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Minimal Tori in S^3 Whose Lines of Curvature Lie in S^2

Dedicated to Professor Morio Obata on his 60th birthday

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Introduction

Let $\varphi: \Sigma \to S^3$ be a minimal immersion of a compact orientable surface Σ into the unit 3-sphere S^3. It is valuable to study the set of such immersions with Σ of given genus. For example, when Σ is of genus 0, i.e., Σ is the 2-sphere, φ must be the totally geodesic immersion of S^2 into S^3 [3] [1] [4].

Assume Σ is the torus. In this case, there is the well-known minimal isometric *embedding* of the flat square torus $S^1(1/\sqrt{2})\times S^1(1/\sqrt{2})$ into S^3 called *the Clifford immersion*. Though there are many minimal immersions of the torus into S^3, they are not embedded. Thus, it is conjectured that the only minimal embedding of the torus into S^3 is the Clifford one [7].

To study this, we consider minimal immersions of a torus into S^3 having the following property:

($*$) Each line of curvature of the immersions lies in some totally geodesic 2-sphere in S^3.

The main theorem of this paper is the following:

THEOREM. (1) *There exist infinitely many minimal immersions of the torus into S^3 satisfying* ($*$).

(2) *A minimal immersion of the torus into S^3 satisfying* ($*$) *is not an embedding provided that it is congruent with the Clifford one.*

§1. Preliminaries.

Let $\varphi: \Sigma \to S^3$ be a smooth immersion of a surface into the unit 3-

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sphere. The first fundamental form of φ is the induced metric $g=\varphi^*\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ is the standard metric of S^3. The second fundamental form h of φ is defined as $h(X, Y)=-\langle\bar{\nabla}_X\nu, Y\rangle$ for all vectors X and Y tangent to φ, where ν is the unit normal vector field of φ and $\bar{\nabla}$ is the canonical connection of S^3.

The existence of isothermal coordinates shows us that there exist local coordinates (u, v) of Σ in which g is written as

$$g=e^{\sigma}(du^2+dv^2)\ , \tag{1.1}$$

where σ is a smooth function of u and v. Write the second fundamental form in these coordinates as

$$h=Ldu^2+2Mdudv+Ndv^2\ , \tag{1.2}$$

where L, M and N are functions of u and v.

The mean curvature of φ is the function H on Σ defined by

$$H=\frac{1}{2}e^{-\sigma}(L+N) \tag{1.3}$$

in the present isothermal coordinates. The immersion φ is called *minimal* when H is identically 0, i.e., $N=-L$ in (1.2).

In these coordinates, the equation of Gauss is

$$-\frac{1}{2}e^{-\sigma}\Delta\sigma=(LN-M^2)e^{-2\sigma}+1\ , \quad \text{where} \quad \Delta=\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\ . \tag{1.4}$$

Consider the complex function f of $z=u+iv$

$$f(z)=M+iN\ . \tag{1.5}$$

When φ is minimal, the equation of Codazzi holds if and only if f is a holomorphic function of z.

§2. Fundamental equation.

Suppose $\varphi\colon \Sigma\to S^3$ be a minimal immersion of the torus. On taking the universal cover of Σ, φ is lifted to the minimal immersion $\tilde{\varphi}\colon \boldsymbol{R}^2\to S^3$. Since the induced metric $\tilde{g}=\tilde{\varphi}^*\langle\ ,\ \rangle$ is conformal to the flat metric of $\boldsymbol{R}^2$ [2], there exist *global* coordinates (u, v) in which the first fundamental form is

$$\tilde{g}=e^{\sigma}(du^2+dv^2)\ , \tag{2.1}$$

where σ is a smooth function on $\boldsymbol{R}^2$ which is invariant by the deck transformations of the cover $R^2 \to \Sigma$, i.e., σ is a doubly periodic function. The second fundamental form of $\tilde{\varphi}$ is written as (1.2), where L, M and N are also doubly periodic functions defined on R^2.

