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A Construction of Transversal Flows for Maximal Markov Automorphisms

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Introduction.

For a strongly mixing transformation defined on a Lebesgue measure space, it is quite frequently the case that each point of the space possesses a neighborhood which can be decomposed into two kinds of fibres (expansive and contractive) and that each fibre is left fixed by the strongly mixing transformation. It was Y. G. Sinai who formulated the concept of transversal fields (transversal fibres) for a transformation and tried to describe this type of phenomenon precisely. Using this behavior of transversal fields, Sinai succeeded in giving a useful sufficient condition for a transformation to be a K-automorphism [1].

Subsequently, generalizing the above results of Sinai I. Kubo gave a useful formulation which could be applied to many concrete situations [2]. M. Kowada pointed out the importance of considering the pair of a transformation and its transversal fields, and investigated a number of properties of such pairs [3], [4].

Most basic examples of transformations possessing transversal fields are ergodic group automorphisms on an n-dimensional torus and Bernoulli shifts. These examples share the following characteristic features. First of all, every such transformation is known to be isomorphic to a Markov automorphism [5]. Secondly, the metric entropy for each of these transformations coincides with the topological entropy. Thirdly, the transversal fields for each of these automorphisms are flows which are ergodic and have the discrete spectrum.

In this paper, we shall show that ergodic Markov automorphisms for which the metric entropy coincides with the topological entropy always possess transversal flows. In § 1, we shall define Markov subshifts [6], maximal Markov automorphisms and transversal flows, and discuss some basic properties. In § 2, we shall prove a representation

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theorem which represents a maximal Markov automorphism as a transformation on some subset D of $[0, 1)\times[0, 1)$. This transformation is a generalization of the well-known Baker's transformation and will be called the generalized Baker's transformation of Markov type. In § 3, we shall construct transversal flows on the subset D as special flows in the sense of Ambrose-Kakutani [7]. The base transformations for the transversal flows constructed above are described in full and are shown to be an analogue of the well known "adding machine" transformations. Finally, in § 4 we shall show that the transversal flows constructed in § 3 are ergodic and have zero entropy if the maximal Markov automorphism is strongly mixing.

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§ 1. Definitions and discussion of basic properties.

In this section we first give the definition of a Markov automorphism and then discuss several notions associated with it.

Let us denote by Z the set of all integers and by N the set of all non-negative integers. We shall call the finite ordered set $s=\{0, 1, 2, \cdots, s-1\}$ the state space, and elements of S is called symbols. We put the discrete topology on the set S and put the product topology on the product spaces S^Z and S^N, which are compact Hausdorff spaces. For each $n\in Z$ or $n\in N$, we shall denote by $\omega(n)$ the n-th coordinate of an element $\omega\in S^Z$ or S^N.

The two-sided shift operator σ on the space S^Z is defined by the relation

$$(\sigma\omega)(n)=\omega(n+1) \quad \text{for} \quad n\in Z\,.$$

The system (S^Z, σ) will be called the two-sided shift on S. The one-sided shift (S^N, σ) on S is defined similarly.

An $s\times s$ matrix M will be called a symbolic matrix if every entry $m(i, j)$ of M equals 0 or 1. An n-tuple $(a(0), a(1), \cdots, a(n-1))$ of symbols $a(k)\in S$ is said to be admissible with respect to a symbolic matrix M if $m(a(k), a(k+1))=1$ holds for $k=0, 1, 2, \cdots, n-2$. A two-sided infinite sequence $\omega=(\cdots, \omega(-1), \omega(0), \omega(1), \cdots)\in S^Z$ is said to be admissible with respect to a symbolic matrix M if $m(\omega(n), \omega(n+1))=1$ holds for all $n\in Z$. We shall denote by X_M the set of all elements in S^Z which are admissible with respect to a fixed symbolic matrix M;

namely,

$$X_M=\{\omega\in S^Z \mid m(\omega(n),\ \omega(n+1))=1 \quad \text{for all } n\in \boldsymbol{Z}\}\ .$$

It is easy to see that X_M is a closed and shift invariant subset of S^Z.

