps S_λ and S_λ' give a relation between π_Λ and $\pi_{\sigma_\lambda,\nu_\lambda}$ as follows.

THEOREM 2.3. (1) *S_λ carries $C^\infty(K, \sigma_\lambda)$ into the kernel of the Schmid operator D* (see [Sc] and [KW], §2) *on $C^\infty(G, \tau_\lambda)$. Moreover, in a $\mathfrak{g}$-equivariant fashion it carries the K-finite vectors of $\pi_{\sigma_\lambda,\nu_\lambda}$ onto the K-finite vectors of π_Λ.* (2) *If $\eta_\lambda=\nu_\lambda=\rho$, then the reducible unitary principal series $\pi_{\sigma_\lambda,\rho}$ is infinitesimally equivalent with the direct sum of the K-finite images of S_λ and S_λ'.*

We note that Theorem 2.2 implies that, if $\eta_\lambda=\nu_\lambda=\rho$, the K-finite assumption in Theorem 2.3 is not necessary. Therefore, if we put

$$A_\lambda=L_\rho\circ S_\lambda \quad \text{and} \quad A_\lambda'=L_\rho\circ S_\lambda', \tag{2.17}$$

it follows from Theorems 2.2 and 2.3 that

$$A_\lambda+A_\lambda'=I, \tag{2.18}$$

where I is the identity operator on $C^\infty(K, \sigma_\lambda)$.

2.4. $G=SU(n, 1)$. We shall consider the case that G/K is hermitian, so $G=SU(n, 1)$ under the assumption that real rank of G is 1. Let

$$\mathfrak{p}^+=\sum_{\alpha\in\Delta_n^+}\mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{p}^-=\sum_{\alpha\in\Delta_n-\Delta_n^+}\mathfrak{g}_\alpha, \tag{2.19}$$

where $\mathfrak{g}_\alpha$ is the root space for α, and let P^+, P^- be the subgroups of G_c

corresponding to $\mathfrak{p}^+$, $\mathfrak{p}^-$ respectively. Then multiplication $P^-\times K_c\times P^+\to G_c$ is one to one, holomorphic, regular and there exists a bounded open subset $\Omega\subset P^+$ such that

$$P^-K_cP^+=P^-K_c\Omega\ . \tag{2.20}$$

Then G acts on Ω by holomorphic automorphism under the definition $z\cdot g=p^+(zg)$ ($z\in\Omega$, $g\in G$), where $p^+(\cdot)$ refers to the P^+ component of an element of $P^-M_cP^+$. Especially, $1\cdot g=1$ for $g\in P^-K_c$ and $G\cap P^-K_c=K$, so $\Omega=G/K$ (cf. [Kn], pp. 225-226). Let $a_t=\exp(t(E_{\alpha_0}+E_{-\alpha_0})/2)$ ($t\in\boldsymbol{R}$). Then we recall that

$$1\cdot a_t=\exp(\operatorname{th} t/2\, E_{\alpha_0}) \quad\text{and}\quad \lim_{t\to\infty}1\cdot a_t=\exp(E_{\alpha_0}) \quad (\text{say } \infty) \tag{2.21}$$

(see [Kn], Corollary in p. 229), $\infty\in\partial\Omega$, the boundary of Ω, and the action of G on Ω is holomorphically extended to $\partial\Omega$. Then since $\operatorname{Ad}(u)a_{-\log(\operatorname{th} t/2)}\in K_c$, we see that

$$1\cdot a_t=\infty\cdot\operatorname{Ad}(u)a_{-\log(\operatorname{th} t/2)}\ . \tag{2.22}$$

In what follows we shall abbreviate the symbols $1\cdot$ and $\infty\cdot$ when we denote functions on Ω and $\partial\Omega$ respectively.

Now let us suppose that $\eta_\lambda=\nu_\lambda=\rho$ and $A_\lambda'(f)\equiv 0$ for $f\in C^\infty(K,\sigma_\lambda)$. Then by (2.14) and (2.18) it follows that

$$\begin{aligned} f(k)&=A_\lambda(f)(k)=L_\rho\circ S_\lambda(f)(k)\\ &=\lim_{a\to\infty}e^{\rho(\log(a))}E_\lambda S_\lambda(f)(a_twk)\ . \end{aligned} \tag{2.23}$$

As shown in [B], a limit of (holomorphic) discrete series is realized on the image of $L_\rho\circ S_\lambda$ equipped with L^2 norm; so $A_\lambda'(f)\equiv 0$ implies that f has a "holomorphic" extension to Ω, which we denote by the same letter (cf. Theorem 12.6 in [KW], [JW] and [KO], §5). On the other hand, $S_\lambda(f)$ is in the kernel of the Schmid operator and thus, of the Dirac operator (cf. [KW], Proposition 3.1, Theorem 6.1 and [NO]). Therefore, (2.22) and (2.23) mean that

$$E_\lambda S_\lambda(f)(a_t)\ \sim\ e^{-\rho(\log(a))}f(\operatorname{Ad}(u)a_{-\log(\operatorname{th} t/2)}w^{-1}) \tag{2.24}$$

as t tends to ∞. Especially, noting the fact that A_λ is a projection operator, we can deduce from Lemma 3.15 in [B] and its proof that the right hand side of (3.40) in [B] also satisfies (2.24) and thus

$$\|S_\lambda(f)(a_t)\|\ \sim\ e^{-\rho(\log(a))}\|f(\operatorname{Ad}(u)a_{-\log(\operatorname{th} t/2)}w^{-1})\| \tag{2.25}$$

as t tends to ∞, where $\|\cdot\|$ denotes the norm of V_λ.

2.5. Orthonormal system of $L^2(K, \sigma)$. Let $K^\wedge$ (resp. $M^\wedge$) denote the set of the equivalence classes of irreducible unitary representations of K (resp. M). For $\tau \in K^\wedge$ and $\sigma \in M^\wedge$ let $[\tau; \sigma]$ denote the multiplicity of σ in the restriction $\tau|M$ of τ to M, and let $K_\sigma^\wedge = \{\tau \in K^\wedge; [\tau; \sigma] \neq 0\}$. In what follows, for simplicity, we suppose that $[\tau; \sigma]=1$ if it is not 0, because this restriction is easily removable. Then for $(\tau, V_\tau) \in K_\sigma^\wedge$ let $d_\tau = \dim \tau$ and let $e_1, e_2, \cdots, e_{d_\tau}$ denote an orthonormal basis of V_τ such that $\{e_i; 1 \leqq i \leqq d_\sigma\}$ ($d_\sigma = \dim \sigma$) is carried by $\tau|M$ according to σ. We put $I_\tau = \{1, 2, \cdots, d_\tau\}$ and $I_\sigma = \{1, 2, \cdots, d_\sigma\}$, and denote the matrix coefficients of τ by $\tau_{ij}(k) = (\tau(k)e_j, e_i)$ ($i, j \in I_\tau$, $k \in K$). Then we define functions on K by

$$\phi_{\tau,j}(k) = \sum_{i \in I_\sigma} \tau_{ij}(k) e_i \qquad (j \in I_\tau) \tag{2.26}$$

and let $\psi_{\tau,j} = (d_\tau/d_\sigma)^{1/2} \phi_{\tau,j}$.