Since $\tilde{\varphi}$ is minimal, the doubly periodic function f in (1.5) is holomorphic on the whole complex plane. Hence by Liouville's theorem, L, M and N must be constant on R^2. Then, by a suitable change of coordinates, we may assume the second fundamental form is diagonalized as

$$h = L(du^2 - dv^2)\ ,$$

where L is a positive constant. Replacing u, v and σ by $u/\sqrt{L}$, $v/\sqrt{L}$ and $\sigma + \log L$ respectively, we have the first fundamental form (2.1) and the second fundamental form

$$h = du^2 - dv^2\ . \tag{2.2}$$

By (2.1) and (2.2), the equation of Gauss (1.4) becomes

$$\Delta\sigma = -4\sinh\sigma\ . \tag{2.3}$$

Conversely, by the fundamental theorem of the theory of surfaces [5], we have the following proposition:

PROPOSITION 2.1. (1) *If $\varphi\colon \Sigma \to S^3$ is a minimal immersion of the torus, and $\tilde{\varphi}\colon R^2 \to S^3$ is the lift of φ to the universal cover of Σ, then there exist coordinates (u, v) of R^2 in which the first and the second fundamental forms of $\tilde{\varphi}$ are written as* (2.1) *and* (2.2) *respectively, and the function σ in* (2.1) *satisfies* (2.3).

(2) *If a smooth function σ on R^2 satisfies* (2.3), *then there exists a minimal immersion $\varphi_\sigma\colon R^2 \to S^3$ whose first and the second fundamental forms are* (2.1) *and* (2.2) *respectively. Moreover, such an immersion is unique up to congruence.*

REMARK. Even if σ in (2.3) is doubly periodic, the corresponding immersion φ_σ is not necessarily doubly periodic. To study minimal immersions of the torus into S^3, we must search for doubly periodic solutions of (2.3) whose corresponding immersions are also doubly periodic.

The trivial solution of (2.3) is $\sigma = 0$. In this case, the corresponding minimal immersion φ_0 is an isometric minimal immersion of R^2 with flat metric which is written explicitly as

$$\varphi_0(u, v) = \left(\frac{1}{\sqrt{2}}\cos\sqrt{2}\,u, \frac{1}{\sqrt{2}}\sin\sqrt{2}\,u, \frac{1}{\sqrt{2}}\cos\sqrt{2}\,v, \frac{1}{\sqrt{2}}\sin\sqrt{2}\,v\right) \in S^3\ ,$$

where $S^3=\{(x^0, x^1, x^2, x^3)\in \boldsymbol{R}^4;\ \sum_{i=0}^{3}(x^i)^2=1\}$. Since φ_0 is doubly periodic, it gives the minimal isometric immersion of the flat torus $\boldsymbol{R}^2/\Gamma$ into S^3, where Γ is the lattice on $\boldsymbol{R}^2$ generated by $\{(0, \sqrt{2}\pi), (\sqrt{2}\pi, 0)\}$. This immersion is called the *Clifford immersion*, which has the following properties:

(1) It is the only isometric minimal immersion of the flat torus into S^3 up to congruence.

(2) The immersion is one-to-one, i.e., it is an embedding.

(3) The area of the immersed torus is $2\pi^2$.

(4) The immersion is given by the first eigenfunctions of the laplacian of $\boldsymbol{R}^2/\Gamma$. In other words, the first eigenvalue of the laplacian of $\boldsymbol{R}^2/\Gamma$ is 2.

§ 3. Lines of curvature.

Suppose $\varphi\colon \boldsymbol{R}^2\to S^3$ be a minimal immersion with the first and the second fundamental forms (2.1) and (2.2) respectively.

Vector fields $\partial/\partial u$ and $\partial/\partial v$ give the principal directions of h, and their integral curves are the lines of curvature of φ. Let

$$c_u(v)=\varphi(u, v)\ , \qquad c_v(u)=\varphi(u, v)\ . \tag{3.1}$$

Then curves c_u and c_v in S^3 are lines of curvature of φ parametrized by v and u respectively. The following lemma is easy to show.