DEFINITION 1.1. For a symbolic matrix M, we call the subsystem (X_M, σ) of the system (S^Z, σ) a Markov subshift corresponding to M. The matrix M will be called the structure matrix of the Markov subshift (X_M, σ).

We can similarly consider a subsystem (X_M^+, σ) (called a one-sided Markov subshift) of the system (S^N, σ) by considering the set $X_M^+=\{\omega\in S^N \mid m(\omega(i), \omega(i+1))=1$ for all $i\in \boldsymbol{N}\}$.

DEFINITION 1.2. A structure matrix M will be called (i) irreducible if for each $i, j\in S$ there exists an integer $n\geqq 1$ such that $m^{(n)}(i, j)\geqq 1$ where $m^{(n)}(i, j)$ denotes the (i, j)-th entry of the matrix M^n,

(ii) aperiodic if there exists an integer $n\geqq 1$ such that $m^{(n)}(i, j)\geqq 1$ for all $i, j\in S$.

For an irreducible structure matrix M, one can construct an invariant Markov measure μ_M for the Markov subshift (X_M, σ) in the following manner (cf. the Perron-Frobenius theorem). Namely, corresponding to the largest positive characteristic value λ we choose a positive characteristic column vector $\boldsymbol{x}=(x_0, x_1, \cdots, x_{s-1})$ and a positive characteristic row vector $\boldsymbol{y}=(y_0, y_1, \cdots, y_{s-1})$ normalized in such a way that $\sum_{i=0}^{s-1} x_i y_i=1$. If we put $p(i, j)=x_j m(i, j)/\lambda x_i$ and $\pi_i=x_i y_i$, then $P=(p(i, j))$ becomes a stochastic matrix and the vector $\pi=(\pi_0, \cdots, \pi_{s-1})$ becomes a row vector of stationary probabilities. The pair (P, π) induces a σ-invariant Markov measure on X_M, which will be denoted by μ_M.

We shall list some properties of the dynamical system (X_M, σ, μ_M) which reflect properties of the matrix M.

PROPOSITION 1.3.

(i) *The dynamical system* (X_M, σ, μ_M) *is ergodic if and only if* M *is irreducible.*

(ii) *The dynamical system* (X_M, σ, μ_M) *is strongly mixing if and only if* M *is aperiodic.*

(iii) *If we denote by* $h_\mu(X_M, \sigma)$ *the entropy of* (X_M, σ) *with respect to a* σ*-invariant measure* μ *and denote by* $h_{\text{top}}(X_M, \sigma)$ *the topological entropy of* (X_M, σ), *then*

(a) $$\sup_\mu h_\mu(X_M, \sigma)=\max_\mu h_\mu(X_M, \sigma)=h_{\mu_M}(X_M, \sigma)\ ,$$

where the supremum and the maximum are taken over all σ-invariant probability measures on X_M.

(b) $$h_{\mu_M}(X_M, \sigma)=h_{\text{top}}(X_M, \sigma)=\log \lambda ,$$

where λ is the largest positive characteristic value of M. And we have

(c) $$h_\mu(X_M, \sigma)<h_{\mu_M}(X_M, \sigma) ,$$

for any σ-invariant probability measure μ on X_M such that $\mu\neq\mu_M$.

Next, we shall give the definition for a transversal flow for an automorphism, which is a special case of a transversal field in the sense of Y. G. Sinai.

DEFINITION 1.4. Let T be an automorphisms of a Lebesgue space $(X, \mathscr{F}, \mu)$. If let $\{Z_t^{(i)}\}(i=1, 2)$ be flows (i.e., one-parameter groups of measure preserving transformation) on $(X, \mathscr{F}, \mu)$. $\{Z_t^{(1)}\}$ (resp. $\{Z_t^{(2)}\}$) is called a contractive (resp. an expansive) transversal flow for the automorphism T with expansive constant λ (resp. $1/\lambda$) if the following conditions are satisfied:

(i) $TZ_t^{(1)}=Z_{\lambda t}^{(1)}T$ (mod 0) holds for every $t\in R$.
(ii) $TZ_t^{(2)}=Z_{t/\lambda}^{(2)}T$ (mod 0) holds for every $t\in R$.
(iii) $h_\mu(X, T)=\log \lambda$.

§2. Representation of Markov subshifts (X_M^+, σ) and (X_M, σ).