LEMMA 2.4. *$\{\psi_{\tau,j}; j \in I_\tau, \tau \in K_\sigma^\wedge\}$ is a complete orthonormal basis of $L^2(K, \sigma)$.*

PROOF. Since

$$\begin{aligned}\phi_{\tau,j}(mk) &= \sum_{i \in I_\sigma} \tau_{ij}(mk) e_i \\ &= \sum_{i,p \in I_\sigma} \tau_{ip}(m) \tau_{pj}(k) e_i \\ &= \sigma(m) \phi_{\tau,j}(k) \qquad (m \in M,\ k \in K)\ ,\end{aligned}$$

and

$$\begin{aligned}(\phi_{\tau,j}, \phi_{\tau',j'}) &= \int_K \Big(\sum_{i \in I_\sigma} \tau_{ij}(k) e_i, \sum_{i' \in I_\sigma} \tau'_{i'j'}(k) e_{i'}\Big) dk \\ &= \sum_{i \in I_\sigma} \int_K \tau_{ij}(k) \tau'_{ij'}(k)^- dk \\ &= \delta_{\tau\tau'} \delta_{jj'} d_\sigma d_\tau^{-1}\ ,\end{aligned}$$

it follows that all $\psi_{\tau,j}$ belong to $L^2(K, \sigma)$ and they are orthonormal each other. Let f be an arbitrary function in $L^2(K, \sigma)$. Then by the Peter-Weyl theorem for $L^2(K)$ (cf. [Su], p. 19) f has a decomposition such as

$$f(k) = \sum_{\tau \in K_\sigma^\wedge} \sum_{i,j \in I_\tau} \sum_{p \in I_\sigma} a_{ijp} \tau_{ij}(k) e_p \qquad (k \in K)\ .$$

Then for $m \in M$

$$f(mk)=\sum_{\tau\in K_{\hat{G}}}\sum_{i,j,q\in I_\tau}\sum_{p\in I_\sigma}a_{ijp}\tau_{iq}(m)\tau_{qj}(k)e_p\ .$$

On the other hand, since f belongs to $L^2(K,\sigma)$, $f(mk)$ must equal

$$\sigma(m)f(k)=\sum_{\tau\in K_{\hat{G}}}\sum_{i',j'\in I_\tau}\sum_{r,s\in I_\sigma}a_{i'j's}\tau_{i'j'}(k)\tau_{rs}(m)e_r\ .$$

So, it follows that $q=s=i'\in I_\sigma$, $i=p\in I_\sigma$ and $a_{pjp}=a_{qjq}$ for all p, $q\in I_\sigma$. Therefore, if we let $a_j=a_{pjp}$,

$$\begin{aligned}f(k)&=\sum_{\tau\in K_{\hat{G}}}\sum_{j\in I_\tau}a_j\sum_{p\in I_\sigma}\tau_{pj}(k)e_p\\&=\sum_{\tau\in K_{\hat{G}}}\sum_{j\in I_\tau}a_j\phi_{\tau,j}(k)\ .\end{aligned}$$

This completes the proof of the lemma. Q.E.D.

§3. *G*-equivariant maps.

We fix a Δ_k^+-dominant integral form λ on $\mathfrak{t}_c$ such that $\Lambda=\lambda-\delta_n+\delta_k$ is Δ^+-dominant and $\eta_\lambda=\nu_\lambda=\rho$ (see (2.15 a, b)). Then $\langle\Lambda,\alpha_0\rangle=0$ and $\langle\Lambda,\beta\rangle\neq0$ for all other positive roots β. Especially, π_Λ is a limit of discrete series of G and $L_\rho\circ S_{\rho,\lambda}$: $L^2(K,\sigma_\lambda)\to L^2(K,\sigma_\lambda)$ is a projection operator (see Theorem 2.2).

Let μ be a Δ^+-dominant integral form on $\mathfrak{t}_c$ and (π,U) a finite dimensional representation of G with lowest weight $-\mu$. Let $d_\pi=\dim U$, $I_\pi=\{1,2,\cdots,d_\pi\}$ and $\mu_i^\sim$ $(i\in I_\pi)$ the weights of π relative to $(\mathfrak{t}_c,\Delta^+)$, that is repeated according to their multiplicities and arranged in increasing order relative to Δ^+; so, $\mu_1^\sim=-\mu$. Let $v_i^\sim$ denote a normalized weight vector corresponding to $\mu_i^\sim$. In the same way let μ_i $(i\in I_\pi)$ denote the weights of π relative to $(\mathfrak{h}_c,\Psi^+)$ that are arranged as above, and v_i $(i\in I_\pi)$ corresponding normalized weight vectors. Then, since $\mu_i^\sim\circ\mathrm{Ad}(u)^{-1}$ and $\pi(u)v_i^\sim$ are respectively a weight and its weight vector with respect to $(\mathfrak{h}_c,\Psi^+)$, we may assume that they coincide with one of, respectively, μ_j and v_j $(j\in I_\pi)$; so we can select $i_0\in I_\pi$ such that $\mu_{i_0}=\mu_1^\sim\circ\mathrm{Ad}(u)^{-1}$ and $v_{i_0}=\pi(u)v_1^\sim$. Since $w\in W$ acts as $+1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$ (see [Kn2], Lemma 4), each $w\mu_i(H)=\mu_i(wHw^{-1})$ $(H\in\mathfrak{h}_c)$ is also one of the weights of π, and thus w acts as a permutation of I_π such as $w\mu_i=\mu_{w(i)}$. Especially, if we denote the matrix coefficients of π by $\pi_{ij}(g)=(\pi(g)v_j,v_i)$ $(i,j\in I_\pi,\ g\in G)$, we see that $\pi_{ij}(wg)=\pi_{w(i)j}(g)$.

Now let us suppose that

$$\text{(A0)}\qquad \lambda-\mu\ \text{is}\ \Delta_k^+\text{-dominant}\ ,$$

and we shall construct a nontrivial G-equivariant map of $C^\infty(G,\sigma_{\lambda-\mu}\times e^{\nu(\lambda-\mu)})$

into $C^\infty(G, V_\lambda)$. Let ξ and β denote the representations $\sigma_\lambda \times e^\rho$ and $\pi|MAN$ respectively, of MAN. For $f \in C^\infty(G, \sigma_{\lambda-\mu} \times e^{\nu(\lambda-\mu)})$ we define

$$f^\sim(k) = \int_M \xi(m) \times \pi(m) \langle f(m^{-1}k), \phi_{\lambda-\mu} \rangle \phi_\lambda \times v_{i_0} dm . \tag{3.1}$$

Then $f^\sim \in C^\infty(K, \sigma_\lambda \times \pi|M)$ and we can extend it to the function on G so that $f^\sim \in C^\infty(G, \xi \times \pi)$ (see [KW], p. 193).

LEMMA 3.1. *The mapping that transfers f in $C^\infty(G, \sigma_{\lambda-\mu} \times e^{\nu(\lambda-\mu)})$ to $f^\sim$ in $C^\infty(K, \sigma_\lambda \times \pi|M)$ is injective.*

PROOF. Let $P_{\lambda-\mu}: V_\lambda \times U \to H_{\lambda-\mu}$ be a nonzero K-intertwining operator. Then by (10.14) in [KW] $P_{\lambda-\mu}(f^\sim(k)) = cf(k)$ $(k \in K)$ with $c \neq 0$, and thus the desired fact is clear. Q.E.D.

