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On Solutions of $x''=-e^{\alpha\lambda t}x^{1+\alpha}$

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§ 1. Introduction.

Consider the second order nonlinear differential equations

$$\frac{d}{dt}\left(t^{\rho}\frac{du}{dt}\right)+t^{\sigma}u^{n}=0 \quad \text{and} \quad \frac{d}{dt}\left(t^{\rho}\frac{du}{dt}\right)-t^{\sigma}u^{n}=0\ , \tag{1.1}$$

which include Fermi-Thomas equation and Emden equation as special cases.

When $\rho\neq 1$, these equations are transformed into

$$\frac{d^2u}{ds^2}+s^{\beta}u^{n}=0\ , \tag{1.2}$$

and

$$\frac{d^2u}{ds^2}-s^{\beta}u^{n}=0\ , \qquad \beta=\frac{\rho+\sigma}{1-\rho} \tag{1.3}$$

respectively by putting $kt^{1-\rho}=s$ where k is a suitable constant, and when $\rho=1$, into

$$\frac{d^2u}{ds^2}+e^{(\sigma+1)s}u^{n}=0 \tag{1.4}$$

and

$$\frac{d^2u}{ds^2}-e^{(\sigma+1)s}u^{n}=0 \tag{1.5}$$

respectively by putting $\log t=s$.

(1.3) was studied in [5], [6] and (1.5) was studied in [8] and the asymptotic behavior of solutions as s approaches the end points of their intervals of definition was investigated in detail. In both cases, this

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was done by transforming the given equation into a certain first order algebraic differential equation.

This method, however, works successfully also for the equation (1.4). This is what we want to show in this paper.

Changing the notations slightly, we rewrite (1.4) in a form

$$x''=-e^{\alpha\lambda t}x^{1+\alpha}\,, \qquad '=\frac{d}{dt} \tag{1.6}$$

$$-\infty<t<\infty\,, \quad 0\leqq x<\infty$$

where α and λ are positive constants and $x^{1+\alpha}$ always represents its non-negative valued branch.

Let $\phi(t)$ be an arbitrary solution of (1.6). Then the transformation

$$y=-\lambda^{-2}e^{\alpha\lambda t}\phi^{\alpha}\,, \qquad z=y' \tag{1.7}$$

changes (1.6) into a first order algebraic differential equation

$$\frac{dz}{dy}=\frac{\alpha^2\lambda^2y^2(y-1)+2\alpha\lambda yz+(\alpha-1)z^2}{\alpha yz}\,. \tag{1.8}$$

This is the same as the equation (8) of [8], which was obtained from

$$x''=e^{\alpha\lambda t}x^{1+\alpha}$$

by putting

$$y=\psi^{-\alpha}\phi^{\alpha}\,, \qquad z=y'$$

where $\psi(t)=\lambda^{2/\alpha}e^{-\lambda t}$ is a particular solution of the above equation and $\phi(t)$ is its arbitrary solution.

Introducing a parameter s, (1.8) can be reduced to a two-dimensional autonomous system

$$\begin{cases} \dfrac{dy}{ds}=\alpha yz \\ \dfrac{dz}{ds}=\alpha^2\lambda^2y^2(y-1)+2\alpha\lambda yz+(\alpha-1)z^2\,. \end{cases} \tag{1.9}$$

Critical points of this system are (0, 0) and (1, 0) if $\alpha\neq 1$, and $(0, c)$ (c: an arbitrary real number) and (1, 0) if $\alpha=1$. Since $\phi(t)>0$, y is always negative. So we consider (1.9) in the half-plane $y<0$ throughout this paper.

With the help of (1.9), we try to find all the solutions $z(y)$ of (1.8). For each $z(y)$, determine $y(t)$ so as to satisfy

$$y' = z(y)$$

which is the second formula of (1.7). Then, finally, $\phi(t)$ can be obtained from the first formula of (1.7). This is the outline of our scheme.

§2. On solutions of (1.8) tending to 0 as $y \to -0$.

Let there be given a solution $z = z(y)$ of (1.8) which tends to 0 as $y \to -0$. Then, if $y(s)$ denotes a solution of the first equation of (1.9), we get a solution $(y(s), z(y(s)))$ of (1.9). Since

$$\frac{dy}{ds} = 0\ , \qquad \frac{dz}{ds} < 0\ ,$$

if $y < 0$ and $z = 0$, $z(y)$ is always positive or negative for y sufficiently close to 0. Hence $y(s) \to -0$ is equivalent to $s \to -\infty$ if $z(y(s)) > 0$ and is equivalent to $s \to \infty$ if $z(y(s)) < 0$.

Now, put

$$z = v_1(y) = \sigma y\ , \tag{2.1}$$

then, if $(y, z) = (y(s), z(y(s)))$ and if $y(s) < 0$, we have

$$\frac{d}{ds}(z - v_1(y)) = -y^2\{\sigma^2 - 2\alpha\lambda\sigma - \alpha^2\lambda^2(y-1)\} < 0 \tag{2.2}$$

on the straight line (2.1). Suppose that

$$\liminf_{y \to -0} \frac{z(y)}{y} = d_1\ , \quad \limsup_{y \to -0} \frac{z(y)}{y} = d_2\ , \qquad d_1 < d_2\ ,$$

then there exists a real number σ such that

$$d_1 < \sigma < d_2\ , \qquad \sigma \neq \alpha\lambda\ . \tag{2.3}$$

However, it follows from the definitions of the inferior limit and the superior limit that, for σ satisfying (2.3), there exists s_1 such that, if $s = s_1$, then

$$\frac{d}{ds}(z - v_1(y)) > 0$$

on the straight line (2.1), where $(y, z) = (y(s), z(y(s)))$. This is contrary to (2.2). Hence,

$$\lim_{y \to -0} \frac{z(y)}{y} = d_1 = d_2\ .$$

Now, put $d=d_1$, then l'Hospital's theorem shows

$$\lim_{y\to-0}\frac{dz}{dy}=d\ .$$

On the other hand, from (1.8), we have

$$\frac{z}{y}\cdot\frac{dz}{dy}=\alpha\lambda^2(y-1)+2\lambda\cdot\frac{z}{y}+\frac{\alpha-1}{\alpha}\left(\frac{z}{y}\right)^2\ .$$

Hence, if $d\neq\pm\infty$, then we have

$$d^2-2\alpha\lambda d+\alpha^2\lambda^2=0 \qquad \text{i.e.}\quad d=\alpha\lambda\ .$$

Thus we have

LEMMA 1. *If $z(y)$ is a solution of (1.8) which tends to 0 as $y\to-0$, then*

$$\lim_{y\to-0}\frac{z(y)}{y}=\alpha\lambda \textit{ or } \pm\infty\ .$$

Furthermore, we can have

LEMMA 2. *There exists one and only one solution $z(y)$ of (1.8) such that*

$$\lim_{y\to-0}\frac{z(y)}{y}=\alpha\lambda\ ,\qquad \lim_{y\to-0}\frac{v(y)}{y}=\lambda \tag{2.4}$$

where $v(y)=y^{-1}z(y)-\alpha\lambda$.

