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A Partial Order of Links

Kouki TANIYAMA

Waseda University

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Abstract. In the previous paper [5] we introduced a new partial order of knots and links, and studied about knots. In this paper we study the partial order of two-component links.

§1. Introduction.

Throughout this paper we work in the piecewise linear category. We use the same definitions, terminology and notation as in [5]. For the definitions of the other standard terms in knot theory, we refer to [4] and [1].

We say that a link L_1 *majorizes* a link L_2, denoted by $L_1 \geqq L_2$, if every projection of L_1 is a projection of L_2, where a projection has no over/under crossing information.

A *mutual crossing point* of a link projection is a crossing point of different components. A *self-crossing point* of a link projection is a crossing point which is not a mutual crossing point.

DEFINITION 1. For a two-component link L, we define the *mutual crossing number* of L, denoted by $\mu(L)$, to be the minimum number of mutual crossing points among all projections of L. We note that $\mu(L)$ is an even number and $\mu(L) \geqq 2|lk(L)|$ where $|lk(L)|$ denotes the absolute value of the linking number of L.

DEFINITION 2. A link L is *prime* if every 2-sphere in S^3 which intersects with L transversally at two points bounds an unknotted ball pair (B^3, B^1).

In this paper we show the following results.

THEOREM 1. *A two-component link L majorizes the Hopf link if*

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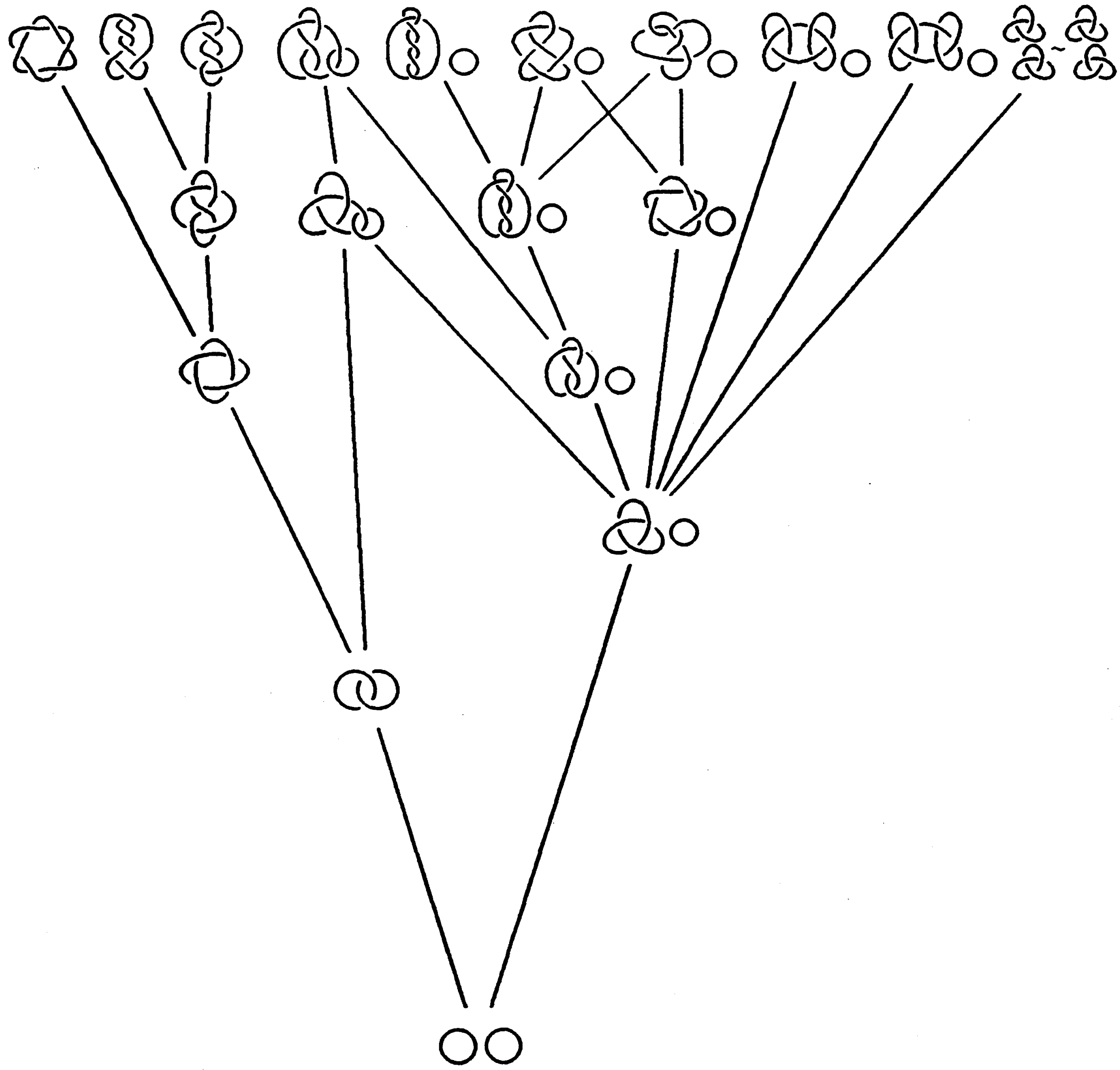