LEMMA 3.1. (1) *The curve c_u has the curvature*

$$\kappa_u=\frac{1}{2}e^{-\sigma/2}\{(\partial_u\sigma)^2+4e^{-\sigma}\}^{1/2}$$

and the torsion

$$\tau_u=e^{-\sigma/2}\left[\left\{\partial_v\left(\frac{e^{-\sigma/2}\partial_u\sigma}{2\kappa_u}\right)\right\}^2+\left\{\partial_v\left(\frac{e^{-\sigma}}{\kappa_u}\right)\right\}^2\right]^{1/2}.$$

(2) *The curve c_v has the curvature*

$$\kappa_v=\frac{1}{2}e^{-\sigma/2}\{(\partial_v\sigma)^2+4e^{-\sigma}\}^{1/2}$$

and the torsion

$$\tau_v=e^{-\sigma/2}\left[\left\{\partial_u\left(\frac{e^{-\sigma/2}\partial_v\sigma}{2\kappa_v}\right)\right\}^2+\left\{\partial_u\left(\frac{e^{-\sigma}}{\kappa_v}\right)\right\}^2\right]^{1/2}.$$

LEMMA 3.2. *Each line of curvature of φ lies in some totally geodesic*

2-sphere in S^3 if and only if σ is the following form:

$$\sigma(u, v)=\log\{U(u)+V(v)\}^2 , \tag{3.2}$$

where U and V are smooth functions on $\boldsymbol{R}$.

PROOF. Suppose σ is as in (3.2). So, it is an easy consequence of Lemma 3.1 that τ_u and τ_v are identically 0 for any u and v. Then each c_u and c_v lies in some totally geodesic 2-sphere in S^3.

Conversely, if each τ_u is identically 0, $\partial_v(e^{-\sigma}/\kappa_u)$ must be identically 0. Hence $4(e^{\sigma}\kappa_u)^2=(\partial_u e^{\sigma/2})^2+4$ must depend only on u. Let $\partial_u e^{\sigma/2}=U(u)$. Then $e^{\sigma/2}=U(u)+V(v)$ for some function $V(v)$ and the conclusion follows. □

PROPOSITION 3.3. *Let φ: $\boldsymbol{R}^2\to S^3$ be a minimal immersion with the first and the second fundamental forms* (2.1) *and* (2.2) *respectively. Then each line of curvature of φ lies in some totally geodesic 2-sphere in S^3 if and only if $\sigma(u, v)$ depends only on one variable u or v.*

PROOF. If σ depends only on u or v, c_u and c_v are curves without torsion because of Lemma 3.1.

Assume each c_u or c_v lies in a totally geodesic S^2. Then σ is written as (3.2). Substituting (3.2) in (2.3), we have

$$U''(U+V)+V''(U+V)-(U')^2-(V')^2=1-(U+V)^4 ,$$

where $U'=dU/du$, $V'=dV/dv$, *etc.* Differentiating this equation by u and v,

$$U'''V'+U'V'''=-12(U+V)^2U'V' .$$

If $U'V'\neq 0$, then

$$\left(\frac{U'''}{U'}\right)+\left(\frac{V'''}{V'}\right)=-12(U+V)^2 .$$

Differentiating the above, we obtain $U'V'=0$. So, $U'V'$ must be identically 0. Hence U or V is a constant function. □

§4. Differential equation.

In this section, we construct a family of minimal immersions of $\boldsymbol{R}^2$ into S^3 whose lines of curvature lie in some totally geodesic 2-spheres in S^3.