For $h \in C^\infty(K, \sigma_\lambda \times \pi|M)$ we define functions h_i by the expansion

$$\begin{aligned} h(g) &= \sum_i h_i(g) \times \pi(g) v_i \\ &= \sum_i [\sum_i h_i(g) \pi_{ji}(g)] v_j . \end{aligned} \tag{3.2}$$

Then each h_i belongs to $C^\infty(G, \sigma_\lambda \times e^\rho)$ and it is uniquely determined by the restriction $(h(k), \pi(k)v_i)$ on K. Here for $f \in C^\infty(G, \sigma_{\lambda-\mu} \times e^{\nu(\lambda-\mu)})$ and $j \in I_\pi$ we define

$$\begin{aligned} S^j_\mu f(g) &= \sum_i S_\lambda(f^\sim{}_i)(g) \pi_{ji}(g) \qquad (g \in G) , \\ A^j_\mu f(k) &= \sum_i A_\lambda(f^\sim{}_i)(k) \pi_{ji}(k) \qquad (k \in K) \end{aligned} \tag{3.3}$$

and also define S'^j_μ and A'^j_μ by replacing S_λ and A_λ with S_λ' and A_λ' respectively (see (2.16) and (2.17)). Then we see that

$$\begin{aligned} S^j_\mu, S'^j_\mu &: C^\infty(K, \sigma_{\lambda-\mu}) \longrightarrow C^\infty(G, V_\lambda) , \\ A^j_\mu, A'^j_\mu &: C^\infty(K, \sigma_{\lambda-\mu}) \longrightarrow C^\infty(K, H_\lambda) . \end{aligned} \tag{3.4}$$

PROPOSITION 3.2. *If $i_0 = d_\pi$, then all S^j_μ and S'^j_μ are G-equivariant.*

PROOF. For simplicity, we denote $\pi_{\sigma_{\lambda-\mu}, \nu_{\lambda-\mu}}(x)f$ by f_x, and let W_j $(j \in I_\pi)$ be the MAN cyclic subspace for $w_j = \phi_\lambda \times v_j$ in $H_\lambda \times U$ and put $U_{i_0+1} = \sum_{j > i_0} W_j$. Then by [KW], pp. 193–194, we see that

$$(f^\sim)(kx) \equiv (f_x)^\sim(k) \mod U_{i_0+1} \qquad (k \in K, x \in G) .$$

Therefore, since $i_0 = d_\pi$ by the hypothesis, it follows that

$$(f^{\sim})(kx)=(f_x)^{\sim}(k) . \tag{3.5}$$

On the other hand,

$$\begin{aligned}(f^{\sim})_x(g)&=f^{\sim}(gx)\\ &=\sum_i f^{\sim}_i(gx)\pi(gx)v_i\\ &=\sum_i\sum_j f^{\sim}_i(gx)\pi_{ji}(gx)v_j\\ &=\sum_{i,j,p} f^{\sim}_i(gx)\pi_{jp}(g)\pi_{pi}(x)v_j\\ &=\sum_j[\sum_i[\sum_p f^{\sim}_p(gx)\pi_{ip}(x)]\pi_{ji}(g)]v_j ,\end{aligned}$$

and thus,

$$(f^{\sim}_x)_i(g)=\sum_p f^{\sim}_p(gx)\pi_{ip}(x) . \tag{3.6}$$

Then by (3.3), (3.5) and (3.6) we see that

$$\begin{aligned}S^j_\mu(f_x)(g)&=\sum_i S_\lambda((f_x)^{\sim}_i)(g)\pi_{ji}(g)\\ &=\sum_i S_\lambda((f^{\sim}_x)_i)(g)\pi_{ji}(g)\\ &=\sum_{i,p} S_\lambda((f^{\sim}_p)_x)(g)\pi_{ip}(x)\pi_{ji}(g)\\ &=\sum_p S_\lambda(f^{\sim}_p)(gx)\pi_{ip}(gx)\end{aligned}$$

by the G-equivariance of S_λ and then

$$=S^j_\mu(f)(gx) .$$

So, we show that S^j_μ is G-equivariant. By the same way we can obtain that S'^j_μ ($j\in I_\pi$) are also G-equivariant. Q.E.D.

§4. Some properties of S^j_μ and A^j_μ.

We keep the notation in §2 and §3, and let f be in $C^\infty(K, \sigma_{\lambda-\mu})$.

LEMMA 4.1. *If $S^1_\mu(f)\equiv 0$ (resp. $S'^1_\mu(f)\equiv 0$), then $S^j_\mu(f)\equiv 0$ (resp. $S'^j_\mu(f)\equiv 0$) for all $j\in I_\pi$.*

PROOF. First we note that for $k\in K$ and $g\in G$

$$\begin{aligned}0=S^1_\mu(f)(kg)&=\sum_{i\in I_\pi} S_\lambda(f^{\sim}_i)(kg)\pi_{1i}(kg)\\ &=\sum_{i,j\in I_\pi}\tau_\lambda(k)S_\lambda(f^{\sim}_i)(g)\pi_{1j}(k)\pi_{ji}(g)\end{aligned}$$

and thus,

$$\sum_{j \in I_\pi} \pi_{1j}(k) S_\mu^j(f)(g) = 0 .$$

Here we recall that v_1 is a lowest weight vector of π and $\pi^*(x) = \pi(\theta(x))^{-1}$ $(x \in G)$. Therefore, it follows that

$$\pi_{1j}(x) = e^{\mu_1(\log H(x))} \pi_{1j}(k(x)) \qquad (x \in G) .$$

Then we can obtain that

$$\sum_{j \in I_\pi} \pi_{1j}(x) S_\mu^j(f)(g) = 0$$

for all x, $g \in G$. Since π is irreducible, the matrix coefficients $\pi_{1j}(x)$ $(x \in G)$ are linearly independent on G, and thus it easily follows that $S_\mu^j \equiv 0$ for all $j \in I_\pi$. Q.E.D.

LEMMA 4.2. *If $S_\mu^j(f) \equiv 0$ (resp. $S'^j_\mu(f) \equiv 0$), then $A_\mu^{w(j)}(f) \equiv 0$ (resp. $A'^{w(j)}_\mu(f) \equiv 0$).*

PROOF. We note that for $a \in A$ and $k \in K$

$$\begin{aligned} 0 = S_\mu^j(f)(aw^{-1}k) &= \sum_{i \in I_\pi} S_\lambda(f^\sim_i)(aw^{-1}k) \pi_{ji}(aw^{-1}k) \\ &= e^{\mu_j(\log(a))} \sum_{i \in I_\pi} S_\lambda(f^\sim_i)(aw^{-1}k) \pi_{w(j)i}(k) . \end{aligned}$$

Therefore, we see that

$$A_\mu^{w(j)}(f)(k) = \lim_{a \to \infty} e^{(\rho - \mu_j)(\log(a))} E_\lambda(S_\mu^j(f)(aw^{-1}k)) = 0 .$$

Q.E.D.

LEMMA 4.3. *If $A_\mu^j(f) \equiv 0$ (resp. $A'^j_\mu(f) \equiv 0$) for all $j \in I_\pi$, then $A_\lambda f^\sim_j \equiv 0$ (resp. $A_\lambda' f^\sim_j \equiv 0$) for all $j \in I_\pi$.*

PROOF. The assumption means that

$$\pi(k)(A_\lambda f^\sim_1(k), A_\lambda f^\sim_2(k), \cdots, A_\lambda f^\sim_{d_\pi}(k))^t \equiv 0 \qquad (k \in K) .$$

Then, applying $\pi(k)^{-1}$ to the both sides, we can obtain the desired result. Q.E.D.

LEMMA 4.4. *If $A_\mu^j(f) \equiv 0$ and $A'^j_\mu(f) \equiv 0$ for all $j \in I_\pi$, then $f \equiv 0$.*

PROOF. By Lemma 3.1 it is enough to show that $f^\sim \equiv 0$. It follows from (2.18), (3.2) and (3.3) that

$$f^{\sim}(k)=\sum_{i,j\in I_\pi} f^{\sim}_i(k)\pi_{ji}(k)v_j$$
$$=\sum_{j\in I_\pi}(A^j_\mu(f)(k)+A'^j_\mu(f)(k))v_j=0 .$$

Q.E.D.