FIGURE 1-1

and only if L is nonsplittable.

THEOREM 2. *A two-component link L majorizes the* (2, 4)*-torus link if and only if* $\mu(L)\geqq 4$.

THEOREM 3. *For a two-component link L the following* (1) *and* (2) *are equivalent*:

(1) *The link L majorizes the Whitehead link.*

(2) *The link L has a two-component prime factor which is not equivalent to any of the* (2, p)*-torus links with* $p\geqq 0$.

THEOREM 4. *For any even number n, there exists an even number m such that if a two-component link L satisfies the inequality* $\mu(L)\geqq m$, *then L majorizes the* (2, n)*-torus link.*

Summarizing the results in this paper and in the previous paper [5], we have a part of the Hasse diagram of $(\mathfrak{L}^2/\sim, \leqq)$ within six-crossings in Fig. 1-1. Here $\mathfrak{L}^2$ means the set of all two-component links, and "$\sim$" means the natural equivalence relation which turns the preordered set $(\mathfrak{L}^2, \leqq)$ into the partially ordered set $(\mathfrak{L}^2/\sim, \leqq)$. Namely, we define the relation $L_1 \sim L_2$ for $L_1, L_2 \in \mathfrak{L}^2$ if $L_1 \leqq L_2$ and $L_1 \geqq L_2$ (see Remark 1 in [5]).

§2. Proof of Theorem 1 and Theorem 2.

PROOF OF THEOREM 1. If L majorizes the Hopf link, then L is nonsplittable since a split link cannot majorize a nonsplittable link. Suppose that L is nonsplittable. Then any projection of L has mutual crossing points. Choose one of them and take a sufficiently small ε-neighbourhood of the point in S^2. By applying Lemma 4 in [5] to the complementary tangle projection of the ε-neighbourhood, we obtain the Hopf link as desired. See Fig. 2-1. $\square$

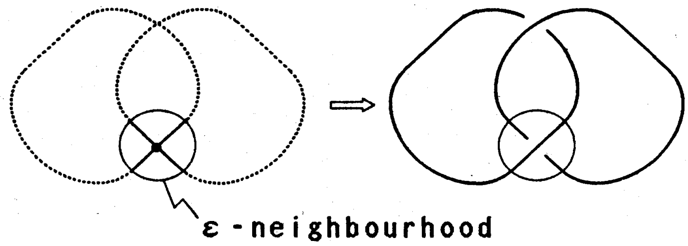


FIGURE 2-1

PROOF OF THEOREM 2. In general, if two-component links L_1 and L_2 satisfy $L_1 \leqq L_2$, then $\mu(L_1) \leqq \mu(L_2)$. Therefore if L majorizes the

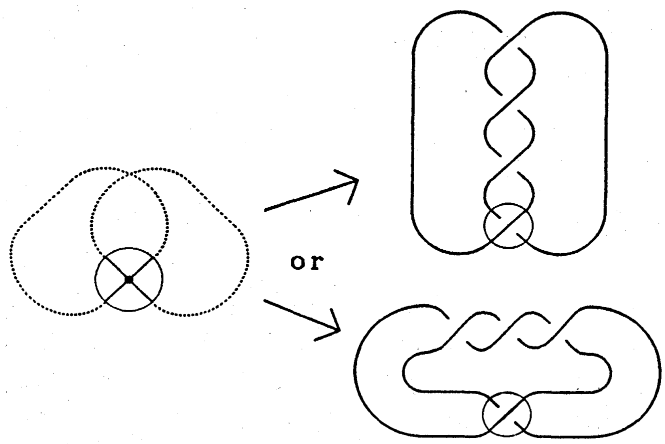


FIGURE 2-2

(2, 4)-torus link, then $\mu(L) \geqq 4$. Suppose that $\mu(L) \geqq 4$. Then Lemma 6 in [5] assures that L majorizes the (2, 4)-torus link. See Fig. 2-2. □