PROOF. Let us transform (1.8) by

$$v=y^{-1}z-\alpha\lambda\ , \tag{2.5}$$

then we have

$$\frac{dv}{dy}=\frac{\alpha^2\lambda^2y-v^2}{\alpha y(v+\alpha\lambda)}\ . \tag{2.6}$$

Using a parameter τ, we can rewrite this into

$$\begin{cases}\dfrac{dy}{d\tau}=\alpha y(v+\alpha\lambda)\\[2mm] \dfrac{dv}{d\tau}=\alpha^2\lambda^2y-v^2\ .\end{cases} \tag{2.7}$$

The matrix representing the linear part of the right-hand side of (2.7) is

$$\begin{bmatrix} \alpha^2\lambda & 0 \\ \alpha^2\lambda^2 & 0 \end{bmatrix}$$

whose eigenvalues are 0 and $\alpha^2\lambda$. Hence, from (2.7), we have

$$\begin{aligned} y &= a_1(Ce^{\alpha^2\lambda\tau}) + a_2(Ce^{\alpha^2\lambda\tau})^2 + \cdots , \\ v &= b_1(Ce^{\alpha^2\lambda\tau}) + b_2(Ce^{\alpha^2\lambda\tau})^2 + \cdots , \end{aligned} \tag{2.8}$$

where C is a constant. Here and henceforth, $\cdots$ denotes a sum of terms whose degrees are greater than the degree of the previous term. Substituting these into (2.7), we have

$$\frac{b_1}{a_1} = \lambda .$$

Hence, from (2.8), we have

$$v(y) = \lambda(y) + \cdots .$$

Therefore, it follows from (2.5) that (1.8) has a solution $z = z(y)$ represented by a convergent power series

$$z = \alpha\lambda y + \lambda y^2 + \cdots$$

in the neighborhood of $y = 0$. This solution satisfies (2.4).

Furthermore, put

$$w = y^{-1}v - \lambda , \tag{2.9}$$

we have from (2.6)

$$\frac{dw}{dy} = \frac{-(\alpha+1)\lambda^2 y - \{2(\alpha+1)\lambda y + \alpha^2\lambda\}w - (\alpha+1)yw^2}{\alpha y(yw + \lambda y + \alpha\lambda)} \tag{2.10}$$

and

$$\begin{cases} \dfrac{dy}{d\tau} = \alpha y(yw + \lambda y + \alpha\lambda) \\ \dfrac{dw}{d\tau} = -(\alpha+1)\lambda^2 y - \{2(\alpha+1)\lambda y + \alpha^2\lambda\}w - (\alpha+1)yw^2 . \end{cases} \tag{2.11}$$

The matrix representing the linear part of the right-hand side of (2.11) is

$$\begin{bmatrix} \alpha^2\lambda & 0 \\ -(\alpha+1)\lambda^2 & -\alpha^2\lambda \end{bmatrix}$$

whose eigenvalues are $\pm\alpha^2\lambda$. Hence $(y, w)=(0, 0)$ is a saddle point of (2.11). On the other hand, it is evident that V

$$\begin{cases} y\equiv 0 \\ w=Ce^{-\alpha^2\lambda\tau} \end{cases} \tag{2.12}$$

is a solution of (2.11), where C is a constant. Therefore (2.11) has no solution tending to the origin except (2.12) and

$$\begin{cases} y=a_1(Ce^{\alpha^2\lambda\tau})+a_2(Ce^{\alpha^2\lambda\tau})^2+\cdots \\ w=b_1(Ce^{\alpha^2\lambda\tau})+b_2(Ce^{\alpha^2\lambda\tau})^2+\cdots \end{cases}$$

where

$$\frac{b_1}{a_1}=-\frac{(\alpha+1)\lambda}{2\alpha^2}\ .$$

Hence we have

$$w=-\frac{(\alpha+1)\lambda}{2\alpha^2}y+\sum_{n=2}^{\infty}c_n y^n\ ,$$

where c_n are determined uniquely. Therefore, from (2.9), we get the solution of (1.8) represented by

$$z(y)=\alpha\lambda y+\lambda y^2-\frac{(\alpha+1)\lambda}{2\alpha^2}y^3+\sum_{n=4}^{\infty}c_{n-2}y^n\ . \tag{2.13}$$

Since there is no solution of (1.8) obtained by (2.12), this is the only solution of (1.8) satisfying (2.4). Thus the proof is completed.

Hereafter, since the solution of (1.8) whose existence has been just shown attains negative values, this solution will be denoted by $\hat{z}_-(y)$. Since (2.6) can be changed into a Briot-Bouquet equation in the neighborhood of $(y, v)=(0, 0)$, we can also prove the existence of $\hat{z}_-(y)$ directly from the theorem (e.g. Theorem 11.1.1 of [3]) which states the existence of the unique holomorphic solution of Briot-Bouquet equations passing through the origin. Moreover we have

LEMMA 3. *If $z(y)$ is a solution of* (1.8) *satisfying*

$$\lim_{y\to-0}\frac{z(y)}{y}=\alpha\lambda\ ,\qquad \lim_{y\to-0}\frac{v(y)}{y}\neq\lambda\ ,$$

where $v(y)=y^{-1}z(y)-\alpha\lambda$, then we have

$$z(y) > \hat{z}_-(y) \tag{2.14}$$

as long as both of these solutions exist.