Now let

$$\Omega'_{\lambda,\mu}=\{f\in C^\infty(K,\sigma_{\lambda-\mu})\ ;\ S'^j_\mu(f)\equiv 0 \text{ for all } j\in I_\pi\} . \tag{4.1}$$

Then we have the following

LEMMA 4.5. *$\Omega'_{\lambda,\mu}$ is G-invariant and S^1_μ is injective on $\Omega'_{\lambda,\mu}$.*

PROOF. This is clear from Proposition 3.2, Lemmas 4.1, 4.2 and 4.4. Q.E.D.

THEOREM 4.6. *Let $G=SU(n,1)$ and suppose that μ satisfies*
(A0) *$\lambda-\mu$ is Δ_k^+-dominant,*
(A1) $\langle\mu,\alpha_0\rangle>0$,
(A2) $i_0=d_\pi$.
Then $S^j_\mu(f)\in L^2(G,V_\lambda)$ for all $f\in\Omega'_{\lambda,\mu}$ and $j\in I_\pi$.

PROOF. Since

$$S^j_\mu(f)(kg)=\tau_\lambda(k)\sum_{p\in I_\pi}\pi_{jp}(k)S^p_\mu(f)(g) \qquad (k\in K,\ g\in G)$$

(cf. the proof of Lemma 4.1), it follows from (2.3 a) that

$$\|S^j_\mu(f)\|^2_{L^2(G,V_\lambda)}=\sum_{p\in I_\pi}\int_K\int_{CL(A^+)}\|S^p_\mu(f)(ak)\|^2D(a)dadk .$$

Therefore, by noting (2.3 b), to obtain the square-integrability it is enough to show that

$$\|S^p_\mu(f)(a_tk)\| \sim e^{-(\rho+\mu_{d_\pi})(\log(a_t))} \qquad (t\to\infty) ,$$

because $\langle\mu_{d_\pi},\alpha\rangle=\langle\mu,\alpha_0\rangle>0$ by (A1) and (A2) (see § 3). Here, for simplicity, we put $d=d_\pi$ and

$$\frac{\langle\mu_i,\alpha\rangle}{\langle\alpha,\alpha\rangle}=n_i ;$$

so $\mu_i(\log(a_t))=n_it$ and $n_{w(i)}=-n_i$. Then by Proposition 3.2 and (3.3) we see that

$$S^p_\mu(f)(a_tk)=S^p_\mu(f_k)(a_t)=S_\lambda((f_k^{\sim})_p)(a_t)e^{n_pt} .$$

Since f belongs to $\Omega'_{\lambda,\mu}$, it follows from Lemmas 4.2 and 4.3 that $A_\lambda'((f_k^\sim)_j)\equiv 0$ for all $j\in I_\pi$. Therefore, we can apply the asymptotic behavior (2.25) to $S_\lambda((f_k^\sim)_p)(a_t)$ and thus, as t tends to ∞, we see that for $r=\operatorname{th} t/2$

$$\begin{aligned}\|S_\lambda((f_k^\sim)_p)(a_t)\| &\sim e^{-\rho(\log(a_t))}\|(f_k^\sim)_p(\mathrm{Ad}(u)a_{-\log(r)}w^{-1})\| \\ &= e^{-\rho(\log(a_t))}\|((f_k^\sim)(\mathrm{Ad}(u)a_{-\log(r)}w^{-1}),\ \pi(\mathrm{Ad}(u)a_{-\log(r)}w^{-1})v_p)\|\end{aligned}$$

(see (3.2)), where we used the fact that $f_k^\sim$ also has a holomorphic extension to Ω which follows from the K-type decomposition of f in $\Omega'_{\lambda,\mu}$ (cf. Lemma 5.3 below). We note that $(f_k^\sim)(\mathrm{Ad}(u)a_{-\log(r)}w^{-1})\to f_k^\sim(w^{-1})$ $(t\to\infty)$ and $\langle\beta,\alpha_0\rangle=\beta(H_{\alpha_0})=0$ for all $\beta\in\Psi_m$ (cf. [He], pp. 221-224). Therefore, it follows from the definition (3.1) of $f_k^\sim$ and (A2) that

$$\|S_\lambda((f_k^\sim)_p)(a_t)\|\sim e^{-\rho(\log(a_t))}(v_d,\ \pi(\mathrm{Ad}(u)a_{-\log(r)})v_{w(p)})\ .$$

Now let (π_n, V_n) $(n\in \boldsymbol{N})$ denote the irreducible representation of $SL(2,\boldsymbol{C})$ with degree $n+1$, that is realized on the homogeneous polynomials of degree n in variables z_1 and z_2 (cf. §6 and [Su], p. 326). Here noting that H_{α_0} and $E_{\pm\alpha_0}$ generates a Lie algebra isomorphic to $\mathfrak{sl}(2,\boldsymbol{C})$, we may deduce that

$$\begin{aligned}(\pi_n(a_{-\log(r)})z_1{}^j z_2{}^{n-j},\ z_2{}^n) &= c(\operatorname{sh}(-\log(r)))^j(\operatorname{ch}(-\log(r)))^{n-j} \\ &\sim c(r^{-1}-r)^j \\ &\sim ce^{-jt}\ ,\end{aligned}$$

as t tends to ∞. Therefore, regarding π as a (reducible) representation of $\mathfrak{sl}(2,\boldsymbol{C})$, we can show that $(\pi(\mathrm{Ad}(u)a_{-\log(r)})v_i,\ v_d)$ $(r=\operatorname{th} t/2)$ equals 0 or behaves asymptotically like $e^{-(n_d-n_i)t}$ $(t\to\infty)$. Then

$$\|S_\lambda(f_k^\sim)_p(a_t)\|\sim e^{-\rho(\log(a_t))}e^{-(n_d+n_p)t}$$

and thus,

$$\|S_\mu^p(f)(a_tk)\|\sim e^{-(\rho+\mu_d)(\log(a_t))}\qquad (t\to\infty)\ .$$

This completes the proof of the theorem. Q.E.D.

§5. Main theorem.

We continue the notation in the previous section. Let $G=SU(n,1)$ and suppose that μ satisfies (A0), (A1) and (A2) in Theorem 4.6.

For $f\in\Omega'_{\lambda,\mu}$ let

$$\|f\|_{\lambda,\mu}=\|S_{\mu}^{1}(f)\|_{L^2(G,V_\lambda)} \tag{5.1}$$

(see Lemma 4.5 and Theorem 4.6), and let $\Omega_{\lambda,\mu}$ denote the completion of $\Omega'_{\lambda,\mu}$ with respect to this norm. Then by Proposition 3.2 and Lemma 4.5 we easily see that

LEMMA 5.1. *$\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}(g)$ $(g\in G)$ preserves $\|\cdot\|_{\lambda,\mu}$ and $\Omega_{\lambda,\mu}$ is G-invariant; so $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$ is a unitary representation of G.*

LEMMA 5.2. *Let H be a G-invariant closed subspace of $\Omega_{\lambda,\mu}$. If ψ_{τ,j_0} belongs to H for some $j_0\in I_\tau$ and $\tau\in K^{\wedge}_{\sigma_{\lambda-\mu}}$, then all $\psi_{\tau,j}$ $(j\in I_\tau)$ belong to H.*

PROOF. By the definition of $\psi_{\tau,j}$ we see that

$$\psi_{\tau,j_0}(k'k)=\sum_{j\in I_\tau}\tau_{jj_0}(k)\psi_{\tau,j}(k') \qquad (k,\ k'\in K) \tag{5.2}$$

and in particular,

$$\psi_{\tau,j}(k)=\int_K\psi_{\tau,j_0}(kk')\tau_{jj_0}(k')^{-}dk' . \tag{5.3}$$

Since H is G-invariant, $\psi_{\tau,j_0}(kk')$ also belongs to H as a function of k. Therefore, by the definition of the Riemann integral and the fact that H is closed, (5.3) means that $\psi_{\tau,j}\in H$ for all $j\in I_\tau$. Q.E.D.