COROLLARY 1. *Let L be a two-component nonsplittable link which has a two-component prime factor that is not equivalent to the Hopf link, then L majorizes the* (2, 4)*-torus link.*

PROOF. We show that $\mu(L) \geqq 4$. Suppose that L has a diagram $\tilde{L}$ with just two mutual crossing points. Then L has the Hopf link as the two-component prime factor. See Fig. 2-3. But the prime decomposition of a link is unique (see [2]), this is a contradiction. □

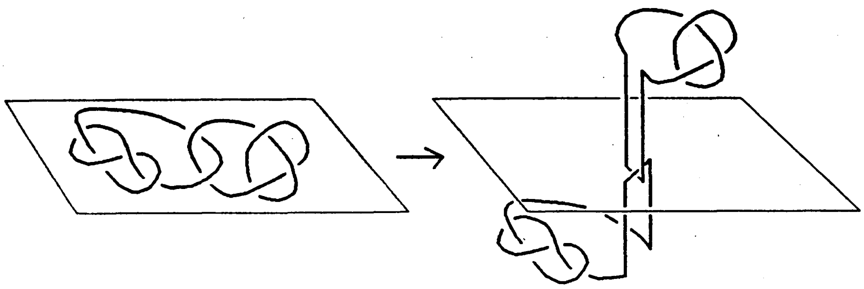


FIGURE 2-3

DEFINITION 3. For a two-component link L, we say that L is a *Hopf link sum* of knots K_1 and K_2, if L is obtained by the connected sum of the Hopf link, K_1 and K_2 where K_1 and K_2 are attached to the different components of the Hopf link. We note that a Hopf link sum of knots has at most four possibilities according to the attaching orientations of knots and ambient spaces.

PROPOSITION 1. (1) *Let $L = K_1 \cup K_2$ be a two-component nonsplittable link. If the knots K_1 and K_2 majorize the knots K_3 and K_4 respectively, then L majorizes any Hopf link sum of K_3 and K_4.*

(2) *Let $L = K_1 \cup K_2$ be a two-component link. If the knots K_1 and K_2 majorize the knots K_3 and K_4 respectively, then L majorizes any split sum of K_3 and K_4.*

PROOF. (1) Let $\hat{L} = \hat{K}_1 \cup \hat{K}_2$ be a projection of L. We choose an arbitrary mutual crossing point P of $\hat{L}$. We add over/under information at P as $\hat{K}_1$ is over $\hat{K}_2$. For the other mutual crossing points, we add over/under information as $\hat{K}_1$ is under $\hat{K}_2$. For the self-crossing points of $\hat{K}_1$ and $\hat{K}_2$, we add over/under information as the resultant knot diagrams $\tilde{K}_3$ and $\tilde{K}_4$ represent the knots K_3 and K_4 respectively. Then the resultant link diagram $\tilde{L} = \tilde{K}_3 \cup \tilde{K}_4$ represents a Hopf link sum of K_3 and K_4. To produce the other Hopf link sums from $\tilde{L}$, we change over/under information at all self-crossing points of $\tilde{K}_3$, or at all mutual crossing points of $\tilde{L}$, or at all self-crossing points of $\tilde{K}_3$ and at all mutual crossing points of $\tilde{L}$.

(2) The proof is similar to that of (1). □

§ 3. Proof of Theorem 3.

It is easily seen that for any even number p, the two-component link obtained by a connected sum of the $(2, p)$-torus link and several knots cannot majorize the Whitehead link. Therefore the condition (2) in Theorem 3 is necessary for the condition (1). We show the converse in this section.

