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On $2p$-fold Transitive Permutation Groups

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Introduction.

In Yoshizawa [6], the following two theorems were proved.

Theorem A. Let p be an odd prime. Let G be a permutation group on a set $\Omega=\{1, 2, \cdots, n\}$ which satisfies the following condition. For any $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ of Ω, a Sylow p-subgroup P of the stabilizer in G of the $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ is nontrivial and fixes exactly $2p+r$ points of Ω, and moreover P is semiregular on the set $\Omega-I(P)$ of the remaining $n-2p-r$ points, where r is independent of the choice of $\alpha_1, \cdots, \alpha_{2p}$ and $0\leqq r\leqq p-2$. Then $n=3p+r$, and there exists an orbit Γ of G such that $|\Gamma| \geqq 3p$ and $G^{\Gamma} \geqq A^{\Gamma}$.

Theorem B. Let p be an odd prime $\geqq 11$. Let G be a permutation group on a set $\Omega=\{1, 2, \cdots, n\}$ which satisfies the following condition. For any $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ of Ω, a Sylow p-subgroup P of the stabilizer in G of the $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ is nontrivial and fixes exactly $3p-1$ points of Ω, and moreover P is semiregular on the set $\Omega-I(P)$ of the remaining $n-3p+1$ points. Then $n=4p-1$, and one of the following two cases holds: (1) There exists an orbit Γ of G such that $|\Gamma|\geqq 3p$ and $G^{\Gamma}\geqq A^{\Gamma}$. (2) G has just two orbits Γ_1 and Γ_2 with $|\Gamma_1|\geqq p$, $|\Gamma_2|\geqq p$ and $|\Gamma_1|+|\Gamma_2|=4p-1$, and G^{Γ_i} is $(|\Gamma_i|-p+1)$-transitive on $\Gamma_i (i=1, 2)$. Moreover, $G^{\Gamma_i}\geqq A^{\Gamma_i}$ if $|\Gamma_i|\geqq p+3$.

In [1], E. Bannai determined all $2p$-fold transitive permutation groups in which the stabilizer of $2p$ points is of order prime to p, where p is an odd prime. By using Theorem A and Theorem B in [6], we will improve it, namely, we will prove the following result.

THEOREM 1. *Let p be an odd prime $\geqq 11$, and let q be an odd prime with $p<q<p+p/3$. Let G be a $2p$-fold transitive permutation group on $\Omega=\{1, 2, \cdots, n\}$. If the order of $G_{1,2,\cdots,2p}$ is not divisible by q, then G is $S_n (2p\leqq n\leqq 2p+q-1)$ or $A_n (2p+2\leqq n\leqq 2p+q-1)$.*

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Besides, by using Theorem 1, we will prove the following result.

THEOREM 2. *Let p be an odd prime $\geqq 11$, and let q be an odd prime with $p<q<p+p/3$. Let G be a $2p$-fold transitive permutation group on $\Omega=\{1, 2, \cdots, n\}$. If $G_{1,2,\cdots,2p}$ has an orbit on $\Omega-\{1, 2, \cdots, 2p\}$ whose length is less than q, then G is $S_n(2p+1\leqq n\leqq 2p+q-1)$ or $A_n(2p+2\leqq n\leqq 2p+q-1)$.*

As an immediate corollary to Theorem 2, we have the following result.

COROLLARY. *Let p be an odd prime $\geqq 11$, and let q be an odd prime with $p<q<p+p/3$. Let D be a $2p$-$(v, k, 1)$ design with $2p<k<2p+q$. If an automorphism group G of D is $2p$-fold transitive on the set of points of D, then D is a $2p$-$(k, k, 1)$ design, namely a trivial design.*

We shall use the same notation as in [4].

§ 1. Proof of Theorem 1.

Let G be a counter example to the theorem.

Let P be a Sylow p-subgroup of $G_{1,2,\cdots,2p}$. Then $P\neq 1$ and P is not semiregular on $\Omega-I(P)$, by [1, Main Theorem] and [2, Theorem 1]. Set $|I(P)|\equiv r(\mathrm{mod}\, p)$, where $0\leqq r\leqq p-1$. We first show that $r\leqq q-p-1$. Suppose, by way of contradiction, that $r\geqq q-p$. Let R be a subgroup of P such that the order of R is maximal among all subgroups of P fixing at least $3p$ points. By Theorem A and Theorem B in [6], we have $|I(R)|=3p+r\geqq 2p+q$, and moreover we have that there exist $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ of $I(R)$ such that $N_G(R)^{I(R)}_{\alpha_1,\cdots,\alpha_{2p}}$ has a q-cycle. This contradicts the assumption of Theorem 1.

Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing at least $4p$ points. (It may be that $Q=1$.) We may assume that $I(Q)=\{1, 2, \cdots, |I(Q)|\}$. Set $N=N_G(Q)^{I(Q)}$. Then N satisfies the following properties.

(i) N is a permutation group on $I(Q)$. $|I(Q)|\geqq 4p$.

(ii) For any $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ of $I(Q)$, the order of $N_{\alpha_1,\cdots,\alpha_{2p}}$ is divisible by p but is not divisible by q.

(iii) For any element x of order p of N fixing at least $2p$ points, $|I(x)|$ must be $2p+r$ or $3p+r$. Moreover, by Theorem A and Theorem B in [6], we have

(iv) N has an element a of order p with $|I(a)|=3p+r$.

We may assume that

$$a=(1)(2)\cdots(3p+r)(3p+r+1, \cdots, 4p+r)\cdots.$$

Set $T=C_N(a)^{I(a)}_{3p+r+1,\cdots,4p+r}$. Then T satisfies the following three properties.

(1) T is a permutation group on $I(a)$. $|I(a)|=3p+r$.

(2) For any p points $\alpha_1, \cdots, \alpha_p$ of $I(a)$, the order of $T_{\alpha_1,\cdots,\alpha_p}$ is divisible by p.

(3) For any p points $\alpha_1, \cdots, \alpha_p$ of $I(a)$, the order of $T_{\alpha_1,\cdots,\alpha_p}$ is not divisible by q.

We will show that such T does not exist. Let $\Delta_1, \cdots, \Delta_s$ be the orbits of T with $|\Delta_i| \geqq p (i=1, \cdots, s)$.

Suppose that $s \geqq 3$. Since $|I(a)|=3p+r \leqq 4p-1$, we have $s=3$. Set $|\Delta_i|=p+k_i (i=1, 2, 3)$. Then, $(p+k_1)+(p+k_2)+(p+k_3) \leqq 3p+r$, and so, $(k_1+1)+(k_2+1)+(k_3+1) \leqq r+3 \leqq q-p+2<p/3+2<p$. We get a contradiction, by (2).

Suppose that $s=2$. We may assume that $|\Delta_1| \geqq |\Delta_2|$. Then, $|\Delta_2| \leqq (3p+r)/2 \leqq (2p+q-1)/2=p+(q-1)/2$. Set $|\Delta_2|=p+k$. Then, $k \leqq (q-1)/2<(2/3)p-1/2$, and so, $p-(k+1)>p/3-1/2$. We take $k+1$ points $\alpha_1, \cdots, \alpha_{k+1}$ from Δ_2. Set $V=T^{\Delta_1}_{\alpha_1,\cdots,\alpha_{k+1}}$. Then by (2), for any $p-k-1$ points $\beta_1, \cdots, \beta_{p-k-1}$ of Δ_1, $V_{\beta_1,\cdots,\beta_{p-k-1}}$ has an element of order p. Assume that V has just two orbits Σ_1 and Σ_2 with $|\Sigma_i| \geqq p (i=1, 2)$. Set $|\Sigma_i|=p+l_i (i=1, 2)$. In this case, $l_i<p (i=1, 2)$. Since $|\Sigma_1|+|\Sigma_2|+|\Delta_2| \leqq |I(a)|=3p+r$, we have that $(p+l_1)+(p+l_2)+(p+k) \leqq 3p+r$. So, $l_1+l_2+2 \leqq r-k+2$. Hence, $p-k-1-(l_1+1)-(l_2+1) \geqq p-k-1-(r-k+2)=p-r-3 \geqq p-(q-p-1)-3=2p-q-2 \geqq 0$. We take l_1+1 points $\gamma_1, \cdots, \gamma_{l_1+1}$ from Σ_1 and l_2+1 points $\delta_1, \cdots, \delta_{l_2+1}$ from Σ_2. Then, the order of $V_{\gamma_1,\cdots,\gamma_{l_1+1},\delta_1,\cdots,\delta_{l_2+1}}$ is not divisible by p, which is a contradiction. Therefore, we may assume that V has the only one orbit Σ with $|\Sigma| \geqq p$. We remark that for any $p-k-1$ points $\eta_1, \cdots, \eta_{p-k-1}$ of Σ, $V^{\Sigma}_{\eta_1,\cdots,\eta_{p-k-1}}$ has an element of order p. Especially, $|\Sigma| \geqq p+(p-k-1)>p+(p/3-1/2)$. Hence $|\Sigma|>q-1/2$. Then we have $|\Sigma| \geqq q$. Suppose that V^{Σ} is imprimitive. Let $\{\Pi_1, \cdots, \Pi_t\}$ be the system of imprimitivity of V^{Σ}. Set $|\Pi_1|=d$. Assume $d \geqq p$. So, we have $t=2$. Moreover, we can see that $2(d-p)+2>p-k-1$. Hence, $2d+k>3p-3$. On the other hand, since $|\Sigma|+|\Delta_2| \leqq 3p+r$, we have $2d+p+k \leqq 3p+r$. So, $2d+k \leqq 2p+r$. Hence $2p+r>3p-3$, which is a contradiction. Therefore, $d<p$. Then, we can see that $d(p-k-1)+dp \leqq |\Sigma|$. Since $|\Sigma|+|\Delta_2| \leqq 3p+r$ and $d \geqq 2$, we have that $2(p-k-1)+2p+(p+k) \leqq 3p+r$. Hence, $5p-k-2 \leqq 3p+r \leqq 2p+q-1$, which is a contradiction. Therefore, V^{Σ} is primitive. Since $p-k-1>p/3-1/2$, by [5, Theorem 13.10], we have $V^{\Sigma} \geqq A^{\Sigma}$. Since $|\Sigma| \geqq q$, this contradicts (3).