In the remainder of this paper, we shall assume that the maximal Markov measure μ_M for the subshift (X_M, σ) is non-atomic and that the dynamical system (X_M, σ, μ_M) is ergodic. These assumptions are equivalent to the assumptions that the structure matrix M is irreducible and that the largest positive characteric value λ of M be greater than 1.

Let $(x_0, x_1, \cdots, x_{s-1})$ be the positive characteristic column vector corresponding to the largest characteristic value λ of M satisfying $\sum_{i=0}^{s-1} x_i=1$. Let us define a right continuous mapping f_M on the unit interval $[0, 1)$ to itself by

$$f_M(t)=\lambda\left(t-\sum_{i=0}^{k-1} x_i-x_k\sum_{i=0}^{j-1}\frac{x_i m(k, i)}{\lambda x_k}\right)+\sum_{i=0}^{j-1} x_i$$

for

$$t\in\left[\sum_{i=0}^{k-1} x_i+x_k\sum_{i=0}^{j-1}\frac{x_i m(k, j)}{\lambda x_k}, \sum_{i=0}^{k-1} x_i+x_k\sum_{i=0}^{i}\frac{x_i m(k, i)}{\lambda x_k}\right)$$

and for $0\leqq k\leqq s-1$, $0\leqq j\leqq s-1$, where we set $\sum_{i=0}^{-1} x_i=0$ for convenience.

Next, we define f_M^0 by $f_M^0(t)=t$ for all $t\in[0, 1)$ and define inductively $f_M^{n+1}(t)=f_M(f_M^n(t))$ for $n=0, 1, 2, \cdots$. Using the f_M^n's, one can define a mapping π_M of the unit interval $[0, 1)$ into the product space S^N in the following way:

$$\pi_M(t)(n)=k \quad \text{if} \quad \sum_{i=0}^{k-1} x_i \leqq f_M^n(t) < \sum_{i=0}^{k} x_i .$$

One can also define a mapping ρ_M of S^N into $[0, \infty)$ by

$$\rho_M(\omega)=\sum_{j=0}^{\omega(0)-1} x_j + x_{\omega(0)} \sum_{j=0}^{\omega(1)-1} \frac{x_j m(\omega(0), j)}{\lambda x_{\omega(0)}} + \cdots$$
$$\cdots + x_{\omega(0)} \frac{x_{\omega(1)} m(\omega(0), \omega(1))}{\lambda x_{\omega(0)}} \times \cdots$$
$$\times \frac{x_{\omega(n)} m(\omega(n-1), \omega(n))}{\lambda x_{\omega(n-1)}} \sum_{j=0}^{\omega(n+1)-1} \frac{x_j m(\omega(n), j)}{\lambda x_{\omega(n)}} + \cdots .$$

If $\omega\in X_M^+$, we see that $\rho_M(\omega)$ is written as

$$\rho_M(\omega)=\sum_{j=0}^{\omega(0)-1} x_j + \frac{1}{\lambda} \sum_{j=0}^{\omega(1)-1} m(\omega(0), j) x_j + \cdots$$
$$+ \frac{1}{\lambda^{n+1}} \sum_{j=0}^{\omega(n+1)-1} m(\omega(n), j) x_j + \cdots$$

and that $0\leqq\rho_M(\omega)\leqq 1$.

The following proposition can be proved easily:

PROPOSITION 2.1.
(i) $\pi_M \circ f_M = \sigma \circ \pi_M$ on $[0, 1)$
(ii) $\rho_M \circ \pi_M(t)=t$ for $t\in[0, 1)$
(iii) $f_M \circ \rho_M = \rho_M \circ \sigma$ on $Y_M^+=\pi_M([0, 1))$.

We can introduce a linear order in the space S^N by considering the lexicographic order; namely, we say $\omega>\omega'$ for $\omega, \omega'\in S^N$ if there exists an integer $n\geqq 0$ such that $\omega(k)=\omega'(k)$ for $0\leqq k<n$ and $\omega(n)>\omega'(n)$. For $i=0, 1, 2, \cdots, s-1$, we shall denote by ω_i the largest element of X_M^+ with respect to the lexicographic order among all the elements of X_M^+ whose 0-th coordinate is i. Namely, $\omega_i=\max_{>}\{\omega\in X_M^+ \mid \omega(0)=i\}$. It is easy to see that $\rho_M(\omega_i)=\sum_{j=0}^{i} x_j$ and that ω_i does not belong to the set Y_M^+.