LEMMA 1. *Let $\hat{L}=\hat{K}_1 \cup \hat{K}_2$ be a two-component link projection on S^2. If there exists a tear drop disk δ with $\partial\delta \subset \hat{K}_1$ together with a successive simple arc $A \subset \hat{K}_1$ as in Fig. 3-1 such that both $\partial\delta \cap \hat{K}_2$ and $A \cap \hat{K}_2$ are nonempty, then* LINK$(\hat{L})$ *contains the Whitehead link. Here* LINK$(\hat{L})$ *is the set of all links which have $\hat{L}$ as their projection.*

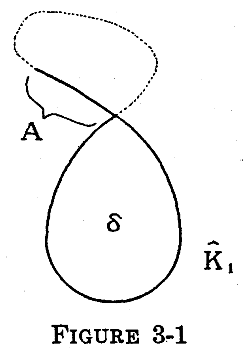
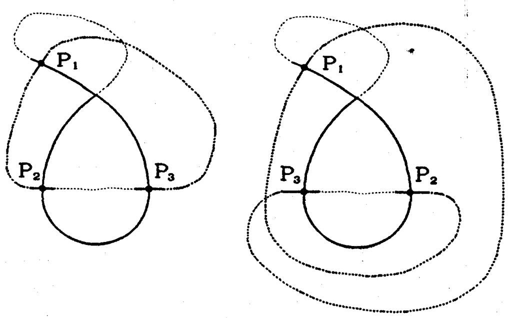


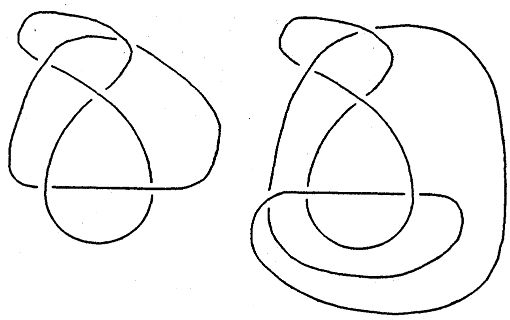
FIGURE 3-1

PROOF. By shortening A if necessary, we may assume that A has just one mutual crossing point P_1. We trace $\hat{K}_2$ from P_1 in two directions



CASE 1 CASE 2

FIGURE 3-2



CASE 1 CASE 2

FIGURE 3-3

and denote the first crossing points with $\partial\delta$ by P_2 and P_3 respectively. Then there are two cases as illustrated in Fig. 3-2. In any case it is easy to make the Whitehead link. See Fig. 3-3. □

LEMMA 2. *Let $\hat{L}$ be a two-component link projection on S^2 which has no self-crossing points. Let $\hat{L}_n$ be the link projection on S^2 as illustrated in Fig.* 3-4 *for each non-negative even number n. If $\hat{L}$ is not ambient isotopic in S^2 to any of the link projections $\hat{L}_n$, then* LINK($\hat{L}$) *contains the Whitehead link.*

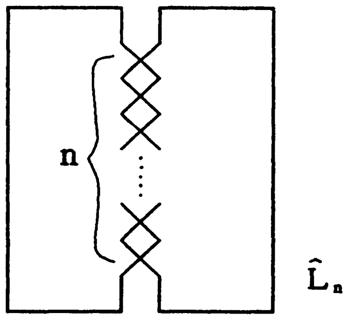


FIGURE 3-4

PROOF. By a straightforward search, we can find a part of $\hat{L}$ which is homeomorphic as a subspace of S^2 to the projection in Fig. 3-5 for some non-negative integer k. Then it is easy to make the Whitehead link. See Fig. 3-6. □