Let φ: $\boldsymbol{R}^2\to S^3$ be one of such immersions. So, by Propositions 2.1 and 3.3, there exist coordinates (u, v) of $\boldsymbol{R}^2$ with the following properties:

(1) The first fundamental form of φ is

$$g=e^{\sigma}(du^2+dv^2)\ , \tag{4.1}$$

(2) the second fundamental form of φ is

$$h=du^2-dv^2\ , \tag{4.2}$$

(3) the function σ depends only on v, and
(4) the function $\sigma(v)$ satisfies the ordinary differential equation:

$$\frac{d^2\sigma}{dv^2}=-4\sinh\sigma\ . \tag{4.3}$$

The equation (4.3) has an integral:

$$\frac{1}{2}\left(\frac{d\sigma}{dv}\right)^2+4\cosh\sigma=4\alpha\ ,$$

where α is an integral constant. Then for each $\alpha\in[1,\infty)$, there exists a unique solution σ_α such that:

$$\frac{1}{2}\left(\frac{d\sigma_\alpha}{dv}\right)^2+4\cosh\sigma_\alpha=4\alpha\ , \tag{4.4}$$

$$\sigma_\alpha(0)=\log a\ ,\quad \text{where}\quad a=\alpha+\sqrt{\alpha^2-1}\ , \tag{4.5}$$

$$\frac{d^2\sigma_\alpha}{dv^2}(0)\leqq 0\ . \tag{4.6}$$

LEMMA 4.1. *The solutions* $\{\sigma_\alpha;\ \alpha\in[1,\infty)\}$ *have the following properties*:

(1) $$\sigma_1=0\ .$$

(2) *For each* $\alpha\in(1,\infty)$, σ_α *is a periodic function with period*

$$T(\alpha)=\frac{2}{\sqrt{a}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-(1-a^{-2})\sin^2 x}}\ .$$

(3) $$\sigma_\alpha(v)=\sigma_\alpha(-v)\ ,\quad \frac{d\sigma_\alpha}{dv}(v)=-\frac{d\sigma_\alpha}{dv}(-v)\ .$$

(4) $$-\log a\leqq\sigma_\alpha\leqq\log a\ .$$

(5) σ_α *is simply decreasing on* $\left[0,\ \frac{T(\alpha)}{2}\right]$ *and increasing on* $\left[\frac{T(\alpha)}{2},\ T(\alpha)\right]$.

PROOF. (1) and (3) are immediate consequences of (4.4).

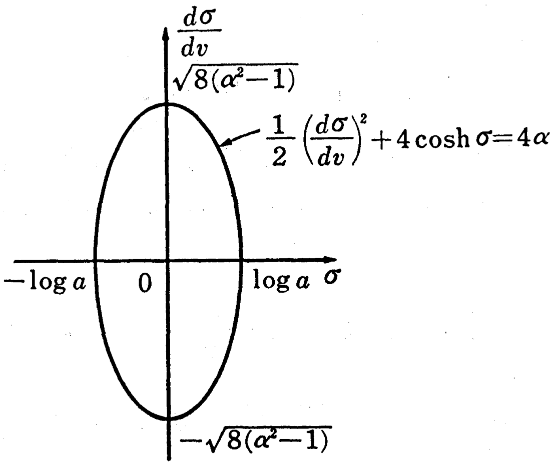


FIGURE 1

Figure 1 is the phase curve of the solution σ_α of the equation (4.4). The tangent vectors $(d\sigma_\alpha/dv, d^2\sigma_\alpha/dv^2)$ of this curve never vanishes, so σ_α is periodic with period

$$T(\alpha)=\int_0^{T(\alpha)} dv=-2\int_{\log\alpha}^{-\log\alpha}\frac{d\sigma_\alpha}{d\sigma_\alpha/dv}$$
$$=2\int_{-\log\alpha}^{\log\alpha}\frac{d\sigma}{\sqrt{8(\alpha-\cosh\sigma)}}$$
$$=\frac{2}{\sqrt{\alpha}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-(1-\alpha^{-2})\sin^2 x}}\ ,$$

and thus (2) is proved.

By Figure 1, (4) and (5) are also proved. □

By Proposition 1.1, there exists the immersion φ_α of $\boldsymbol{R}^2$ into S^3 defined by (4.1), (4.2) and $\sigma=\sigma_\alpha$. Since $\sigma_1=0$, the immersion φ_1 is the Clifford immersion.