PROPOSITION 2.2.
(1) *The closure of* Y_M^+ *in the product space* S^N *is* X_M^+.
(2) $\pi_M(t)<\pi_M(s)$ *holds if and only if* $0\leqq t<s<1$.

(3) *If* $\omega<\omega'$, *then* $\rho_M(\omega)\leqq\rho_M(\omega')$ *holds for* $\omega, \omega'\in X_M^+$.

(4) *The subset* $E=X_M^+-Y_M^+$ *of* X_M^+ *is countable and the mapping* ρ_M *is* $1-1$ *on* Y_M^+ *onto* $[0, 1)$. *Furthermore, for every* $t\in\rho_M(E)$, *the inverse image* $\rho_M^{-1}(\{t\})$ *consists of exactly two points* $\pi_M(t)$ *and* $\sup_{s<t}\pi_M(s)$.

PROOF. From the definition of π_M it follows that for any $n>0$ there exists a $t_n\in[0, 1)$ such that

$$\pi_M(t_n)(k)=\omega(k) \quad \text{for} \quad 0\leqq k\leqq n .$$

This implies that there exists a sequence $\{\omega_n\}$ of elements of Y_M^+ converging to ω. This proves the assertion (1). The assertions (2) and (3) follow easily from the definition of π_M and ρ_M.

We now proceed to prove the assertion (4). We first note that for any $t\in[0, 1)$

$$\rho_M^{-1}(t)=\{\omega\in X_M^+ \mid \sup_{s<t}\pi_M(s)\leqq\omega\leqq\inf_{s>t}\pi_M(s)\} .$$

Since the function f_M^n is right continuous for each $n\geqq 0$, we have $\inf_{s>t}\pi_M(s)=\pi_M(t)$. As is shown from an argument similar to the proof of the assertion (1), the relations $\omega\in X_M^+$ and $\omega<\pi_M(t)$ imply that $\omega\leqq\sup_{s<t}\pi_M(s)$. Consequently, $\rho_M^{-1}(t)=\{\sup_{s<t}\pi_M(s), \pi_M(t)\}$.

Assume now $\omega=\sup_{s<t}\pi_M(s)<\pi_M(t)=\omega'$. Then, there exists an $n>0$ such that $\omega(k)=\omega'(k)$ for $0\leqq k<n$, $\omega(n)<\omega'(n)$ and $\omega(n)=\max\{l \mid m(\omega(n-1), l)=1 \text{ and } l<\omega'(n)\}$. From the fact that $\rho_M(\omega)=\rho_M(\omega')=t$ it follows that

$$\begin{aligned} x_{\omega(n)}&+\frac{1}{\lambda}\sum_{j=0}^{\omega'(n+1)-1} m(\omega'(n), j)x_j+\frac{1}{\lambda^2}\sum_{j=0}^{\omega'(n+2)-1} m(\omega'(n+1), j)x_j+\cdots \\ &=\frac{1}{\lambda}\sum_{j=0}^{\omega(n+1)-1} m(\omega(n), j)x_j+\frac{1}{\lambda^2}\sum_{j=0}^{(n+2)-1} m(\omega(n+1), j)x_j+\cdots . \end{aligned}$$

Therefore,

$$\rho_M(\sigma^n\omega')-\sum_{j>\omega(n)}^{\omega'(n)-1} x_j=\rho_M(\sigma^n\omega) .$$

On the other hand,

$$\rho_M(\sigma^n\omega')\in\left[\sum_{j=0}^{\omega'(n)-1} x_j\, , \quad \sum_{j=0}^{\omega'(n)} x_j\right) ,$$

while

$$\rho_M(\sigma^n\omega)\in\left[\sum_{j=0}^{\omega(n)-1} x_j\, , \quad \sum_{j=0}^{\omega(n)} x_j\right] ,$$

from which it follows that $\rho_M(\sigma^n\omega')=\sum_{j=0}^{\omega'(n)-1} x_j$ and

$$\rho_M(\sigma^n\omega)=\sum_{j=0}^{\omega(n)} x_j. \quad \text{This implies that } \sigma^n\omega=\omega_{\omega(n)}\in X_M^+-Y_M^+=E .$$

The set E can be written as

$$E=\{\omega\in X_M^+ \mid \sigma^n\omega=\omega_i \quad \text{for some } n\in N \text{ and some } i\in\{0, 1, \cdots, s-1\}\}$$

and therefore, it is countable. This proves the assertion (4). We will state as a remark the following fact established in the proof of assertion (4).

