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Harmonic Bloch and BMO Functions on the Unit Ball in Several Variables

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Introduction.

Harmonic Bloch functions on the unit ball are those harmonic functions h for which the quantity $|\nabla h(x)|(1-|x|^2)$ is bounded (for x in the ball). We prove that the space of harmonic Bloch functions on the unit ball is isomorphic to the space of harmonic BMO functions on the unit ball as Banach spaces. In this proof, we use the stochastic theory to give a good estimate (inequality (1.2) in Theorem).

§1. Preliminaries and the main theorem.

Let D_n be the unit ball in the n dimensional Euclidian space ($n \geqq 2$) and $H(D_n)$ the space of real harmonic functions on D_n.

A function h in $H(D_n)$ is said to be a *harmonic Bloch function* if

$$\|h\|_{H,n} = \sup_{x \in D_n} \frac{1}{2}(1-|x|^2)|\nabla h(x)| < \infty ,$$

where $|\nabla h(x)| = \{\sum_{i=1}^{n} (\partial h(x)/\partial x_i)^2\}^{1/2}$. The space of harmonic Bloch functions is denoted by $B_H(D_n)$.

Let $p \geqq 1$. A function h in $H(D_n)$ is said to be a *harmonic BMO$_p$ function* if

$$\|h\|_{p,n} = \sup \left\{ \frac{1}{|B|} \int_B |h(x)-h(b)|^p dx \right\}^{1/p} < \infty$$

where the supremum is taken over all balls B in D_n, $|B| = \int_B dx$, and b is

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the center of B. The space of harmonic BMO_p functions is denoted by $BMOH_p(D_n)$.

The spaces $B_H(D_n)$ and $BMOH_p(D_n)$ are Banach spaces (modulo constant) with norms $\|\ \|_{H,n}$ and $\|\ \|_{p,n}$, respectively. For any $p\geqq 1$, we prove that $B_H(D_n)$ is isomorphic to $BMOH_p(D_n)$ as Banach spaces. In fact, we show the following theorem.

THEOREM. *Let $p\geqq 1$. Then there is a positive constant $c(p, n)$, depending on p and n, such that for every h in $H(D_n)$*

$$\frac{1}{c(p, n)}\|h\|_{p,n}\leqq\|h\|_{H,n}\leqq c(p, n)\|h\|_{p,n} . \tag{1.1}$$

In particular, when $p=2$ we have

$$\left(\frac{1}{\alpha(n)}\right)^{1/2}\|h\|_{2,n}\leqq\|h\|_{H,n}\leqq(n(n+2))^{1/2}\|h\|_{2,n} , \tag{1.2}$$

where

$$\alpha(n)=2n\int_0^1\log\frac{1}{1-r^2}\, r^{n-1}dr .$$

We note that $\alpha(2m)=1+1/2+\cdots+1/m$.

In the special case when $n=2$, the above result was essentially obtained by Coifman, Rochberg and Weiss [3], and Gotoh [6].

§2. Proof of Theorem.

By the John-Nirenberg type inequality, (1.1) follows from (1.2) (see, for example, [4, Ch. VII]). So we only show (1.2).

To prove the left inequality of (1.2), we need some lemmas. By an elementary calculation, we have the following lemma.

LEMMA 1. *For a point b in D_n and a positive number a with $a<1-|b|$, put $T(x)=ax+b$ for x in D_n. Then we have*

$$a(1-|x|^2)\leqq 1-|T(x)|^2 .$$

Let (B_t) be the n dimensional Brownian motion starting at the origin and let $T(r)$ be the stopping time $\inf\{t>0;\ |B_t|\geqq r\}$. By the Ito formula, we obtain the following lemma.

LEMMA 2. *If $0<r<1$, then we have*

$$E\left[\int_0^{T(r)}\left(\frac{1}{1-|B_s|^2}\right)^2 ds\right]\leqq\frac{1}{2}\log\frac{1}{1-r^2}\ ,$$

where $E[\quad]$ *is the expectation.*

PROOF. Let $g(x)=(1/2)\log(1/(1-|x|^2))$ for x in D_n. Then we have

$$\begin{aligned}\Delta g(x)&=\frac{n-(n-2)|x|^2}{(1-|x|^2)^2}\\&\geqq\frac{2}{(1-|x|^2)^2}\ ,\end{aligned}$$

where Δ is the usual Laplacian. By the Ito formula (see [5, p. 68]), we have

$$\begin{aligned}E\left[\int_0^{T(r)}\left(\frac{1}{1-|B_s|^2}\right)^2 ds\right]&\leqq E\left[\int_0^{T(r)}\frac{1}{2}\Delta g(B_s)ds\right]\\&=E[g(B_{T(r)})]=\frac{1}{2}\log\frac{1}{1-r^2}\ .\end{aligned}$$

Thus Lemma 2 is proved.