REMARK. Though the period of the Clifford immersion in the direction v is $\sqrt{2}\pi$, $\lim_{\alpha\downarrow 1} T(\alpha)=\pi$. This shows that the Clifford immersion is *isolated* in the family $\{\varphi_\alpha\}$ as an immersion of the torus.

Consider the lines of curvature of φ_α,

$$c_u^\alpha(v)=\varphi_\alpha(u, v)\ , \qquad c_v^\alpha(u)=\varphi_\alpha(u, v)\ .$$

Since they lie in some totally geodesic 2-spheres in S^3, we may consider each of c_u^α and c_v^α as a curve in $S^2\subset\boldsymbol{R}^3$. By Lemma 3.1, we obtain the following lemma.

LEMMA 4.2. (1) *c_v^α is the curve in S^2 with the curvature*

$$\kappa_v^\alpha=\sqrt{2\alpha e^{-\sigma_\alpha}-1}\ .$$

(2) *c_v^α is a small circle with radius $e^{\sigma_\alpha/2}/\sqrt{2\alpha}$ in $\boldsymbol{R}^3$.*

(3) *φ_α gives a minimal immersion of the cylinder whose fundamental domain is*

$$\left\{(u,\ v);\ 0\leqq u<\sqrt{\frac{2}{\alpha}}\pi\right\}\subset\boldsymbol{R}^2\ .$$

(4) *c_u^α is the curve in S^2 with the curvature*

$$\kappa_u^\alpha=e^{-\sigma_\alpha}\ .$$

(5) *The curves c_u^α are congruent with each other.*

§5. Existence of minimal tori.

In this section, we prove the first part of the main theorem.

Let σ_α and φ_α be as in the previous section. Then by Lemma 4.2 (3), φ_α gives an immersion of the cylinder.

Assume φ_α gives an immersion of the torus and c_u^α never closes up in S^3. Then the image of c_u^α is dense in the image of φ_α. On the other hand, the image of c_u^α lies in some totally geodesic 2-sphere, then the image of φ_α lies in the 2-sphere. This is impossible. Hence φ_α gives an immersion of the torus if and only if the curve c_u^α is closed with some integral times of the period of σ_α.

The first part of the main theorem is an immediate consequence of the following proposition:

PROPOSITION 5.1. *There exist countably many α's in $(1,\ \infty)$ such that the curve c_u^α is closed with period $k_\alpha T(\alpha)$ for some positive number $k_\alpha\geqq 2$.*

We shall prove this later.

Take $\alpha\in(1,\ \infty)$, and let $T=T(\alpha)$ and $\sigma=\sigma_\alpha$. Consider $c=c_u^\alpha|_{[0,T(\alpha)]}$ as a curve in $S^2\subset\boldsymbol{R}^3$. Let $\kappa=\kappa_u^\alpha=e^{-\sigma}$ be the curvature of c as the curve in S^2 and $\tilde{\kappa}=\sqrt{\kappa^2+1}$ that of c as the curve in $\boldsymbol{R}^3$. In the rest of this section, we take the arc length s as the parameter of c instead of v. To begin with, we have the following lemma:

LEMMA 5.2.

(1) $$\text{Length of } c=\int_0^T e^{\sigma/2}dv=\pi\ .$$

(2) $$\int_0^\pi \kappa ds=\pi\ .$$

(3) $$\int_0^\pi \tilde{\kappa} ds<2\pi\ ;$$

PROOF. Since $\|dc/dv\|=e^{\sigma/2}$,

$$\begin{aligned}\text{length of } c &=\int_0^T e^{\sigma/2}dv\\ &=2\int_{-\log\alpha}^{\log\alpha}\frac{e^{\sigma/2}d\sigma}{\sqrt{8(\alpha-\cosh\sigma)}} \qquad \text{(by (4.4))}\\ &=\pi\ ,\end{aligned}$$

then (1) is proved.