PROOF. Since $v=v(y)$ satisfies (2.6) and

$$\lim_{y\to -0} v(y)=0 ,$$

we have

$$\frac{dv(y)}{dy} > \frac{\alpha\lambda^2}{v(y)+\alpha\lambda} > 0$$

for y sufficiently close to 0. Hence we have $v(y)<0$ since $y<0$. Consequently

$$\frac{dv(y)}{dy} > \lambda .$$

Take y_0 and y ($y<y_0<0$) sufficiently close to 0. Then, integrating the both sides of this inequality from y to y_0, we have

$$v(y_0)-v(y) > \lambda(y_0-y) .$$

As $y_0 \to -0$, we have

$$\frac{v(y)}{y} \geqq \lambda .$$

Therefore, from the uniqueness of $\hat{z}_-(y)$,

$$\limsup_{y\to -0} \frac{v(y)}{y} > \lambda . \tag{2.15}$$

Hence, if $\hat{v}_-(y)=y^{-1}\hat{z}_-(y)-\alpha\lambda$, then there exists a sequence $\{y_n\}$ with $y_n \to -0$ such that

$$\frac{v(y_n)}{y_n} > \frac{\hat{v}_-(y_n)}{y_n} .$$

Therefore (2.14) holds if $y=y_n$. Thus we can complete this proof by the uniqueness of solutions of (1.8).

§3. The phase portraits of (1.9).

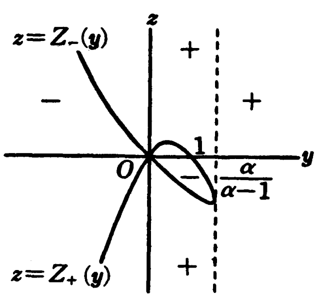
From (1.9), we have

$$\frac{dz}{ds}=\alpha^2\lambda^2y^2(y-1)+2\alpha\lambda y\sigma+(\alpha-1)\sigma^2 \tag{3.1}$$

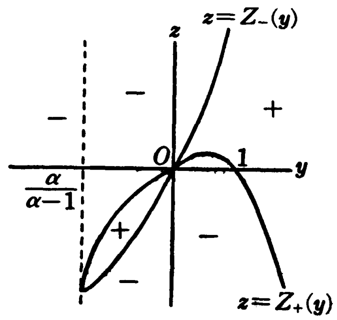
on the straight line $z=\sigma$. If $\alpha\neq 1$ and $dz/ds=0$, then $\sigma=Z_+(y)$ or $Z_-(y)$ where

$$Z_+(y)=\frac{-1+\sqrt{(1-\alpha)y+\alpha}}{\alpha-1}\alpha\lambda y\ , \qquad Z_-(y)=\frac{-1-\sqrt{(1-\alpha)y+\alpha}}{\alpha-1}\alpha\lambda y\ .$$

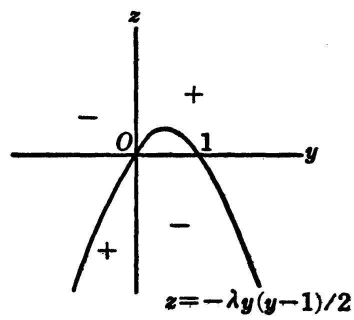
If $\alpha=1$ and $dz/ds=0$, then $\sigma=-\lambda y(y-1)/2$. Therefore, the signature of dz/ds can be shown as in the following figures:



The case when $\alpha>1$
FIGURE 1.1



The case when $0<\alpha<1$
FIGURE 1.2



The case when $\alpha=1$
FIGURE 1.3

In these figures, $\pm$ denote the signs of dz/ds.

Now we claim that, if $(y(s), z(s))$ denotes an arbitrary solution of (1.9) defined in the region $y<0$, then $z(s)\not\to\infty$ as s increases and $z(s)\not\to-\infty$ as s decreases. From (1.9) we have $dy/ds<0$ if $y<0$, $z>0$, and $dy/ds>0$ if $y<0$, $z<0$. If $\alpha>1$, then the straight line $z=\sigma y$ intersects with the curve $z=Z_+(y)$ in the region $y<0$ if $\sigma>Z'_+(0)$, and intersects with the curve $z=Z_-(y)$ in the same region if $\sigma<Z'_-(0)$, since

$$\lim_{y\to-\infty} Z'_+(y)=\infty\ , \qquad \lim_{y\to-\infty} Z'_-(y)=-\infty\ .$$

Hence it follows from (2.2) that the solution $(y(s), z(s))$ of (1.9) which is placed initially in the region $z>Z_-(y)$ or in the region $z<Z_+(y)$ tends to the region $Z_-(y)<z<Z_+(y)$ as s increases or as s decreases respectively. Hence we have our claim. If $\alpha=1$, then we have our claim from the same reasoning and Fig. 1.3. If $0<\alpha<1$, then Fig. 1.2 shows our claim.

Furthermore, from (1.9), we have

$$\frac{dy}{ds}=0\ , \qquad \frac{dz}{ds}<0\ ,$$

if $y<0$ and $z=0$. Hence, in the yz-plane, a solution of (1.9) intersects with the negative part of the y-axis once at most.

Moreover, we can claim that a solution $(y(s), z(s))$ of (1.9) satisfies neither

(3.2) $$(y(s), z(s)) \to (-\infty, c)\,, \qquad c \geqq 0$$

as s increases, nor

(3.3) $$(y(s), z(s)) \to (-\infty, c)\,, \qquad c \leqq 0$$

as s decreases. In fact if there exists a solution $(y(s), z(s))$ of (1.9) satisfying (3.2), then

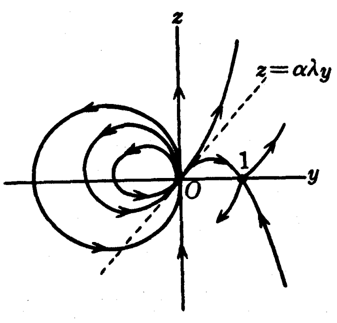
$$\frac{z(s)}{y(s)} \to -0$$

as s increases. Since $dy/ds<0$ in this case, $z(s)$ is a single-valued function of $y(s)$. Therefore it follows from (1.8) that

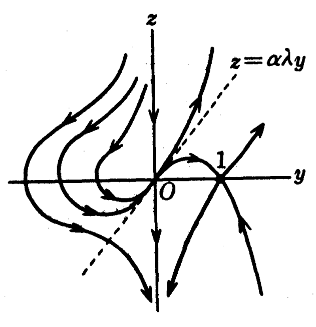
$$\frac{z}{y}\cdot\frac{dz}{dy} = \alpha\lambda^2(y-1) + 2\lambda\cdot\frac{z}{y} + \frac{\alpha-1}{\alpha}\left(\frac{z}{y}\right)^2 \to -\infty$$

as s increases. Hence we have $dz/dy \to \infty$ which implies $z(s) \to -\infty$ as s increases. This contradicts (3.2). In the similar manner, we can show that there is no solution of (1.9) satisfying (3.3).