Therefore, we have $s=1$. By (2), for any p points $\alpha_1, \cdots, \alpha_p$ of Δ_1, $T^{\Delta_1}_{\alpha_1,\cdots,\alpha_p}$ has an element of order p. Hence, we have that $2p \leqq |\Delta_1| \leqq 3p+r$.

Assume that T^{Δ_1} is imprimitive. Let $\{\Gamma_1, \cdots, \Gamma_v\}$ be the system of imprimitivity of T^{Δ_1}, and let $|\Gamma_1|=f$. If $f\geqq p$, then $v=2$ or 3, and we get a contradiction by using the similar argument to that of the above case $s=2, 3$ respectively. Hence, we have $f<p$. So, we can see that $pf+pf\leqq|\Delta_1|$. But, this is a contradiction, since $f\geqq 2$. Therefore, T^{Δ_1} is primitive. Then by [5, Theorem 13.10], we have $T^{\Delta_1}\geqq A^{\Delta_1}$, which is a contradiction by (3).

§2. **Proof of Theorem 2.**

Let G be a counter example to the theorem. Let Δ be an orbit of $G_{1,2,\cdots,2p}$ on $\Omega-\{1, 2, \cdots, 2p\}$ such that $|\Delta|<q$. By [4], we have $2\leqq|\Delta|<q$.

Let Q be a Sylow q-subgroup of $G_{1,2,\cdots,2p}$. Then $Q\neq 1$, by Theorem 1. By the lemma of Witt [5, Theorem 9.4] and Theorem 1, we have $N_G(Q)^{I(Q)}\geqq A^{I(Q)}$. So, $N_G(Q)_{1,2,\cdots,2p}^{I(Q)-\{1,2,\cdots,2p\}}\geqq A^{I(Q)-\{1,2,\cdots,2p\}}$. Hence we have $I(Q)=\Delta\cup\{1, 2, \cdots, 2p\}$, since $I(Q)\supset\Delta$. This shows that $I(Q)$ is independent of the choice of Sylow q-subgroup Q of $G_{1,2,\cdots,2p}$ and is uniquely determined by $G_{1,2,\cdots,2p}$. Let R be a subgroup of Q such that the order of R is maximal among all subgroups of Q fixing more than $|I(Q)|$ points. We may assume that $I(R)=\{1, 2, \cdots, |I(R)|\}$. Set $N=N_G(R)^{I(R)}$ and $|\Delta|=l$. Then N satisfies the following properties.

(i) N is a permutation group on $I(R)$. $|I(R)|\equiv 2p+l \pmod q$.

(ii) For any $2p$ points $\alpha_1, \cdots, \alpha_{2p}$ of $I(R)$, a Sylow q-subgroup S of $N_{\alpha_1,\cdots,\alpha_{2p}}$ satisfies that $S\neq 1$, $|I(S)|=2p+l$, S is semiregular on $I(R)-I(S)$, and $I(S)$ is independent of the choice of Sylow q-subgroup S of of $N_{\alpha_1,\cdots,\alpha_{2p}}$ and is uniquely determined by $N_{\alpha_1,\cdots,\alpha_{2p}}$.