Similarly, (2) is true because

$$\begin{aligned}\int_0^\pi \kappa ds&=\int_0^T e^{-\sigma}e^{\sigma/2}dv\\ &=2\int_{-\log\alpha}^{\log\alpha}\frac{e^{-\sigma/2}d\sigma}{\sqrt{8(\alpha-\cosh\sigma)}}\\ &=-2\int_{\log\alpha}^{-\log\alpha}\frac{e^{\rho/2}d\rho}{\sqrt{8(\alpha-\cosh\rho)}}\\ &=\pi\ .\end{aligned}$$

Finally, by Lemma 5.2,

$$\begin{aligned}\int_0^\pi \tilde{\kappa} ds&=\int_0^\pi \sqrt{\kappa^2+1}\,ds\\ &<\int_0^\pi \kappa ds+\int_0^\pi ds\\ &=2\pi\ ,\end{aligned}$$

so (3) is proved. □

This lemma leads the following:

LEMMA 5.3. *If the curve c is closed with period k_α-times that of the period of the metric $e^{-\sigma/2}$, then $k_\alpha\geqq 2$.*

PROOF. If $k_\alpha=1$, the total curvature of the closed curve c as a curve in $\boldsymbol{R}^3$ is

$$\int_0^\pi \tilde{\kappa} ds<2\pi$$

by Lemma 5.1. On the other hand, by Fenchel's theorem [5], the total curvature of a closed space cannot be less than 2π. This is impossible. □

Let e (resp. n) be the unit tangent vector (resp. the unit normal vector) of c as a curve in S^2. So, $(c(s), e(s), n(s))$ forms the moving frame of R^3 along c. Define $F(\alpha)$ to be the orthogonal matrix which changes the frame $(c(0), e(0), n(0))$ to $(c(\pi), e(\pi), n(\pi))$. So $F(\alpha)$ is a continuous curve in $SO(3)$ parametrized by $\alpha \in (1, \infty)$.

Each orthogonal matrix $A \in SO(3)$ is conjugate to a matrix

$$R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Let $\theta(\alpha)$ be a continuous function such that $F(\alpha)$ is conjugate to $R(\theta(\alpha))$. In terms of F, the curve c is closed with period k_α-times that of the metric $e^{\sigma/2}$ if and only if $F(\alpha)^{k_\alpha}$ is the identity matrix. This condition is equivalent to

$$k_\alpha\theta(\alpha)\equiv 0 \mod 2\pi . \tag{5.1}$$

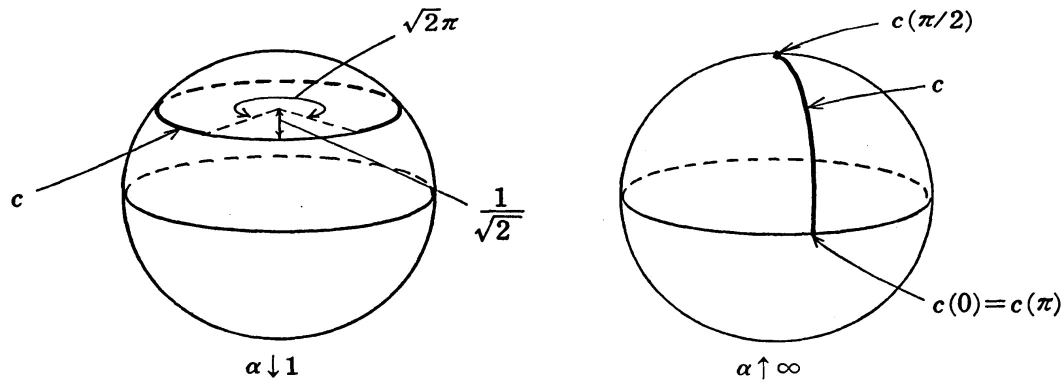
To prove Proposition 5.1, we see the behavior of the curve $F(\alpha)$ when α tends to 1 and ∞.

LEMMA 5.4.

(1) $$\lim_{\alpha\downarrow 1}\theta(\alpha)=\sqrt{2}\,\pi .$$

(2) $$\lim_{\alpha\uparrow\infty}\theta(\alpha)=\pi .$$

PROOF. The curve c converges to the small circle with radius $1/\sqrt{2}$ in R^3 as $\alpha\downarrow 1$.