REMARK 1. An element $\omega\in X_M^+$ belongs to $X_M^+-Y_M^+$ if and only if there exists an integer n and a state $i\in S$ such that $\sigma^n\omega=\omega_i$.

In order to construct an f_M-invariant measure on $[0, 1)$, let us define a function $h(t)$ on the unit interval $[0, 1)$ as follows:

$$h(t)=y_k \quad \text{if} \quad t\in\left[\sum_{j=0}^{k-1} x_j\,,\quad \sum_{j=0}^{k} x_j\right),$$

where $(y_0, y_1, \cdots, y_{s-1})$ is the positive characteristic row vector of M corresponding to the largest positive characteristic value λ satisfying the condition $\sum_{j=0}^{s-1} x_j y_j=1$.

We define the measure ν_M on the unit interval $[0, 1)$ by putting

$$\nu_M(A)=\int_A h(t)dt$$

for any Borel set A. Let $P=\{p_j\}_{j=0}^{s-1}$ be the partition of the unit interval $[0, 1)$ into sets $p_j=[\sum_{i=0}^{j-1} x_i, \sum_{i=0}^{j} x_i)$, $0\leqq j\leqq s-1$. Then the relation $\nu_M(f_M^{-1} A)=\nu_M(A)$ is easily verified for every atom of the partition $\bigvee_{k=0}^{n} f_M^{-k}(P)$, $n\in N$. Since $\bigvee_{k=0}^{\infty} f_M^{-k}(P)$ generates the σ-algebra of Borel subsets of $[0, 1)$, we can conclude that ν_M is an f_M-invariant measure, and hence, that the system $([0, 1), f_M, \nu_M)$ gives an endomorphism.

THEOREM 2.1. (*Representation Theorem*) *Suppose the maximal Markov endomorphism* (X_M^+, σ, μ_M) *is non-atomic and ergodic, then the system* (X_M^+, σ, μ_M) *is isomorphic to the system* $([0, 1), f_M, \nu_M)$ *under the isomorphism* ρ_M. *More precisely the following assertions hold*:

(i) $\mu_M\circ\rho_M=\nu_M$.

(ii) *The map* ρ_M: $Y_M^+\to[0, 1)$ *is one-to-one and onto, and hence* ρ_M: $X_M^+\to[0, 1]$ *is an isomorphism* (mod 0).

(iii) $\rho_M\circ\sigma=f_M\circ\rho_M$ *on* Y_M^+.

Before we prove a representation theorem for the maximal Markov automorphism, we need several notations. Let N' be the set $\{-1, -2, -3, \cdots, -n, \cdots\}$ and we denote by $M^*=(m^*(i, j))(0\leqq i, j\leqq s-1)$ the transpose of the structure matrix M. We now define X_M^- and $X_{M^*}^+$ as

$$X_M^-=\{(\cdots, \omega(-n), \cdots \omega(-2), \omega(-1)) \in S^{N'} \mid m(\omega(-n-1), \omega(-n))=1 \text{ for each positive integer } n\}$$

and

$$X_{M^*}^+=\{(\omega(-1), \omega(-2), \cdots, \omega(-n), \cdots) \in S^{N'} \mid m^*(\omega(-n), \omega(-n-1))=1 \text{ for each positive integer } n\} .$$