We put

$$K^{\wedge}_{\lambda,\mu}=\{\tau_\xi\in K^{\wedge}_{\sigma_{\lambda-\mu}}\ ;\ \xi>s_0(\lambda)+\mu=\lambda-\alpha_0+\mu\} . \tag{5.4}$$

Then we see the following

LEMMA 5.3. *Let f be in $\Omega_{\lambda,\mu}$. Then f has a decomposition such as*

$$f=\sum a_{\tau,j}\psi_{\tau,j} ,$$

where $j\in I_\tau$ and $\tau\in K^{\wedge}_{\lambda,\mu}$.

PROOF. We shall give attention to the right K-type decomposition of f (see Lemma 2.4); it follows from (3.1) that f and $f^{\sim}$ have the same K-types which appear in their decompositions, and from (3.2) that $f^{\sim}$ and $f^{\sim}_{i}$ have the difference of the K-types of π. So $f^{\sim}_{i}$ and f have the difference of the K-types of π. Then the assumption implies that $S'^{j}_{\mu}(f)\equiv 0$ for all $j\in I_\pi$, and thus it follows from Lemmas 4.2 and 4.3 that $A_\lambda'(f^{\sim}_{i})\equiv 0$ for all $i\in I_\pi$. Therefore, the desired result follows from Theorem 12.6 in [KW]. Q.E.D.

LEMMA 5.4. *Let* $\tau=\tau_{\lambda+\mu}$. *Then* $\psi_{\tau,j}\in\Omega_{\lambda,\mu}$ *for all* $j\in I_\pi$, *and in particular,* $\Omega_{\lambda,\mu}\neq\{0\}$.

PROOF. By the same argument in the proof of Lemma 5.3 we see that highest weights of the K-types which appear in the decomposition of $(\psi_{\tau,j})^{\sim}_i$ $(i\in I_\pi)$ are greater than or equal to λ. Therefore, S_λ' vanishes all $(\psi_{\tau,j})^{\sim}_i$ (see [KW], Theorem 12.6). This means that $S'^i_\mu(\psi_{\tau,j})\equiv 0$ for all i, $j\in I_\pi$ and thus $\psi_{\tau,j}\in\Omega'_{\lambda,\mu}$ for all $j\in I_\pi$. Q.E.D.

LEMMA 5.5. S^1_μ *is injective on* $\Omega_{\lambda,\mu}$.

PROOF. This is clear from Lemma 4.5. Q.E.D.

THEOREM 5.6. *Let* $G=SU(n,1)$ *and suppose that a* Δ_k^+*-dominant integral form* λ *satisfies* $\eta_\lambda=\nu_\lambda=\rho$ *and a* Δ^+*-dominant integral form* μ *does* (A0), (A1) *and* (A2) *respectively. Then* $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}},\Omega_{\lambda,\mu})$ *is an irreducible unitary representation of* G, *whose matrix coefficients are square-integrable on* G.

PROOF. We obtained in Lemma 5.1 that $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}},\Omega_{\lambda,\mu})$ is a unitary representation of G, so we shall prove the irreducibility. Let H be a nonzero G-invariant, closed subspace of $\Omega_{\lambda,\mu}$ and let f be a nonzero element in H. Then by Lemma 5.5 there exists a point $g_0\in G$ for which $S^1_\mu(f)(g_0)\neq 0$. Since S^1_μ is G-equivariant (see Proposition 3.2) and H is G-invariant, by replacing f with f_{g_0}, we may assume that $S^1_\mu(f)(e)\neq 0$, that is,

$$
\begin{aligned}
S^1_\mu(f)(e)&=\sum_{i\in I_\pi} S_\lambda(f^{\sim}_i)(e)\pi_{1i}(e)\\
&=S_\lambda(f^{\sim}_1)(e)\\
&=\int_K \tau_\lambda(k)^{-1}f^{\sim}_1(k)dk\neq 0\ .
\end{aligned}\tag{5.5}
$$

Here we recall that f can be written as $f=\sum a_{\tau,j}\psi_{\tau,j}$, where $\tau\in \hat{K}_{\lambda,\mu}$ (see Lemma 5.3); so the highest weight of τ is greater than or equal to $\lambda+\mu$ and thus, by the same argument in the proof of Lemma 5.3 highest weights of K-types which appear in the decomposition of $f^{\sim}_1$ are greater than or equal to λ. Therefore, if we put $\tau_0=\tau_{\lambda+\mu}$, we see that

$$
\int_K \tau_\lambda(k)^{-1}(\psi_{\tau,j})^{\sim}_1(k)dk=0 \qquad (\tau\neq\tau_0)\ .
$$

Then (5.5) implies that $a_{\tau_0,j_0}\neq 0$ for some $j_0\in I_\pi$.

On the other hand, it follows that

$$\int_K \tau_{0j_0j_0}(k')^- f(kk')dk' = \sum_{\tau,j} a_{\tau,j} \int_K \tau_{0j_0j_0}(k')^- \sum_m \tau_{mj}(k')dk' \psi_{\tau,m}(k)$$
$$= a_{\tau_0,j_0}\psi_{\tau_0,j_0}(k) .$$

Here we recall that $a_{\tau_0,j_0} \neq 0$ and f is in a closed, G-invariant subspace H. Therefore, applying the proof of Lemma 5.2, we can deduce that $\psi_{\tau_0,j_0} \in H$ and thus, $\psi_{\tau_0,j} \in H$ for all $j \in I_\pi$ by Lemma 5.2. If $H \neq \Omega_{\lambda,\mu}$, by replacing H with the orthogonal complement H' of H in the above argument, we can also deduce that $\psi_{\tau_0,j} \in H'$ for all $j \in I_\pi$. This contradicts the fact that $H \cap H' = \{0\}$; so we see that $H = \Omega_{\lambda,\mu}$ and thus the representation is irreducible.

Now we shall consider the linear functional L on $\Omega_{\lambda,\mu}$ defined by

$$L(f) = \langle S^1_\mu(f)(e), e_1\rangle$$

for $f \in \Omega_{\lambda,\mu}$. Then there exists a ϕ in $\Omega_{\lambda,\mu}$ for which

$$\langle S^1_\mu(f)(e), e_1\rangle = (f, \phi) ,$$

and thus, by Proposition 3.2, it follows that

$$(\phi_x, \phi) = L(\phi_x) = \langle S^1_\mu(\phi_x)(e), e_1\rangle = \langle S^1_\mu(\phi)(x), e_1\rangle \qquad (x \in G) .$$

Therefore, the matrix coefficient (ϕ_x, ϕ) belongs to $L^2(G)$ (see Theorem 4.6). Since the representation is irreducible and unitary, it follows from Theorem 1 in [V], p. 435 that all matrix coefficients are square-integrable on G.