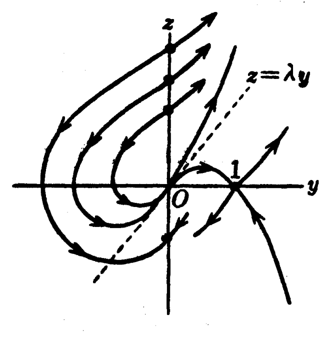
Consequently, we can draw the phase portraits of (1.9).



The case when $\alpha>1$
FIGURE 2.1



The case when $0<\alpha<1$
FIGURE 2.2



The case when $\alpha=1$
FIGURE 2.3

Here the phase portraits of (1.9) of the region $y>0$ can be drawn by the results of [8]. Notice that $(y, z)=(0, 1/((1-\alpha)s+c))$ (c is a constant) is a solution of (1.9). Also notice that it is necessary in drawing Fig. 2.3 to show the existence of a solution of (1.8) such that

(3.4) $$\lim_{y\to-0} z(y) = c$$

where c is a non-zero constant. Suppose that $\alpha=1$. Then, from (1.8), we have

$$\frac{dz}{dy}=\frac{\lambda^2 y(y-1)+2\lambda z}{z}. \tag{3.5}$$

Put $z=1/\zeta$, then

$$\frac{d\zeta}{dy}=-\lambda^2 y(y-1)\zeta^3-2\lambda\zeta^2.$$

Since this equation has the unique solution $\zeta\equiv 0$ satisfying initial condition $\zeta(y_0)=0$ for a certain y_0, (3.5) does not have a solution unbounded as y tends to a finite value. Therefore, if $z(y)$ denotes a nonnegative solution of (1.8), then we can conclude from Fig. 1.3 that $z(y)$ satisfies (3.4). Furthermore, if $z(y)$ denotes a nonpositive solution of (1.8) whose curve lies under the curve representing the solution $\hat{z}_-(y)$ whose existence is shown in Lemma 2 and if

$$\lim_{y\to -0} z(y)=0,$$

then it follows from Lemma 3 that

$$\lim_{y\to -0}\frac{z(y)}{y}\neq\lambda.$$

Therefore, it follows from Lemma 1 that

$$\lim_{y\to -0}\frac{z(y)}{y}=\infty. \tag{3.6}$$

Hence, from (3.5), we have

$$\lim_{y\to -0}\frac{dz(y)}{dy}=2\lambda,$$

which contradicts (3.6). Consequently, all solutions of (1.8) whose curves lie under the curve representing $\hat{z}_-(y)$ satisfy (3.4).

Finally, notice that, if $(y(s), z(s))$ denotes a solution of (1.9), then

$$y(s)\to -0$$

as $s\to\pm\infty$. In fact, if

$$(y(s), z(s))\to(c, \pm\infty), \qquad -\infty<c<0,$$

then we can get from this a solution $z(y)$ of (1.8) such that

(3.7) $$\lim_{y\to c} z(y)=\pm\infty\ .$$

However, if we put $z=1/\zeta$, then we have from (1.8)

(3.8) $$\frac{d\zeta}{dy}=-\frac{\alpha^2\lambda^2y^2(y-1)\zeta^3+2\alpha\lambda y\zeta^2+(\alpha-1)\zeta}{\alpha y}\ .$$

Since this has a solution $\zeta\equiv 0$, it follows from the uniqueness of solutions of (3.8) that (3.7) is impossible.

§ 4. Asymptotic behavior of solutions of (1.6) I.

Let $\phi(t)$ be a positive solution of (1.6) whose domain will be denoted by (ω', ω) in the sequel. In this section, we consider the asymptotic behavior of $\phi(t)$ as $t\to\omega$.

Since we have $\phi''(t)<0$ from (1.6), the limit of $\phi(t)$ as $t\to\omega$ always exists. Now, suppose that

(4.1) $$\lim_{t\to\omega}\phi(t)\neq 0\ .$$

Then there exist constants c_1 and T_1 such that $t\geqq T_1$ implies

$$\phi(t)>c_1>0\ .$$

Since there exists a constant γ such that

$$e^{\alpha\lambda t}>\gamma>0$$

if $t\geqq T_1$, we have

$$\phi''(t)<-c_2<0\ ,\qquad c_2=\gamma c_1^{1+\alpha}>0$$

for $t\geqq T_1$. Therefore

$$\phi'(t)-\phi'(T_1)<-c_2(t-T_1)\ .$$

Hence, if $\omega=\infty$, then

$$\lim_{t\to\omega}\phi'(t)=-\infty\ .$$

Namely, for an arbitrary negative number R, there exists a number T_2 such that $t\geqq T_2$ implies

$$\phi'(t)<R<0\ .$$

Hence we have

$$\phi(t)-\phi(T_2)<R(t-T_2) .$$

Therefore

$$\lim_{t\to\omega}\phi(t)=-\infty .$$

This is a contradiction. Thus we suppose that $\omega\neq\infty$. Then it follows from (4.1) that

$$\lim_{t\to\omega}\phi(t)=\infty .$$

Therefore we can take T_3 such that $\phi'(T_3)>0$. Since $\phi''(t)<0$, we have

$$\phi'(t)<\phi'(T_3)$$

if $t\geqq T_3$. Integrating the both sides,

$$\phi(t)-\phi(T_3)<\phi'(T_3)(t-T_3) .$$

As $t\to\omega$, we have

$$\phi'(T_3)(\omega-T_3)=\infty ,$$

namely $\omega=\infty$, which is a contradiction. Thus we have

$$\lim_{t\to\omega}\phi(t)=0 . \tag{4.2}$$

It follows from (4.2) that there exists a number T_4 such that $\phi'(T_4)<0$. Since $\phi''(t)<0$, we have