The space X_M^- can be identified naturally with $X_{M^*}^+$. The shift operator σ on $X_{M^*}^+$ is defined by

$$(\sigma\omega)(-n)=\omega(-n-1) \quad \text{for every positive integer } n .$$

Let $Q=(q(i, j))$ be the transition matrix and let $\pi'=(\pi_0', \pi_1', \cdots, \pi_{s-1}')$ be the stationary probability vector associated with the structure matrix M^* constructed by the Perron-Frobenius theorem. Then, it is easy to see that

$$\pi_i=\pi_i' \quad \text{for each } i , \quad 0\leqq i\leqq s-1 ,$$

and

$$q(i, j)=\frac{\pi_j}{\pi_i}p(j, i)=\frac{y_j m^*(i, j)}{\lambda y_i}$$

for each pair (i, j), $0\leqq i, j\leqq s-1$. Here, $(y_0, y_1, \cdots, y_{s-1})$ denotes the positive characteristic column vector of M^* with $\sum_{i=0}^{s-1} y_i=1$; $(y_0, y_1, \cdots, y_{s-1})$ is considered to be the positive characteristic row vector of M corresponding to the largest positive characteristic value. As was done for (X_M^+, σ), the σ-invariant measure μ_{M^*}, the set $Y_{M^*}^+$ and the mappings $\pi_{M^*}, \rho_{M^*}, f_{M^*}$ can be constructed for the one-sided Markov subshift $(X_{M^*}^+, \sigma)$, so that the system $(X_{M^*}^+, \sigma)$ is isomorphic with the endomorphism $([0, 1), f_{M^*}, \nu_{M^*})$.

We will now represent the space X_M as a kind of product of the spaces $X_{M^*}^+$ and X_M^+. Let us define the following two spaces:

$$X_{M^*}^+\otimes X_M^+=\{(\omega, \omega') \in S^{N'}\times S^N \mid \omega \in X_{M^*}^+, \omega' \in X_M^+ \quad \text{and} \quad m(\omega(-1), \omega'(0))=1\} ,$$

$$Y_{M^*}^+\otimes Y_M^+=\{(\omega, \omega') \in S^{N'}\times S^N \mid \omega \in Y_{M^*}^+, \omega' \in Y_M^+ \quad \text{and} \quad m(\omega(-1), \omega'(0))=1\} .$$

The space X_M of all M-admissible two-sided sequences can be naturally identified with the space $X_{M^*}^+\otimes X_M^+$ and the shift operator σ on X_M can be identified with the operator (denoted again by σ) on $X_{M^*}^+ \otimes X_M^+$ defined by

$$\sigma(\omega, \omega') = (\omega'(0) \cdot \omega, \sigma\omega') ,$$

where the symbol $\cdot$ denotes the concatination, i.e.,

$$u \cdot v = (u(1), u(2), \cdots, u(n), v(1), v(2), \cdots, v(m)) ,$$

if $u = (u(1), u(2), \cdots, u(n))$ and $v = (v(1), v(2), \cdots, v(m))$ (m may be infinite). We note that the inverse σ^{-1} of σ on $X_{M^*}^+ \otimes X_M^+$ is given by

$$\sigma^{-1}(\omega, \omega') = (\sigma\omega, \omega(-1) \cdot \omega') .$$

In what follows we shall call the system $(X_{M^*}^+ \otimes X_M^+, \sigma)$ the two-sided Markov subshift associated with the structure matrix M. It follows easily from Proposition 2.2 that the closure of $Y_{M^*}^+ \otimes Y_M^+$ in the product space $S^{N'} \times S^N$ is $X_{M^*}^+ \otimes X_M^+$ and that the set $X_{M^*}^+ \otimes X_M^+ - Y_{M^*}^+ \otimes Y_M^+$ is countable. The mapping $\rho_{M^*} \times \rho_M$ defined on $X_{M^*}^+ \otimes X_M^+$ by

$$(\rho_{M^*} \times \rho_M)(\omega, \omega') = (\rho_{M^*}(\omega), \rho_M(\omega'))$$

is one-to-one on the set $Y_{M^*}^+ \otimes Y_M^+$ and the image $D = (\rho_{M^*} \times \rho_M)(Y_{M^*}^+ \otimes Y_M^+) \subset [0, 1) \times [0, 1)$ can be represented as

$$D = \bigcup_{0 \leq j, k \leq s-1} \left\{ Q_j \times P_k \,\middle|\, Q_j = \left[\sum_{i=0}^{j-1} y_i ,\ \sum_{i=0}^{j} y_i\right), \quad P_k = \left[\sum_{j=0}^{k-1} x_j ,\ \sum_{j=0}^{k} x_j\right), \quad m(j, k) = 1 \right\} .$$