This completes the proof of the theorem. Q.E.D.

REMARK 5.7. If we start the argument with $\Delta^{+\prime}$ instead of Δ^+, we can obtain another class of the discrete series of G.

§6. Examples.

We shall apply Theorem 5.6 to the cases of $SU(1, 1)$ and $SU(2, 1)$, and check up on the representations $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$.

6.1. Let $SU(1, 1)$ be the subgroup of $SL(2, \boldsymbol{C})$ which leaves invariant the hermitian form $-|z_1|^2+|z_2|^2$. Then the discrete series of G is originally realized on the L^2 weighted Bergman space on the unit disc $D=\{z \in \boldsymbol{C};\ |z|<1\}$ (cf. [Su], p. 237); actually, let $m \in \frac{1}{2}\boldsymbol{Z}$ and $|m| \geqq 1$, then for $m \geqq 1$ the Bergman space $A_{2,m-1}(D)$ is defined by

$$A_{2,m-1}(D)=\Big\{F\colon D\to \boldsymbol{C}\,;\ F \text{ is holomorphic on } D \text{ and}$$

$$\|F\|_{2,m-1}=\left[\int_D |F(z)|^2(1-|z|^2)^{2m-2}dz\right]^{1/2}<\infty\Big\} \tag{6.1}$$

and for $m\leqq -1$, $A_{2,m-1}(D)$ is made up of conjugate holomorphic functions on D with finite norm, where we replace m by $|m|$. Let $T_m(g)$ $(g\in G)$ denote the operator on $A_{2,m-1}(D)$ defined by

$$\begin{aligned} &T_m(g)F(z)=J(g^{-1}, z)^{-2m}F(g^{-1}\cdot z) \qquad (m\geqq 1)\,,\\ &T_m(g)F(z)=[\operatorname{conj} J(g^{-1}, z)]^{-2|m|}F(g^{-1}\cdot z) \qquad (m\leqq -1)\,, \end{aligned} \tag{6.2}$$

where $J(g, z)=\beta^- z+\alpha^-$ and

$$g\cdot z=\frac{\alpha z+\beta}{\beta^- z+\alpha^-} \qquad \text{for} \quad g=\begin{bmatrix}\alpha & \beta\\ \beta^- & \alpha^-\end{bmatrix} \quad \text{and} \quad z\in D\,. \tag{6.3}$$

Then the representations $(T_m, A_{2,m-1}(D))$ $(m\in \frac{1}{2}\boldsymbol{Z}$ and $|m|\geqq 1)$ of G are irreducible and unitary. They are called the holomorphic and antiholomorphic discrete series, respectively for $m\geqq 1$ and for $m\leqq -1$; they exhaust the whole discrete series of G (cf. [Su], p. 290).

Let $\mu=\frac{1}{2}n\alpha_0$ $(n\in \boldsymbol{N})$ and V_n the vector space of all homogeneous polynomials of degree n in variables z_1 and z_2, and let $\pi_n(g)$ $(g\in G)$ denote the operator on V_n defined by

$$\pi_n(g)\phi(z)=\phi(z\cdot g)\,, \tag{6.4}$$

where $z\cdot g=(az_1+cz_2,\ bz_1+dz_2)$ for $g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $z=(z_1, z_2)$. Then (π_n, V_n) $(n\in \boldsymbol{N})$ is a finite dimensional representation of G with lowest weight $-\mu$; $d_{\pi_n}=\dim V_n=n+1$ and $\{v_j{}^\sim=[(j-1)!(n+1-j)!]^{-1/2}z_1{}^{j-1}z_2{}^{n+1-j};\ 1\leqq j\leqq n+1\}$ is the set of normalized weight vectors with respect to the compact Cartan subgroup $K=SO(2)$ of G (see § 3 and [Su], p. 326). Then we see that μ satisfies the conditions (A0), (A1) and (A2) in Theorem 5.6.

By comparing the infinitesimal characters and the lowest K-types, we see that the representation $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$ $(\lambda=\rho,\ \mu=\frac{1}{2}n\alpha_0)$ constructed in Theorem 5.6 is equivalent to the antiholomorphic discrete series $(T_m, A_{2,m-1}(D))$ for $m=-\frac{1}{2}(n+1)$. Therefore, there exists an intertwining operator between $\Omega_{\lambda,\mu}$ and $A_{2,m-1}(D)$. In fact, we can obtain the intertwining operator by applying the Fourier transform associated with a discrete series, which was investigated in [K] and [K2]; for $f\in L^2(G)$ the Fourier transform $F_m(f)$ associated with T_m is defined by

$$F_m(f)(z)=\int_G f(g)T_m(g^{-1})1(z)dg \qquad (z\in D)\ , \tag{6.5}$$

where 1 is the constant function on D taking the value 1. Some basic properties of F_m are summarized as follows. Let ψ be the normalized matrix coefficient of T_m corresponding to the lowest K-type of T_m. Then $F_m(f)=F_m(\psi*f)\in A_{2,m-1}(D)$ and $F_m\colon \psi*L^2(G)\to A_{2,m-1}(D)$ is bijective and norm preserving (see [K], Theorem 5.2). On the other hand, since $\dim\tau_\lambda=1$, it follows from Theorem 4.6 that $S_\mu^1(f)\in L^2(G)$ for $f\in\Omega_{\lambda,\mu}$. Therefore, we can obtain a composition map

$$F\circ S_\mu^1\ :\ \Omega_{\lambda,\mu}\longrightarrow A_{2,m-1}(D)\ , \tag{6.6}$$

and it is G-equivariant (see Proposition 3.2 and (6.5)).

THEOREM 6.1. *Let* $\lambda=\rho$, $\mu=\frac{1}{2}n\alpha_0$ *and* $m=-(n+1)/2$ $(n\geqq 1)$. *Then the G-equivariant map* $F_m\circ S_\mu^1$ *is an intertwining operator between* $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$ *and* $(T_m, A_{2,m-1}(D))$; *that is, it is bijective and*

$$c2^{-n}\|f\|_{\lambda,\mu}=n\|F_m\circ S_\mu^1(f)\|_{2,m-1} \qquad \textit{for} \quad f\in\Omega_{\lambda,\mu}\ ,$$

where c is a constant which does not depend on f and n.

Before giving the proof we note the following

LEMMA 6.2. *Let* $\pi=\pi_n$ *and* $C_j^n=\int_K \pi_{1j}(k_\theta)e^{-in\theta/2}d\theta$ $(1\leqq j\leqq n+1)$. *Then*

$$\sum_{i=1}^{n+1}|C_j^n|^2=2^{-n}\ .$$

PROOF. We note that u and $\mathrm{Ad}(u)$ (see § 2) are respectively given by

$$u=2^{-1/2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

and

$$\mathrm{Ad}(u)k_\theta=\begin{bmatrix}\cos\frac{1}{2}\theta & i\sin\frac{1}{2}\theta\\ i\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta\end{bmatrix}$$

for $k_\theta=\mathrm{diag}(e^{i\theta/2}, e^{-i\theta/2})$. Therefore, by substituting

$$\pi_{ij}(k_\theta)=(\pi(k_\theta)v_j,\ v_i)=(\pi(\mathrm{Ad}(u)k_\theta)v_j^\sim,\ v_i^\sim)\ ,$$

where $v_j^\sim=[(j-1)!\,(n+1-j)!\,]^{-1/2}z_1^{\,j-1}z_2^{\,n+1-j}$ $(1\leqq j\leqq n+1)$, we can obtain the desired result from combinatorial calculation. Q.E.D.