$$\phi'(t)<\phi'(T_4) ,$$

if $t\geqq T_4$. Hence, if $t\geqq T_4$, then

$$\phi(t)-\phi(T_4)<\phi'(T_4)(t-T_4) ,$$

which implies

$$\omega\neq\infty . \tag{4.3}$$

Now we shall obtain an analytical expression of $\phi(t)$ in the neighborhood of $t=\omega$. For this, we transform (1.6) into (1.8) by (1.7). From (4.2) and (4.3), we have

$$\lim_{t\to\omega}y=0 . \tag{4.4}$$

Furthermore, if t is sufficiently close to ω, then $z>0$. In fact, if $z\leqq 0$, then we have

$$\frac{dy}{dt} \leqq 0$$

from (1.7). This contradicts (4.4). Hence it follows from Lemma 1 that

$$\lim_{t \to \omega} \frac{z}{y} = -\infty .$$

Hence, put

$$\eta = -y , \qquad w = \eta z^{-1} , \tag{4.5}$$

then we have

$$\eta \to 0 , \qquad w \to 0 ,$$

as $t \to \omega$. Moreover, from (1.8) and (4.5), we have

$$\eta \frac{dw}{d\eta} = \alpha\lambda^2(\eta+1)w^3 + 2\lambda w^2 + \frac{w}{\alpha} . \tag{4.6}$$

This is a Briot-Bouquet equation. Hence we have

$$w = \sum_{m+n>0} w_{mn}^{(1)} \eta^m \{\eta^{1/\alpha}(C \log \eta + \Gamma)\}^n , \qquad w_{01}^{(1)} = 1 \tag{4.7}$$

in the neighborhood of $\eta = 0$, where Γ is an arbitrary constant, $w_{mn}^{(1)}$ are constants and C is a constant such that $C=0$ if $1/\alpha$ is not an integer. Reviewing the processes of the formal transformations for obtaining solutions of Briot-Bouquet equations, we have

$$w_{m0}^{(1)} = 0 \qquad (m = 1, 2, \cdots) . \tag{4.8}$$

If $1/\alpha$ is not an integer, then we have from (4.7)

$$w = \sum_{m+n>0} w_{mn}^{(1)} \Gamma^n \eta^{m+(n/\alpha)} . \tag{4.9}$$

Since $\eta > 0$ and $w > 0$ if t is sufficiently close to ω, it follows from (4.9) that $\Gamma > 0$. From (1.7), (4.5) and (4.9),

$$-1 = \sum_{m+n>0} w_{mn}^{(1)} \Gamma^n \eta^{m+(n/\alpha)-1} \eta' .$$

Integrating the both sides,

$$\omega - t = \alpha \Gamma \eta^{1/\alpha} \{1 + \sum_{m+n>0} w_{mn}^{(2)} \eta^m (\Gamma \eta^{1/\alpha})^n\} .$$

Put

$$\Gamma\eta^{1/\alpha}=\zeta ,$$

then

$$\alpha^{-1}(\omega-t)=\zeta\{1+\sum_{m+n>0} w_{mn}^{(2)}(\Gamma^{-1}\zeta)^{\alpha m}\zeta^{n}\} .$$

Therefore, from Smith's lemma (Lemma 1 of [6] or [7]), we have

$$\zeta=\alpha^{-1}(\omega-t)\{1+\sum_{m+n>0} w_{mn}^{(3)}(\alpha^{-1}(\omega-t))^{\alpha m}(\alpha^{-1}(\omega-t))^{n}\} .$$

Hence

$$\eta^{1/\alpha}=\alpha^{-1}\Gamma^{-1}(\omega-t)\{1+\sum_{m+n>0} w_{mn}^{(4)}(\omega-t)^{\alpha m+n}\} .$$

It follows from (1.7) that

$$\phi(t)=\alpha^{-1}\lambda^{2/\alpha}\Gamma^{-1}e^{-\lambda\omega}(\omega-t)\{1+\sum_{m+n>0}\phi_{mn}(\omega-t)^{\alpha m+n}\} \tag{4.10}$$

holds in the neighborhood of $t=\omega$, where ϕ_{mn} are constants.

Now, we can summarize the above results as follows:

THEOREM 1. *If $\phi(t)$ denotes an arbitrary solution of* (1.6) *and its domain is denoted by (ω', ω), then ω is finite and*

$$\lim_{t\to\omega}\phi(t)=0 .$$

Moreover, if $1/\alpha$ is not an integer, then $\phi(t)$ can be represented by an analytical expression of the form

$$\phi(t)=A(\omega-t)\{1+\sum_{m+n>0}\phi_{mn}(\omega-t)^{\alpha m+n}\}$$

in the neighborhood of $t=\omega$, where A and ϕ_{mn} are constants.

§5. Asymptotic behavior of solutions of (1.6) II.

In this section, we consider the asymptotic behavior of $\phi(t)$ as $t\to\omega'$.

Let $\phi(t, a, b)$ be a solution of (1.6) satisfying an initial condition

$$x(t_0)=a , \qquad x'(t_0)=b , \tag{5.1}$$

where t_0 is a fixed constant. If $y(t)$ and $z(t)$ are defined by replacing $\phi(t)$ by $\phi(t, a, b)$ in (1.7), then we have

$$z=y'=\alpha y\left(\lambda+\frac{\phi'(t, a, b)}{\phi(t, a, b)}\right) . \tag{5.2}$$

Moreover, if $y_0=y(t_0)$ and $z_0=z(t_0)$, then we have

$$y_0=-\lambda^{-2}e^{\alpha\lambda t_0}a^{\alpha}\ , \qquad z_0=\alpha y_0\left(\lambda+\frac{b}{a}\right)\ . \tag{5.3}$$