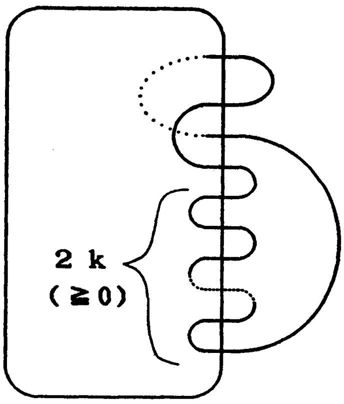


FIGURE 3-5

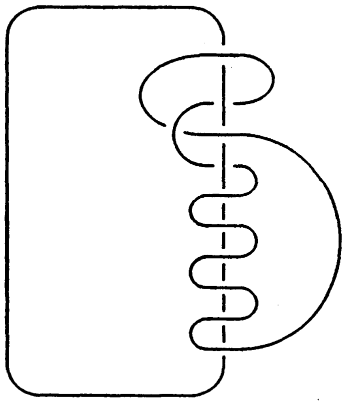


FIGURE 3-6

PROOF OF THEOREM 3. Let L be a two-component link which satisfies the condition (2), and let $\hat{L}=\hat{K}_1\cup\hat{K}_2$ be a projection of L. We denote by n the number of the mutual crossing points of $\hat{L}$. We note that $n\geqq 4$ by the proof of Corollary 1. Let us divide $\hat{K}_1$ into n parts

$\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_n$, where each $\hat{\alpha}_i$ is the image of a sub-arc of the circle and its end points are mutual crossing points. Similarly, we divide $\hat{K}_2$ into $\hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_n$. For each $i \in \{1, 2, \cdots, n\}$, let $\hat{\alpha}'_i \subset \hat{\alpha}_i$ (resp. $\hat{\beta}'_i \subset \hat{\beta}_i$) be a simple arc obtained from $\hat{\alpha}_i$ (resp. $\hat{\beta}_i$) by a series of eliminations of a tear drop disk. See for example Fig. 3-7.

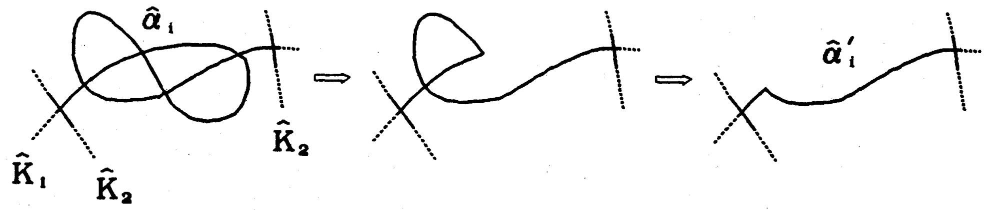


FIGURE 3-7

We set $\hat{K}'_1 = \cup_{i=1}^n \hat{\alpha}'_i$, $\hat{K}'_2 = \cup_{i=1}^n \hat{\beta}'_i$ and $\hat{L}' = \hat{K}'_1 \cup \hat{K}'_2$. Then we have $\mathrm{LINK}(\hat{L}) \supset \mathrm{LINK}(\hat{L}')$ by Lemma 3 in [5]. If $\hat{K}'_1$ or $\hat{K}'_2$ has self-crossing points, then we can apply Lemma 1 to conclude that $\mathrm{LINK}(\hat{L}')$ contains the Whitehead link. Therefore the rest is the case that both $\hat{K}'_1$ and $\hat{K}'_2$ are simple closed curves on S^2. Then by Lemma 2, it is sufficient to check the case that $\hat{L}'$ is ambient isotopic to $\hat{L}_n$ in Fig. 3-4. Then from the condition (2), there exist numbers i and j with $i \neq j$, $i, j \in \{1, 2, \cdots, n\}$ such that $\hat{\alpha}_i \cap \hat{\alpha}_j \neq \emptyset$ or $\hat{\beta}_i \cap \hat{\beta}_j \neq \emptyset$. Then we can apply Lemma 5 in [5] to conclude that $\mathrm{LINK}(\hat{L})$ contains the Whitehead link. See Fig. 3-8. $\square$

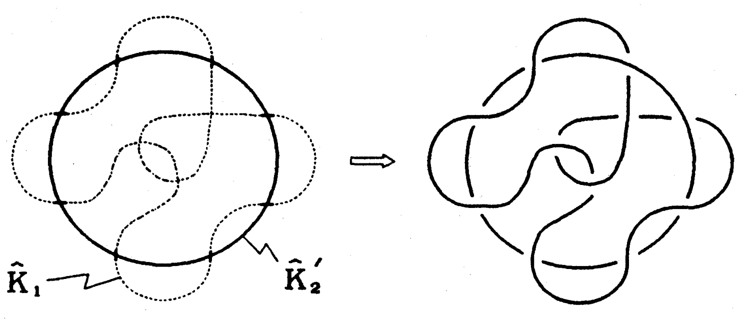


FIGURE 3-8

§ 4. Proof of Theorem 4.