PROOF OF THEOREM 6.1. First we shall prove the equation of the norm. Since $\dim \tau_\lambda = 1$, it follows from the proof of Theorem 5.6 that for $f \in \Omega_{\lambda,\mu}$ $S^1_\mu(f)(x) = \langle S^1_\mu(f)(x), e_1\rangle$ $(x \in G)$ is a matrix coefficient of the discrete series T_m $(m = -\frac{1}{2}(n+1))$; so it is a linear combination of the normalized matrix coefficients of T_m (see (3.2) in [Ka]). In particular, it follows from Lemma 3.1 and Theorem 5.2 in [Ka] that

$$n\|F_m \circ S^1_\mu(f)\|_{2,m-1} = c\|\psi * S^1_\mu(f)\|_{L^2(G)} = c\|E_m(S^1_\mu(f))\|_{L^2(G)} ,$$

where $E_m(f)(x) = \int_K e^{im\theta/2} f(k_\theta x) d\theta$ $(x \in G)$. Here we note that

$$\begin{aligned} E_m(S^1_\mu(f))(x) &= \int_K e^{im\theta/2} \sum_{i \in I_\pi} S_\lambda(f^\sim{}_i)(k_\theta x)\pi_{1i}(k_\theta x) d\theta \\ &= \int_K e^{-in\theta/2} \sum_{i \in I_\pi} S_\lambda(f^\sim{}_i)(x) \sum_{j \in I_\pi} \pi_{1j}(k_\theta)\pi_{ji}(x) d\theta \\ &= \sum_{j \in I_\pi} C^n_j S_\mu{}^j(f)(x) . \end{aligned}$$

Therefore, as in the proof of Theorem 4.6 we can deduce that

$$\begin{aligned} \|E_m(S^1_\mu(f))\|_{L^2(G)} &= c_\pi \sum |C^n_j|^2 \sum \|S^j_\mu(f)\|_{L^2(G)} \\ &= c_\pi 2^{-n} \|f\|_{\lambda,\mu} \qquad \text{(by Lemma 6.2)} . \end{aligned}$$

This is nothing but the desired equation. Especially, $F_m \circ S^1_\mu$ is injective and the image is closed in $A_{2,m-1}(D)$. Since the map $F_m \circ S^1_\mu$ is G-equivariant, the image must be G-invariant. Therefore, noting the irreducibility of T_m we see that the image coincides with $A_{2,m-1}(D)$, so the surjectivity of $F_m \circ S^1_\mu$ is obtained.

This completes the proof of the theorem. Q.E.D.

REMARK 6.3. (1) The representation stated in Remark 5.7 corresponds to the holomorphic discrete series and Theorem 6.1 holds with $m = \frac{1}{2}(n+1) \geqq 1$.

(2) When $\mu = 0$ $(n = 0, m = \pm\frac{1}{2})$, Theorem 6.1 also holds if we replace $A_{2,m-1}(D)$ by the Hardy space $H^2(D)$ for $m = \frac{1}{2}$ and the conjugation for $m = -\frac{1}{2}$. In this case, $F_{\pm 1/2}$ are defined by using the limits of discrete series $T_{\pm 1/2}$ (cf. [Su], Chap. V, § 2). Especially, S^1_μ, and S'^1_μ coincide with S_ρ and S_ρ' respectively; so this case is nothing but the classical theory of the Szegö operator (cf. [Ru] and [Ra], p. 178).

(3) Let $G = SU(n, 1)$ and suppose that the lowest K-type of the discrete series $(\pi_{\sigma_{\lambda-\mu}, \nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$ is of one dimensional. Then it is possible

to generalize Theorem 6.1 as a relation between $\Omega_{\lambda,\mu}$ and the L^2 weighted Bergman space on G/K. Actually, by using the Fourier transform associated with a discrete series (see [K2]), we can obtain the generalization by the same argument as above.

6.2. Let $G=SU(2,1)$ be the subgroup of $SL(3, \boldsymbol{C})$ leaving the hermitian form $|z_1|^2+|z_2|^2-|z_3|^2$ invariant; $K=S(U(2)\times U(1))$ and

$$A=\left\{a_t=\begin{bmatrix}\mathrm{ch}\, t & & \mathrm{sh}\, t\\ & 1 & \\ \mathrm{sh}\, t & & \mathrm{ch}\, t\end{bmatrix};\ t\in \boldsymbol{R}\right\}. \tag{6.7}$$

Then $\mathfrak{g}_c=\mathfrak{sl}(3, \boldsymbol{C})=\{X\in M_{33}(\boldsymbol{C});\ \mathrm{tr}(X)=0\}$ and

$$\mathfrak{t}_c=\{T_{a,b}=\mathrm{diag}(a, b, c)\,;\ a+b+c=0,\ a, b, c\in \boldsymbol{C}\}\,. \tag{6.8}$$

Let Δ_0^+ be the positive root system of $(\mathfrak{g}_c, \mathfrak{t}_c)$ requiring that

$$\alpha(T_{1,0})>0 \qquad \text{for} \quad \alpha\in\Delta_0^+ \tag{6.9}$$

and let α_1, α_2 be the simple roots in Δ_0^+. Let Λ_1 and Λ_2 be the basic highest weights defined by

$$\Lambda_1=\frac{2\alpha_1+\alpha_2}{3}\,,\quad \Lambda_2=\frac{\alpha_1+2\alpha_2}{3}\,. \tag{6.10}$$

Then $2\langle\Lambda_i, \alpha_j\rangle/\langle\alpha_j, \alpha_j\rangle=\delta_{ij}$ $(1\leqq i, j\leqq 2)$; Λ_1 and Λ_2 span $\mathfrak{t}_c^*$.

As obtained in §7 in [W] each element in $G^\wedge$, the set of all equivalence classes of irreducible unitary representations of G, is parametrized as π_Λ, where $\Lambda=k_1\Lambda_1+k_2\Lambda_2$ $(k_1, k_2\in \boldsymbol{C})$. Actually, the discrete series and the limit of discrete series are parametrized by a pair of integers k_1 and k_2 satisfying the following conditions (see [W], pp. 183-184);

the holomorphic discrete series (HD):	$k_1+k_2<-2,\ k_1<0,\ k_2\geqq 0$
the antiholomorphic discrete series (AHD):	$k_1+k_2<-2,\ k_2<0,\ k_1\geqq 0$
the nonholomorphic discrete series (NHD):	$k_1+k_2<-2,\ k_1<-1,\ k_2<-1$
the limits of discrete series (LD1):	$k_1+k_2=-2,\ k_1>-1$
the limits of discrete series (LD2):	$k_1+k_2=-2,\ k_2>-1.$

Then $G^\wedge$ consists of the representations listed above combined with the irreducible unitary principal series, the extra representations and the trivial representation.