Conversely we have

Lemma 4. (I) *Let (y_0, z_0) be given such that $z_0>0$ and let $z_+(y)$ be a solution of* (1.8) *such that $z_+(y_0)=z_0$. Define $y(t)$ by*

$$\frac{dy}{dt}=z_+(y)\ , \qquad y(t_0)=y_0\ .$$

Then, if $z_-(y)$ is a nonpositive solution of (1.8) *which is equal to $z_+(y)$ at a point $(\tilde{y}, 0)$ on the y-axis, there exists a number t_1 such that*
(i) $\lim_{t\to t_1+0} y(t)=\tilde{y}$,
(ii) *$y(t)$ can be continued in the interval $t<t_1$ uniquely by*

$$\frac{dy}{dt}=z_-(y)\ , \qquad y(t_1)=\tilde{y}\ . \tag{5.4}$$

(II) *If (y_0, z_0) is given such that $z_0<0$, then, interchanging signs, we can get the similar conclusion.*

(III) *Let (y_0, z_0) be given such that $z_0=0$ and let $z_\pm(y)$ be solutions of* (1.8) *such that*

$$z_+(y)\geqq 0\ , \quad z_-(y)\leqq 0\ , \quad z_\pm(y_0)=0\ .$$

Then $y(t)$ can be defined uniquely by

$$\frac{dy}{dt}=\begin{cases} z_+(y) & \text{if} \quad t>t_0 \\ 0 & \text{if} \quad t=t_0 \\ z_-(y) & \text{if} \quad t<t_0 \end{cases} \tag{5.5}$$
$$y(t_0)=y_0\ .$$

Proof. (I) If $(f(t), g(t))$ is a solution of

$$\begin{cases} \dfrac{dy}{dt}=z \\ \dfrac{dz}{dt}=\dfrac{\alpha^2\lambda^2y^2(y-1)+2\alpha\lambda yz+(\alpha-1)z^2}{\alpha y} \end{cases} \tag{5.6}$$
$$y(t_0)=y_0\ , \qquad z(t_0)=z_0\ ,$$

then we have

$$(f(t),\ g(t))=(y(t),\ z_+(y(t)))$$

in a region where $z_+(y(t))\neq 0$. Since the phase portraits of (5.6) in $y<0$ can be also shown by Fig. 2.1–Fig. 2.3, there exists a number t_1 such that $g(t_1)=0$ and $g(t)\neq 0$ if $t\neq t_1$. Hence we have

$$f(t_1)=\lim_{t\to t_1+0} y(t)=\tilde{y}\ , \qquad z_+(\tilde{y})=\lim_{t\to t_1+0} z_+(y(t))=0\ .$$

Moreover $f(t)$ satisfies (5.4), since $z_-(y)$ is equal to $g(f^{-1}(y))$. Hence it suffices to define $y(t)=f(t)$ in the region $t<t_1$ for the continuation of $y(t)$. If $u(t)$ is another solution of (5.4), then $(u(t),\ z_-(u(t)))$ satisfies (5.6) and

$$u(t_1)=\tilde{y}\ , \qquad z_-(u(t_1))=0\ .$$

Therefore, from the uniqueness of solutions of (5.6), we have $f(t)=u(t)$.

(II) This can be shown by the similar reasoning.

(III) If $(f(t),\ g(t))$ denotes a solution of (5.6), then $(f(t),\ g(t))$ satisfies (5.5), since $z_\pm(y)$ is equal to $g(f^{-1}(y))$ if $t\neq t_0$. Thus it suffices to put $y(t)=f(t)$ for completing the proof.

Let $\hat{z}_-(y)$ be a nonpositive solution of (1.8) whose existence was shown in Lemma 2 and let $\hat{z}_+(y)$ be a solution of (1.8) which is nonnegative and connects with $\hat{z}_-(y)$ at a point of the y-axis whose coordinate will be denoted by $(y_*,\ 0)$. Define $y(t)$ by

$$\frac{dy}{dt}=\hat{z}_-(y)\ . \tag{5.7}$$

Then, since $\hat{z}_-(y)$ is represented by (2.13) in the neighborhood of $y=0$, we have

$$\frac{dy}{dt}=\alpha\lambda y(1+o(1))$$

as $y\to -0$. Hence, noticing that $y<0$,

$$-y(1+o(1))=Ce^{\alpha\lambda t}\ , \qquad C>0\ . \tag{5.8}$$

Therefore, as $y\to -0$, we have

$$t\to -\infty\ .$$

Furthermore, from (5.8),

$$y=\sum_{n=1}^{\infty}\tilde{a}_n(-Ce^{\alpha\lambda t})^n\ , \qquad \tilde{a}_1=1\ .$$

Hence it follows from (1.7) that

$$\phi(t)=\lambda^{2/\alpha}C^{1/\alpha}\sum_{n=0}^{\infty}\hat{a}_n(C)e^{\alpha\lambda nt}\ , \qquad \hat{a}_0(C)=1\ , \tag{5.9}$$

where $\hat{a}_n(C)$ $(n=1, 2, \cdots)$ are constants depending on C. Consequently $\omega'=-\infty$ and, as $t\to-\infty$, we have

$$\phi(t)\to\lambda^{2/\alpha}C^{1/\alpha}>0\ . \tag{5.10}$$

Now, let γ be a curve defined by $z=\hat{z}_+(y)$ and $z=\hat{z}_-(y)$ and let D be a region surrounded by γ and the z-axis.

If $y_*<y_0$, then, from (5.3), we have

$$0<a<\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}\ .$$

If a is fixed so as to satisfy this, then a straight line $y=y_0$ intersects with γ at two points. Hence, it follows from (5.3) that, if (y_0, z_0) is equal to the coordinate of the intersection, then b attains two values which will be denoted by b_1 and b_2, where $b_1>b_2$. From Lemma 4, we can determine $y_i(t)$ $(i=1, 2)$ uniquely by

$$\frac{dy}{dt}=\begin{cases}\hat{z}_+(y) & \text{if} \quad t>t(b_i)\\ 0 & \text{if} \quad t=t(b_i)\\ \hat{z}_-(y) & \text{if} \quad t<t(b_i)\end{cases} \tag{5.11}$$
$$y(t_0)=y_0\ ,$$

where $t(b_i)$ is a number such that

$$\hat{z}_+(y(t(b_i)))=\hat{z}_-(y(t(b_i)))=0\ .$$

Hence, through (1.7), we can get two solutions $\phi(t, a, b_1)$ and $\phi(t, a, b_2)$ of (1.6) which satisfy (5.9) and (5.10). Here we show that

$$\lim_{t\to-\infty}\phi(t, a, b_1)<\lim_{t\to-\infty}\phi(t, a, b_2)\ . \tag{5.12}$$