Let $\pi_{M^*} \times \pi_M$ be the mapping from D onto $Y_{M^*}^+ \otimes Y_M^+$ defined by

$$(\pi_{M^*} \times \pi_M)(x, x') = (\pi_{M^*}(x), \pi_M(x')) ,$$

and $f_{M^*} \times f_M$ be the mapping from D to D defined by

$$(f_{M^*} \times f_M)(x, x') = (y, f_M(x')) ,$$

where y is the unique number in the set

$$\left[\sum_{j=0}^{\pi_M(x')(0)-1} y_j ,\ \sum_{j=0}^{\pi_M(x')(0)} y_j\right) \cap f_{M^*}^{-1}(x) .$$

It is easy to see that $f_{M^*} \times f_M$ is a one-to-one map of D onto itself. From Proposition 2.1, we obtain the following:

PROPOSITION 2.3.
(1) $(\rho_{M^*} \times \rho_M) \circ (\pi_{M^*} \times \pi_M)(x, x') = (x, x')$ *for* $(x, x') \in D$.
(2) $(\pi_{M^*} \times \pi_M) \circ (f_{M^*} \times f_M) = \sigma \circ (\pi_{M^*} \times \pi_M)$.

(3) $(f_{M^*}\times f_M)\circ(\rho_{M^*}\times\rho_M)=(\rho_{M^*}\times\rho_M)\circ\sigma$ on $Y^+_{M^*}\otimes Y^+_M$.

We note that in the construction of the Markov transition probabilities $p(i, j)$ (resp. $q(i, j)$) associated with the structure matrices M (resp. M^*), we adopted the normalization $\sum_{i=0}^{s-1} x_i=1$ and $\sum_{i=0}^{s-1} y_i=1$, instead of the normalization $\sum_{i=0}^{s-1} x_iy_i=1$ as was done in §1. Therefore, the stationary probability vector $(\pi_0, \pi_1, \cdots, \pi_{s-1})$ for the shift invariant maximal Markov measure $\mu_{M^*}\times\mu_M$ defined on $X^+_{M^*}\otimes X^+_M$ satisfies the equality

$$\pi_i=x_iy_i\Big/\sum_{i=0}^{s-1} x_iy_i \quad 0\leqq i\leqq s-1 .$$

Let $h(x, x')$ be the constant function on D with the value $(\sum_{i=1}^{s-1} x_iy_i)^{-1}\lambda^{-1}$ and define a measure $\nu_{M^*}\times\nu_M$ on D by putting $(\nu_{M^*}\times\nu_M)(A)=\int_A h(x, x')dxdx'$ for every Borel subset A of D. Then, for a cylinder set $[\omega(-1), \cdots, \omega(-j)]\times[\omega'(0), \cdots, \omega'(k)]$ of $X^+_{M^*}\otimes X^+_M$ we have

$$\begin{aligned}
&(\nu_{M^*}\times\nu_M)(\rho_{M^*}\times\rho_M)([\omega(-1), \cdots, \omega(-j)]\times[\omega'(0), \cdots, \omega'(k)])\\
&=(\nu_{M^*}\times\nu_M)\Bigg(\Bigg[\sum_{i=0}^{\omega(-1)-1} y_i+\frac{1}{\lambda}\sum_{i=0}^{\omega(-2)-1} m^*(\omega(-1), i)y_i+\\
&\qquad\cdots+\frac{1}{\lambda^{j-1}}\sum_{i=0}^{\omega(-j)-1} m^*(\omega(-j+1), i)y_i ,\\
&\qquad\sum_{i=0}^{\omega(-1)-1} y_i+\cdots+\frac{1}{\lambda^{j-1}}\sum_{i=0}^{\omega(-j)} m^*(\omega(-j+1), i)y_i\Bigg)\\
&\qquad\times\Bigg[\sum_{i=0}^{\omega'(0)-1} x_i+\frac{1}{\lambda}\sum_{i=0}^{\omega'(1)-1} m(\omega'(0), i)x_i+\cdots\\
&\qquad\cdots+\frac{1}{\lambda^k}\sum_{i=0}^{\omega'(k)-1} m(\omega'(k-1), i)x_i ,\\
&\qquad\sum_{i=0}^{\omega'(0)-1} x_i+\frac{1}{\lambda}\sum_{i=0}^{\omega'(1)-1} m(\omega'(0), i)x_i+\cdots\\
&\qquad\cdots+\frac{1}{\lambda^k}\sum_{i=0}^{\omega'(k)} m(\omega'(k-1), i)x_i\Bigg)\Bigg)\\
&=\frac{1}{\left(\sum_{i=0}^{s-1} x_iy_i\right)\lambda}\frac{y_{\omega(-j)}}{\lambda^{j-1}}\frac{x_{\omega'(k)}}{\lambda^k} ,
\end{aligned}$$