Behavior of c when α tends to 0 and ∞.

FIGURE 2

By Lemma 5.2, the length of c is π independent of α, so the angle between $(c(0), e(0), n(0))$ and $(c(\pi), e(\pi), n(\pi))$ tends to $\sqrt{2}\pi$ as $\alpha \downarrow 1$ (Figure 2). Then (1) is true.

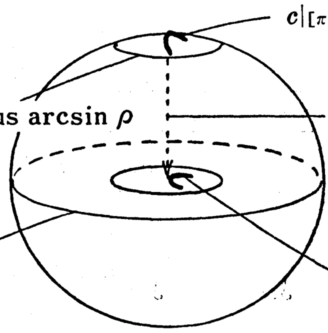


FIGURE 3

To prove (2), we consider c as a curve in $\boldsymbol{R}^3$ such that $c(\pi/2)$ is the north pole (0, 0, 1) of the unit sphere, and E denotes the equator of the unit sphere as in Figure 3. Let ρ be a small positive number. So the curve $c_0 = c|_{[0,\pi/2-\rho]}$ and $c_1 = c|_{[\pi/2+\rho,\pi]}$ converge to the great circles in S^2 with length $\pi/2-\rho$ as $\alpha \uparrow \infty$ because σ tends to ∞ and $\kappa = e^{-\sigma}$ tends uniformly to 0.

Let $\tilde{c}$ be the orthogonal projection of $c|_{[\pi/2-\rho,\pi/2+\rho]}$ to the plane containing E, and $\tilde{k}$ the total curvature of $\tilde{c}$. For sufficiently small ρ, $\tilde{k}$ is nearly equal to the total curvature of $c|_{[\pi/2-\rho,\pi/2+\rho]}$. Then by Lemma 5.2 and the fact that the curvature of c is concentrated in $s=\pi/2$ as $\alpha \uparrow \infty$, we have

$$\lim_{\alpha \uparrow \infty} \tilde{k} = \pi + \delta(\rho) ,$$

where $\lim_{\rho \downarrow 0} \delta(\rho) = 0$. So the rotation number of $\tilde{c}$ tends to $1/2 + \delta'(\rho)$ as $\alpha \uparrow \infty$, where $\lim_{\rho \downarrow 0} \delta' = 0$.

Hence, the curve c converges to a curve consisting of two great arcs of length $\pi/2$ which meet at north pole with angle π. This shows that $\lim_{\alpha \uparrow \infty} \theta(\alpha) = \pi$ and (2) is proved. □

PROOF OF PROPOSITION 5.1. By Lemma 5.4, there exist countably many α's in $(1, \infty)$ such that $\theta(\alpha)/2\pi$ are rational numbers. For such α, the lines of curvature c are closed in $[0, k_\alpha \pi]$. Moreover, by Lemma 5.3, $k_\alpha \geqq 2$. □

§6. Proof of non-embeddedness.

In this section, we prove the last part of the main theorem. This is the immediate consequence of the following proposition.

PROPOSITION 6.1. *If the curve c in the previous section is closed in S^2 with period k-times that of its metric, where $k \geqq 2$, then c must have a self-intersection.*

PROOF. Assume c has no self-intersection. So, c bounds a simply connected domain Ω of S^2 such that the normal vector field of c is the inward normal of $\partial\Omega$. By Gauss-Bonnet theorem for a domain of a surface [5], we have

$$\int_{\Omega} 1 dv + \int_{\partial\Omega} \kappa ds = 2\pi ,$$

where dv is the canonical area element of S^2. On the other hand, the total curvature of $\partial\Omega$ is

$$\int_{\partial\Omega} \kappa ds = k \int_0^{\pi} \kappa ds = k\pi ,$$

because of Lemma 5.2 (3). Then,

$$\text{Area of } \Omega = \int_{\Omega} 1 dv = (2-k)\pi \leqq 0 .$$

This is impossible. □

This completes the proof of the main theorem.

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