Since z_0 depends on b, we here write $z_0(b)$ instead of z_0. Then we have from (5.3)

$$z_0(b_1)<0<z_0(b_2)\ .$$

Hence it follows from Fig. 2.1–Fig. 2.3 that there exists a positive number h such that

$$y_1(t_0)=y_2(t_0-h)\ .$$

Furthermore, there exists a positive constant τ such that $y_1(t_0-\tau)$ is in the region where the power series representing the left-hand side of (5.8) converges. Since the right-hand side of (5.11) does not depend on t, we have

$$y_1(t_0-\tau)=y_2(t_0-h-\tau) .$$

Therefore, from (5.8), we have

$$C_1e^{\alpha\lambda(t_0-\tau)}=-y_1(t_0-\tau)(1+o(1))=-y_2(t_0-h-\tau)(1+o(1))=C_2e^{\alpha\lambda(t_0-h-\tau)} ,$$

where C_i $(i=1, 2)$ are constants corresponding to C of (5.8). Hence

$$C_1=C_2e^{-\alpha\lambda h}<C_2 .$$

Consequently, we obtain (5.12) from (5.10).

If $a=\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}$, then $(y_*, 0)$ is the only point at which the straight line $y=y_0$ touches γ. Therefore, it follows from (5.3) that, if $b=-a\lambda$, then $\phi(t, a, b)$ is a solution of (1.6) which satisfies (5.9) and (5.10).

THEOREM 2. (I) *If* $0<a<\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}$, *then there exist two numbers* b_1 *and* b_2 *with* $b_1>b_2$ *such that* $\phi(t, a, b_i)$ $(i=1, 2)$ *can be continued up to* $t=-\infty$ *and*

$$0<\lim_{t\to-\infty}\phi(t, a, b_1)<\lim_{t\to-\infty}\phi(t, a, b_2)<\infty .$$

Moreover, $\phi(t, a, b_i)$ *can be represented by an analytical expression of the form*

$$\phi(t, a, b_i)=A_i\sum_{n=0}^{\infty}\phi_n^{(i)}e^{\alpha\lambda nt}$$

in the neighborhood of $t=-\infty$, *where* A_i *and* $\phi_n^{(i)}$ *are constants and* $\phi_0^{(i)}=1$.

(II) *If* $a=\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}$ *and* $b=-a\lambda$, *then* $\phi(t, a, b)$ *can be continued up to* $t=-\infty$ *and*

$$0<\lim_{t\to-\infty}\phi(t, a, b)<\infty .$$

Moreover, $\phi(t, a, b)$ *can be represented by an analytical expression of the form*

$$\phi(t, a, b)=A\sum_{n=0}^{\infty}\phi_n e^{\alpha\lambda nt}$$

in the neighborhood of $t=-\infty$, *where* A *and* ϕ_n *are constants and* $\phi_0=1$.

Next, we consider the case when the initial value (y_0, z_0) is placed in D. Suppose that

$$0<a<\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}, \qquad b_2<b<b_1 .$$

Then, from (5.3), we have $(y_0, z_0)\in D$. Now, we take the solution $(y(s), z(s))$ of (1.9) passing through the point (y_0, z_0). From $(y(s), z(s))$, we can get a solution $z_+(y, b)$ of (1.8) in the region $z>0$ and a solution $z_-(y, b)$ of (1.8) in the region $z<0$. Hence, it follows from Lemma 4 that there exists a number $t(b)$ such that $y=y(t, b)$ can be defined by

$$\frac{dy}{dt}=\begin{cases} z_+(y, b) & \text{if} \quad t>t(b) \\ 0 & \text{if} \quad t=t(b) \\ z_-(y, b) & \text{if} \quad t<t(b) \end{cases} \tag{5.13}$$
$$y(t_0)=y_0 ,$$

uniquely. By using $y(t, b)$, we get the solution $\phi(t, a, b)$ of (1.6) which satisfies (5.1) from (1.7).

On the other hand, it follows from Lemma 1 that

$$\lim_{y\to-0} z_-(y, b)=0 ,$$

$$\lim_{y\to-0} \frac{z_-(y, b)}{y}=\alpha\lambda-0 . \tag{5.14}$$

Hence, it follows from Lemma 2 and (2.15) that

$$\limsup_{y\to-0} \frac{v_-(y)}{y}>\lambda \tag{5.15}$$

where

$$v_-(y, b)=y^{-1}z_-(y, b)-\alpha\lambda .$$

Moreover, from (5.2) and (5.14), we have

$$\lim_{t\to\omega'} \frac{\phi'(t, a, b)}{\phi(t, a, b)}=-0 .$$

Since $\phi(t, a, b)>0$ and $\phi''(t, a, b)<0$, we have

$$\phi'(t, a, b)<0 . \tag{5.16}$$

Hence, as $t\to\omega'$, $\phi(t, a, b)$ does not tend to 0. Therefore, if $-\infty<\omega'<t_0$, then

$$\lim_{t \to \omega'} \phi(t, a, b) = \infty .$$

However, since $\phi''(t, a, b) < 0$, there exists a positive number T_1 such that, if $t \leqq T_1$, then

$$\phi'(t, a, b) > \phi'(T, a, b) .$$

Integrating the both sides,

$$\phi(T_1, a, b) - \phi(t, a, b) > \phi'(T_1, a, b)(T_1 - t) .$$

Hence, as $t \to \omega'$, we have a contradiction $\omega' = -\infty$. Thus we have

$$\omega' = -\infty . \tag{5.17}$$

From (1.7) and (5.2),

$$y^{-1} v_-(y, b) = -\alpha \lambda^2 \frac{\phi'(t, a, b)}{e^{\alpha \lambda t} \phi(t, a, b)^{1+\alpha}} . \tag{5.18}$$