Let x be an element of order q of $N_{1,\cdots,2p}$. Then, we may assume that

$$x=(1)(2)\cdots(2p+l)(2p+l+1, \cdots, 2p+l+q)\cdots .$$

Set $T=C_N(x)_{2p+l+1,\cdots,2p+l+q}^{I(x)}$. Then T satisfies the following properties.

(1) T is a permutation group on $I(x)$. $|I(x)|=2p+l$.

(2) For any $2p-q$ points $\alpha_1, \cdots, \alpha_{2p-q}$ of $I(x)$, a Sylow q-subgroup M of $T_{\alpha_1,\cdots,\alpha_{2p-q}}$ is a cyclic group of order q generated by a q-cycle, and $|I(M)|=2p-q+l$. Moreover, $I(M)$ is independent of the choice of Sylow q-subgroup M of $T_{\alpha_1,\cdots,\alpha_{2p-q}}$ and is uniquely determined by $T_{\alpha_1,\cdots,\alpha_{2p-q}}$.

Suppose that T is primitive. By [5, Theorem 13.9], we have $T\geqq A^{I(x)}$, which contradicts (2).

Next suppose that T is imprimitive. Let $\{\Gamma_1, \cdots, \Gamma_s\}$ be the system of imprimitivity of T, and let $|\Gamma_1|=d$. Assume that $d<q$. Then, we can see that $d(2p-q)+dq\leqq 2p+l$. Since $2\leqq d$, we have $2(2p-q)+2q\leqq$

$2p+l$. So, we have $2p-l\leqq 0$, which is a contradiction. Therefore $d\geqq q$, and so, we have $s=2$. Set $d=q+k$. Then, we can see that $2(k+1)>2p-q$. On the other hand, we have $2(q+k)=2p+l$. Hence, $2p+l-2q+2>2p-q$. Thus $l=q-1$, and so, we have $k=(2p-q-1)/2$. We take $2p-q-k$ points $\alpha_1, \cdots, \alpha_{2p-q-k}$ from Γ_1. Since $(q+k)-(2p-q-k)<q$, we have that for any k points $\beta_1, \cdots, \beta_k$ of Γ_2, $T^{\Gamma_2}_{\alpha_1,\cdots,\alpha_{2p-q-k},\beta_1,\cdots,\beta_k}$ has a q-cycle. So, $T^{\Gamma_2}_{\alpha_1,\cdots,\alpha_{2p-q-k}}$ is a k-transitive group, by [3, Lemma 6]. Hence, $T^{\Gamma_2}_{\alpha_1,\cdots,\alpha_{2p-q-k}}\geqq A^{\Gamma_2}$ by [5, Theorem 13.9]. We take $2p-q$ points $\gamma_1, \cdots, \gamma_{2p-q}$ from Γ_1. By considering the Sylow q-subgroups of $T_{\gamma_1,\cdots,\gamma_{2p-q}}$, we get a contradiction by (2).

Therefore, T is an intransitive group. Suppose that T has just two orbits Γ_1 and Γ_2 with $|\Gamma_i|\geqq q (i=1, 2)$. Set $|\Gamma_i|=q+k_i (i=1, 2)$. Then, we can see that $(k_1+1)+(k_2+1)>2p-q$. On the other hand $(q+k_1)+(q+k_2)\leqq 2p+l-1$. So, we have $2p-q-2<2p-2q+l-1$. Thus $q-1<l$, which is a contradiction. Therefore, we may assume that T has the only one orbit Γ with $|\Gamma|\geqq q$. We take a point α from $I(x)-\Gamma$ and $2p-q$ points $\beta_1, \cdots, \beta_{2p-q}$ from Γ. Let $\langle a\rangle$ be a Sylow q-subgroup of $T_{\beta_1,\cdots,\beta_{2p-q}}$, where a is a q-cycle. We take a point γ from $I(x)-I(a)$. Let $\langle b\rangle$ be a Sylow q-subgroup of $T_{\beta_1,\cdots,\beta_{2p-q-1},\gamma}$, where b is a q-cycle. Since $\langle a\rangle$ and $\langle b\rangle$ are Sylow q-subgroups of $T_{\alpha,\beta_1,\cdots,\beta_{2p-q-1}}$, we get a contradiction by (2).

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