We first state a version of Ramsey's theorem in [3].

THEOREM (Ramsey). *For any natural number n, there exists a natural number m such that if the natural number m' is greater than or equal to m, then the number m' satisfies the following condition* $(*)$:

(*) *For any permutation* $(\sigma(1), \sigma(2), \cdots, \sigma(m'))$ *of* $(1, 2, \cdots, m')$ *either* (a) *or* (*b*) *holds*:

(a) *There exist numbers* $i_1, i_2, \cdots, i_n \in \{1, 2, \cdots, m'\}$ *with* $i_1 < i_2 < \cdots < i_n$ *such that* $\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_n)$.

(b) *There exist numbers* $i_1, i_2, \cdots, i_n \in \{1, 2, \cdots, m'\}$ *with* $i_1 < i_2 < \cdots < i_n$ *such that* $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_n)$.

The proof of Theorem 4 may be summarized as follows. Let $\hat{L}$ be a two-component link projection which has m' mutual crossing points with $m' \geqq m$, where m is the number in Ramsey's theorem for the even number n. We will show an algorithm of adding over/under information to $\hat{L}$ so that the resultant link is a closed 2-braid. The algorithm has some flexibility which has an effect on the linking number of the resultant link. Then Ramsey's theorem ensures us the existence of an algorithm for obtaining a closed 2-braid with the linking number the half of n. Therefore the resultant link is the $(2, n)$-torus link as desired.

ALGORITHM. Let $\hat{L} = \hat{K}_1 \cup \hat{K}_2$ be a two-component link projection which has m' mutual crossing points with $m' \geqq m$. Let P_0 be a mutual crossing point of $\hat{L}$. Let $f: S_1^1 \cup S_2^1 \to S^2$ be a general position map of two disjoint circles S_1^1 and S_2^1 with $f(S_1^1) = \hat{K}_1$ and $f(S_2^1) = \hat{K}_2$. Let $P_1 \subset S_1^1$ and $Q_1 \subset S_2^1$ be the preimage of P_0. We give an orientation to each circle. Along the orientation, we denote the preimages on S_1^1 (resp. on S_2^1) of the mutual crossing points of $\hat{L}$ by $P_1, P_2, \cdots, P_{m'}$ (resp. $Q_1, Q_2, \cdots, Q_{m'}$).

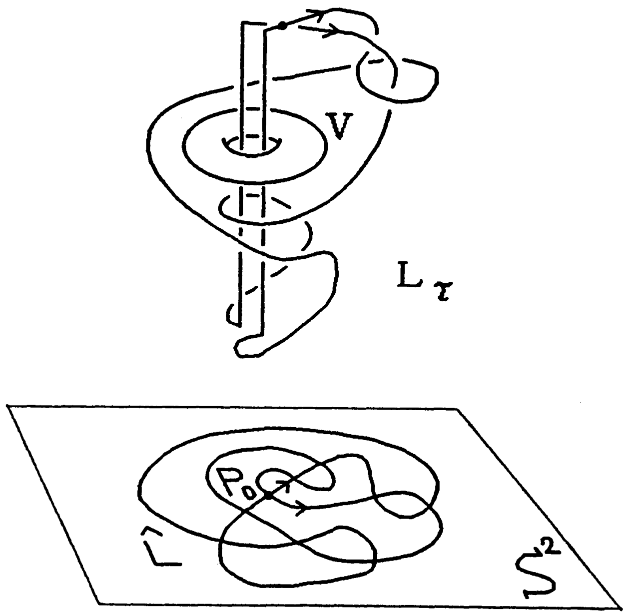


FIGURE 4-1

Let the map τ: $\{P_1, P_2, \cdots, P_{m'}, Q_1, Q_2, \cdots, Q_{m'}\} \to \{1, 2, \cdots, 2m'\}$ be a bijection which satisfies the condition that if $i<j$ for $i, j \in \{1, 2, \cdots, m'\}$, then $\tau(P_i)<\tau(P_j)$ and $\tau(Q_i)<\tau(Q_j)$. Let us call such a map a *good bijection*. Then there exists a smooth function h_τ: $S_1^1 \cup S_2^1 \to \boldsymbol{R}^1$ which satisfies the following conditions:

(1) h_τ has maximal points at P_1 and Q_1, minimal points at P_1' and Q_1' which are close behind P_1 and Q_1 respectively, and no other critical points.