Fix $n\geqq 2$. For short, we denote the Bloch norm $\|\quad\|_{H,n}$ by $\|\quad\|_H$ and the BMO norm $\|\quad\|_{p,n}$ by $\|\quad\|_p$. Also, we denote D_n by D and ∂D denotes its boundary.

PROOF of the left inequality of (1.2). Let B be the ball in D with the center b and the radius a. Put $T(x)=ax+b$ for x in D. Then

$$\frac{1}{|B|}\int_B|h(x)-h(0)|^2dx=\frac{1}{|D|}\int_D|h(T(x))-h(T(0))|^2dx\ ,$$

and by Lemma 1,

$$\begin{aligned}\|h\circ T\|_H^2&=\sup\frac{1}{2}(1-|x|^2)|\nabla(h\circ T)(x)|\\&=\sup\frac{1}{2}(1-|x|^2)a|\nabla h(T(x))|\\&\leqq\sup\frac{1}{2}(1-|T(x)|^2)|\nabla h(T(x))|\\&\leqq\|h\|_H^2\ .\end{aligned}$$

Hence it suffices to prove that

(2.1) $$\frac{1}{|D|}\int_D |h(x)-h(0)|^2 dx \leqq n\|h\|_H^2 .$$

Let $d\pi$ be the normalized Lebesgue measure on ∂D. In polar coordinates, we have

(2.2) $$\frac{1}{|D|}\int_D |h(x)-h(0)|^2 dx = n\int_0^1\int_{\partial D} |h(rx)-h(0)|^2 d\pi(x)\, r^{n-1} dr .$$

Since $B_{T(r)}$ is uniformly distributed on the sphere $\{|x|=r\}$, by the Ito formula and Lemma 2, we have

(2.3) $$\begin{aligned}\int_{\partial D} |h(rx)-h(0)|^2 d\pi(x) &= E[|h(B_{T(r)})-h(B_0)|^2] \\ &= E\left[\int_0^{T(r)} |\nabla h(B_s)|^2 ds\right] \\ &\leqq 4\|h\|_H^2 E\left[\int_0^{T(r)} \left(\frac{1}{1-|B_s|^2}\right)^2 ds\right] \\ &\leqq 2\|h\|_H^2 \log \frac{1}{1-r^2} .\end{aligned}$$

Combining (2.2) and (2.3), we have

$$\begin{aligned}\frac{1}{|D|}\int_D |h(x)-h(0)|^2 dx &\leqq 2n\|h\|_H^2 \int_0^1 \log\left(\frac{1}{1-r^2}\right) r^{n-1} dr \\ &\leqq \alpha(n)\|h\|_H^2 ,\end{aligned}$$

where $\alpha(n)=2n\int_0^1 \log(1/(1-r^2))r^{n-1}dr$. Therefore we obtain (2.1) which proves the left inequality of (1.2).

To prove the right inequality of (1.2), we need the following lemma.

LEMMA 3. *Let f be a continuous function on $\bar{D}$. If f is a harmonic function on D, then we have*

$$|\nabla f(0)|^2 \leqq n^2 \int_{\partial D} |f(y)|^2 d\pi(y) .$$

PROOF. Let $k_y(x)=(1-|x|^2)/|x-y|^n$. By Durrett [5, p. 36], we have

$$f(x)=\int_{\partial D} k_y(x) f(y) d\pi(y) .$$

Hence we obtain

$$\begin{aligned}|\nabla f(0)|^2 &\leqq \int_{\partial D} |\nabla k_y(0)|^2 |f(y)|^2 d\pi(y) \\ &= \int_{\partial D} \frac{|f(y)|^2 n^2}{|y|^{2n+2}} d\pi(y) \\ &= n^2 \int_{\partial D} |f(y)|^2 d\pi(y) .\end{aligned}$$

Thus we verify the lemma.

PROOF of the right inequality of (1.2). Let B be the ball in D such that the center is b and the radius is $1-|b|$. For $0<r<1-|b|$, put $T_r(x)=rx+b$. Then by Lemma 3, we have

$$\begin{aligned}\|h\|_2^2 &\geqq \frac{1}{|B|}\int_B |h(x)-h(b)|^2 dx \\ &= \frac{n}{(1-|b|)^n}\int_0^{1-|b|}\int_{\partial D} |h(T_r(x))-h(T_r(0))|^2 d\pi(x)\, r^{n-1} dr \\ &\geqq \frac{|\nabla h(b)|^2}{n(1-|b|^n)}\int_0^{1-|b|} r^{n+1} dr \\ &= \frac{|\nabla h(b)|^2(1-|b|)^2}{n(n+2)} \\ &\geqq \frac{1}{n(n+2)}\left(\frac{1}{2}(1-|b|^2)|\nabla h(b)|\right)^2 .\end{aligned}$$

This completes the proof of the right inequality of (1.2).

REMARK. The left inequality of (1.1) can be also proved without using the Ito formula. But our stochastic proof gives a better constant. Inequality (2.3), the key estimate in this paper, owes to the Ito formula.

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