However, since (5.16) holds, $\lim_{t \to \omega'} \phi(t, a, b)$ exists. If $0 < \lim_{t \to \omega'} \phi(t, a, b) < \infty$, then it follows from (5.17) that

$$\lim_{t \to -\infty} e^{\alpha \lambda t} \phi(t, a, b)^{1+\alpha} = 0 .$$

Hence, if $\lim_{t \to -\infty} \phi'(t, a, b) = 0$, then, applying l'Hospital's theorem to (5.18), we have

$$\lim_{y \to \infty} y^{-1} v_-(y, b) = \lambda .$$

This is contrary to (5.15). Hence we have

$$\lim_{t \to -\infty} \phi'(t, a, b) \neq 0 .$$

Namely, for all t, there exists a positive number ε such that

$$\phi'(t, a, b) < -\varepsilon < 0 .$$

This implies

$$\lim_{t \to -\infty} \phi(t, a, b) = \infty ,$$

which contradicts our hypothesis. Thus we have

$$\lim_{t \to -\infty} \phi(t, a, b) = \infty .$$

THEOREM 3. *If* $0 < a < \lambda^{2/\alpha} e^{-\lambda t_0} (-y_*)^{1/\alpha}$ *and* $b_2 < b < b_1$, *then* $\phi(t, a, b)$ *can* *be continued up to* $t=-\infty$ *and*

$$\lim_{t\to-\infty}\phi(t, a, b)=\infty .$$

Finally, we consider the case when (y_0, z_0) is in the outside of the closure of D. In this case, one of the following three statements holds;
(i) $\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}\geqq a>0$ and $b>b_1$,
(ii) $\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}\geqq a>0$ and $b<b_2$,
(iii) $\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}<a$.
Here it is supposed that, if $a=\lambda^{2/\alpha}e^{-\lambda t_0}(-y_*)^{1/\alpha}$, then $b_1=b_2=-a\lambda$.

In the same manner as in the previous case, we take a solution $(y(s), z(s))$ of (1.9) passing through the point (y_0, z_0) and get the solutions $z_+(y, b)$ and $z_-(y, b)$ of (1.8) which are nonnegative and nonpositive respectively. Furthermore, since we can define $y(t, b)$ uniquely so as to satisfy (5.13), we can get the solution $\phi(t, a, b)$ of (1.6) which satisfies (5.1) from (1.7).

Since we showed in Lemma 3 that $z=\hat{z}_-(y)$ is the minimal solution of (1.8) satisfying $\lim_{y\to-0}y^{-1}z=\alpha\lambda$, we have from Lemma 1

$$\lim_{y\to-0}\frac{z_-(y, b)}{y}=\infty .$$

Hence, from (5.2), we have

$$\lim_{t\to\omega'}\frac{\phi'(t, a, b)}{\phi(t, a, b)}=\infty .$$

Since $\phi(t, a, b)>0$, there exists a constant T_1 such that $t<T_1$ implies

$$\phi'(t, a, b)>0 . \tag{5.19}$$

Hence, as $t\to\omega'$, $\phi'(t, a, b)$ does not tend to 0, for $\phi''(t, a, b)<0$. Namely, there exist constants ε and T_2 such that $t\leqq T_2$ implies

$$0<\varepsilon<\phi'(t, a, b) .$$

Therefore

$$\varepsilon(T_2-t)<\phi(T_2, a, b)-\phi(t, a, b) .$$

From this we can deduce a contradiction

$$\lim_{t\to-\infty}\phi(t, a, b)=-\infty .$$

Hence we have $\omega'\neq-\infty$. Consequently, from (5.19),

$$\lim_{t \to \omega'} \phi(t, a, b) = 0 .$$

Therefore, from (1.7), we have

$$\lim_{t \to \omega'} y(t, b) = 0 .$$

Hence, if we put

$$\eta = -y , \qquad w = \eta z^{-1} ,$$

then we get (4.6) from (1.8). In the same manner as in § 4, we have an analytical expression of the solution $\phi(t, a, b)$ of (1.6), if $1/\alpha$ is not an integer.

THEOREM 4. *Suppose that a and b satisfy one of the following statements;*
(i) $\lambda^{2/\alpha} e^{-\lambda t_0}(-y_*)^{1/\alpha} \geqq a > 0$ *and* $b > b_1$,
(ii) $\lambda^{2/\alpha} e^{-\lambda t_0}(-y_*)^{1/\alpha} \geqq a > 0$ *and* $b < b_2$,
(iii) $\lambda^{2/\alpha} e^{-\lambda t_0}(-y_*)^{1/\alpha} < a$,
where $b_1 = b_2 = -a\lambda$ *if* $a = \lambda^{2/\alpha} e^{-\lambda t_0}(-y_*)^{1/\alpha}$. *Let* (ω', ω) *denote the domain of* $\phi(t, a, b)$. *Then* ω' *is finite and*

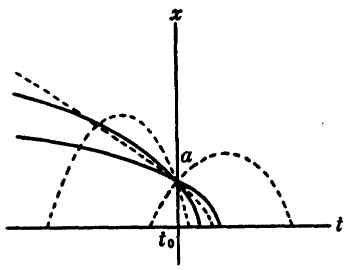
$$\lim_{t \to \omega'} \phi(t, a, b) = 0 .$$

Furthermore, if $1/\alpha$ *is not an integer, then* $\phi(t, a, b)$ *can be represented by an analytical expression of the form*

$$\phi(t, a, b) = A(t - \omega')\{1 + \sum_{m+n>0} \phi_{mn}(t - \omega')^{\alpha m + n}\}$$

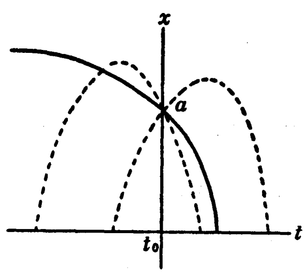
in the neighborhood of $t = \omega'$, *where* A *and* ϕ_{mn} *are constants.*

From the above theorems, we can draw figures which represent the asymptotic behavior of solutions of (1.6) as follows:



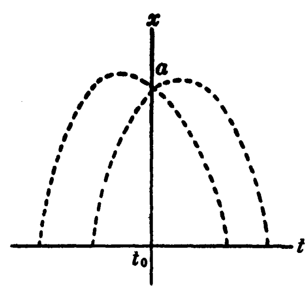
$0 < a < \lambda^{2/\alpha} e^{-\lambda t_0}(-y_*)^{1/\alpha}$

FIGURE 3.1



$\lambda^{2/\alpha} e^{-\lambda t_0}(-y_*)^{1/\alpha} = a$

FIGURE 3.2



$\lambda^{2/\alpha} e^{-\lambda t_0}(-y_*)^{1/\alpha} < a$

FIGURE 3.3

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