Now we shall check up on the representations $(\pi_{\sigma_{\lambda-\mu},\nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$ obtained in Theorem 5.6. First we replace the positive root system Δ_0^+ with

$$\Delta^+=\{-\alpha_1,\ \alpha_2,\ -\alpha_3\}=s_1s_2\Delta_0^+\ , \tag{6.11}$$

where s_i is the reflection in $\mathfrak{t}_c^*$ with respect to α_i $(1\leqq i\leqq 2)$. Then $\alpha_0=-\alpha_3$ is the positive noncompact simple root and

$$u_\alpha=\exp(\tfrac{1}{4}\pi(E_\alpha-E_{-\alpha}))=\sqrt{2}^{-1}\begin{bmatrix}1 & & -1\\ & \sqrt{2} & \\ 1 & & 1\end{bmatrix} \tag{6.12}$$

(see § 2). Therefore, the Cayley transform $\mathrm{Ad}(u_{\alpha_0})$ carries $\mathfrak{t}_c$ to

$$\mathfrak{h}_c=\left\{H_{u,v}=\begin{bmatrix}-u/2 & & v/2\\ & u & \\ v/2 & & -u/2\end{bmatrix};\ u,\ v\in C\right\}\ . \tag{6.13}$$

Actually,

$$\mathrm{Ad}(u_{\alpha_0})(T_{a,b})=H_{u,v}\ ;\qquad u=b,\ v=2a+b\ , \tag{6.14}$$

and if we put $\beta_i=\mathrm{Ad}(u_{\alpha_0})\alpha_i$ $(1\leqq i\leqq 3)$, we see that

$$\begin{aligned}\beta_1(H_{u,v})&=-3u+v\ ,\\ \beta_2(H_{u,v})&=3u+v\ ,\\ \beta_3(H_{u,v})&=2v\ .\end{aligned} \tag{6.15}$$

Therefore, the positive roots system Ψ^+ of $(\mathfrak{g}_c,\ \mathfrak{h}_c)$ defined in § 2 is given by

$$\Psi^+=\{\beta_1,\ \beta_2,\ \beta_3\}\ . \tag{6.16}$$

We note that the representation π_Λ in [W] corresponds to $\pi_{-\Lambda-\delta_k+\delta_n}$ in our notation, and then, $\lambda=-\Lambda$ (see § 2.3). Therefore, the limit of discrete series π_Λ $(\nu_\lambda=\rho)$ in § 2.3 corresponds to (LD1) in [W] because $\lambda=-(k_1\Lambda_1+k_2\Lambda_2)$ is dominant with respect to $\Delta_k^+=\{-\alpha_1\}$, so $k_1\geqq 0$, and $\nu_\lambda=\rho$ implies that $k_1+k_2=-2$.

Let $\pi=\pi_\mu$ be a finite dimensional representation of G with lowest weight $-\mu\in\mathfrak{t}_c^*$ with respect to Δ^+. In order to apply Theorem 5.6 to $SU(2,1)$ we have to determine the set of μ satisfying the conditions:

$$\begin{aligned}&\text{(A0)}\qquad \lambda-\mu \text{ is } \Delta_k^+\text{-dominant}\ ,\\ &\text{(A1)}\qquad \langle\mu,\ \alpha_0\rangle>0\ ,\\ &\text{(A2)}\qquad i_0=d_{\pi_\mu}\ .\end{aligned} \tag{6.17}$$

We recall that (A2) implies that $\mu_{i_0}=-\mu\circ\mathrm{Ad}(u_{\alpha_0})^{-1}\in\mathfrak{h}_c^*$ is the highest weight of π_μ with respect to Ψ^+ (see §3). Then, by the classification of finite dimensional representations of $\mathfrak{sl}(3, C)$ (cf. [AS], p. 1231), we see that μ satisfies (A2) if and only if

$$\mu = -m\Lambda_1 \qquad (m=0, 1, 2, \cdots)\,. \tag{6.18}$$

Suppose that μ is of this form. Then μ satisfies (A1) for $m>0$ and (A0) for $m \leqq k_1$ when $\lambda = -k_1\Lambda_1 - k_2\Lambda_2$, $k_1+k_2=-2$ and $k_1 \geqq 0$; so the set of μ satisfying (6.17) is given by

$$\{\mu = -m\Lambda_1;\ 1 \leqq m \leqq k_1\} \tag{6.19}$$

for the above λ. Therefore, we conclude that the representations $(\pi_{\sigma_{\lambda-\mu}, \nu_{\lambda-\mu}}, \Omega_{\lambda,\mu})$ correspond to the antiholomorphic discrete series with lowest K-type $\lambda+\mu$ ($\pi_{\Lambda+\mu}$ in [W]), and they exhaust the whole (AHD) in the list.

Similarly, if we start the argument with $\Delta^+ = s_2 s_1 \Delta_0^+$ instead of $s_1 s_2 \Delta_0^+$, we can obtain the holomorphic discrete series (HD) in the list (see Remark 5.7). However, we cannot obtain the nonholomorphic discrete series (NHD) in our method, because μ has to satisfy the condition (6.19).

References

[AS] J. P. ANTOINE and D. SPEISER, Characters of irreducible representations of simple groups, I, J. Math. Physics, **5** (1964), 1226–1234.

[B] B. BLANK, Boundary behavior of limits of discrete series representations of real rank one semisimple groups, Pacific J. Math., **122** (1986), 299–318.

[GTKS] J. E. GILBERT, P. A. TOMAS, R. A. KUNZE and R. J. STANTON, Higher gradients and representations of Lie groups, *Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. 2*, Wadsworth Math. Series, (1983), 416–436.

[HC] HARISH-CHANDRA, Discrete series for semisimple Lie groups II, Acta Math., **116** (1966), 1–111.

[He] S. HELGASON, *Differential Geometry and Symmetric Spaces*, Academic Press (1962).

[JW] K. JOHNSON and N. R. WALLACH, Composition series and intertwining operators for the spherical principal series, I, Trans. Amer. Math. Soc., **229** (1977), 137–173.

[K] T. KAWAZOE, Discrete part of L^p functions on $SU(1,1)$, Keio Res. Rep., **10** (1986).

[K2] T. KAWAZOE, A transform on classical bounded symmetric domains associated with a holomorphic discrete series, Tokyo J. Math., **12** (1989), 269–297.

[Kn] A. W. KNAPP, Bounded symmetric domains and holomorphic discrete series, Pure Appl. Math., **8** (1972), 211–246, Marcel Dekker.

[Kn2] A. W. KNAPP, Weyl group of a cuspidal parabolic, Ann. Sci. École Norm. Sup. (4), **8** (1975), 275–294.

[KO] A. W. KNAPP and K. OKAMOTO, Limits of holomorphic discrete series, J. Func. Anal., **9** (1972), 375–409.

[KS] A. W. KNAPP and E. M. STEIN, Intertwining operators for semisimple groups, Ann. of Math., **93** (1971), 489–578.

[KW] A. W. KNAPP and N. R. WALLACH, Szegö kernels associated with discrete series, Invent. Math., **34** (1976), 163–200.

[NO] M. S. NARASIMHAN and K. OKAMOTO, An analogue of the Borel-Weil-Bott theorem for hermitian symmetric pairs of non-compact type, Ann. of Math., **91** (1970), 486–511.

[Ra] R. M. RANGE, *Holomorphic Functions and Integral Representations in Several Complex Variables*, GTM, **108** (1986), Springer-Verlag.

[Ru] W. RUDIN, *Function Theory in the Unit Ball of* C^n, GmW, **241** (1980), Springer-Verlag.

[Sc] W. SCHMID, On the realization of the discrete series of a semisimple Lie group, Rice Univ. Studies, **56**, No. 2 (1970), 99–108.

[Su] M. SUGIURA, *Unitary Representations and Harmonic Analysis*, Wiley (1975).

[V] V. S. VARADARAJAN, *Harmonic Analysis on Real Reductive Groups*, Lecture Notes in Math., **576** (1977), Springer-Verlag.

[W] N. R. WALLACH, On the Selberg trace formula in the case of compact quotient, Bull. Amer. Math. Soc., **87** (1976), 171–195.

[Z] G. ZUCKERMAN, Tensor products of finite and infinite dimensional representations of semisimple Lie groups, Ann. of Math., **106** (1977), 265–308.

Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA 223, JAPAN