(2) If $\tau(E)<\tau(F)$ for $E, F \in \{P_1, P_2, \cdots, P_{m'}, Q_1, Q_2, \cdots, Q_{m'}\}$, then $h_\tau(E)>h_\tau(F)$.

Let L_τ be the link represented by the map $(f\times h_\tau)$: $S_1^1 \cup S_2^1 \to S^2 \times \boldsymbol{R}^1 \subset S^3$, where we regard $S^2\times \boldsymbol{R}^1$ as $S^3 -$ {the north pole, the south pole}. We note that the link L_τ is uniquely determined by the bijection τ. Let V be a solid torus as illustrated in Fig. 4-1. Then the link L_τ is "monotone" in the solid torus $cl(S^3-V)$, and hence L_τ is a closed 2-braid. Thus L_τ is a $(2, p)$-torus link for some even number p.

PROOF OF THEOREM 4. Let $(\sigma(1), \sigma(2), \cdots, \sigma(m'))$ be a permutation of $(1, 2, \cdots, m')$ defined by the equation $f(P_i)=f(Q_{\sigma(i)})$ for all $i\in\{1, 2, \cdots, m'\}$. By Ramsey's theorem and by reversing the orientation of S_2^1 if necessary, we may suppose that there exist numbers $i_1, i_2, \cdots, i_n \in \{1, 2, \cdots, m'\}$ with $i_1<i_2<\cdots<i_n$ such that $\sigma(i_1)<\sigma(i_2)<\cdots<\sigma(i_n)$. Let us denote the sign $\varepsilon=\pm 1$ of each mutual crossing point of $\hat{L}$ as in Fig. 4-2.

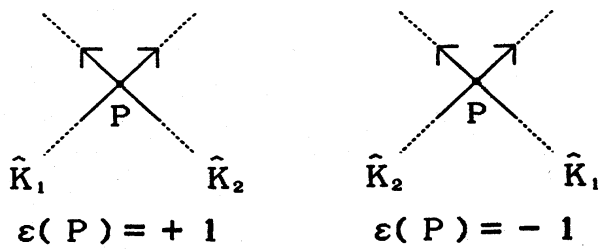


FIGURE 4-2

Let τ_1 be an arbitrary good bijection which satisfies that $\tau_1(P_{i_j})+\varepsilon(f(P_{i_j}))=\tau_1(Q_{\sigma(i_j)})$ for $j\in\{1, 2, \cdots, n\}$. Let τ_2 be a good bijection defined by the following: $\tau_2(P_{i_j})=\tau_1(Q_{\sigma(i_j)})$ and $\tau_2(Q_{\sigma(i_j)})=\tau_1(P_{i_j})$ for each $j\in\{1, 2, \cdots, n\}$, and $\tau_2(E)=\tau_1(E)$ for the other.

Then at least one of $|lk(L_{\tau_1})|$ and $|lk(L_{\tau_2})|$ is greater than or equal to the half of n, since $|lk(L_{\tau_1})-lk(L_{\tau_2})|=n$. Let τ_0 be the good bijection defined by $\tau_0(P_i)=i$ and $\tau_0(Q_i)=m'+i$ for $i\in\{1, 2, \cdots, m'\}$. Then $|lk(L_{\tau_0})|=0$. Every good bijection is obtained from τ_0 by a finite sequence of exchanging the values of P_i and Q_j which differ by one, and an exchange changes the linking number of the associated link of the good bijection by one. Therefore, by the existence of the intermediate value, there is a good bijection τ_3 such that $|lk(L_{\tau_3})|$ equals the half of n. Then the link L_{τ_3} is the $(2, n)$-torus link. This completes the proof of Theorem 4. □

COROLLARY 2. *There is an ascending chain in the partially ordered set* $(\mathfrak{L}^2/\sim, \leqq)$.

In fact, there is a strictly increasing infinite sequence of even numbers $n_1=0$, $n_2=2$, $n_3=4$, $n_4=6$, n_5, $n_6, \cdots$ such that the $(2, n_{i+1})$-torus link majorizes the $(2, n_i)$-torus link for all natural number i.

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Present Address:
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY
OHKUBO, SHINJUKU-KU, TOKYO 169, JAPAN