while

$$\begin{aligned}
&\mu_{M^*}\times\mu_M([\omega(-1), \cdots, \omega(-j)]\times[\omega'(0), \cdots, \omega'(k)]\\
&=\pi_{\omega(-j)}p(\omega(-j), \omega(-j+1))\cdots p(\omega'(k-1), \omega'(k)))\\
&=\frac{y_{\omega(-j)}x_{\omega'(k)}}{\left(\sum_{i=0}^{s-1} x_iy_i\right)\lambda^{j+k}} .
\end{aligned}$$

Thus, $\mu_{M^*}\times\mu_M$ and $(\nu_{M^*}\times\nu_M)\circ(\rho_{M^*}\times\rho_M)$ coincide on cylinder sets in $X_{M^*}^+\otimes X_M^+$, and therefore, we conclude that

$$\mu_{M^*}\times\mu_M=(\nu_{M^*}\times\nu_M)\circ(\rho_{M^*}\times\rho_M) .$$

Summarizing the discussion above, we obtain the following theorem corresponding to Theorem 2.1.

THEOREM 2.2. *(Representation Theorem) If the maximal Markov automorphism (X_M, σ, μ_M) (which is isomorphic to $(X_{M^*}^+\otimes X_M^+, \sigma, \mu_{M^*}\times\mu_M)$) is non-atomic and ergodic, then it is isomorphic to the system $(D, f_{M^*}\times f_M, \nu_{M^*}\times\nu_M)$ under the isomorphism $\rho_{M^*}\times\rho_M$.*

In the sequel we shall call the automorphism $(D, f_{M^*}\times f_M, \nu_{M^*}\times\nu_M)$ the Baker's transformation of Markov type."

§ 3. A construction of transversal flows.

In this section we shall construct a transversal flow for the maximal Markov automorphism in the form of a special flow (Ambrose-Kakutani flow) with the base transformation defined on the unit interval [0, 1).

Let $(X_{M^*}^+, \sigma)$ be the Markov subshift associated with a structure matrix M^* and denote by $W_{M^*}^{(n)}$ the set of all the words of length n admissible with respect to M^*. Namely,

$$W_{M^*}^{(n)}=\{(\omega(-1), \cdots, \omega(-n))\mid m^*(\omega(-i), \omega(-i-1))=1 \quad \text{for} \quad 1\leqq i\leqq n-1\} .$$

Next, we introduce another linear order $\succ$ in $W_{M^*}^{(n)}$ for each $n=1, 2\cdots$ as follows. For $(\omega(-1), \cdots, \omega(-n))$ and $(\omega'(-1), \cdots, \omega'(-n))$ in $W_{M^*}^{(n)}$, we define $\succ$ as

$$(\omega(-1), \cdots, \omega(-n))\succ(\omega'(-1), \cdots, \omega'(-n)) ,$$

if there exists $j(0\leqq j\leqq n-1)$ such that $\omega(-n+k)=\omega'(-n+k)$ for $0\leqq k<j-1$ and $\omega(-n+j)>\omega'(-n+j)$ hold. Using this linear order $\succ$ we introduce a stopping time $\tau(\omega)$ for ω in $Y_{M^*}^+$ in the following manner: We define $\tau(\omega)$ to be the smallest positive integer n for which there exists an element $(\omega^*(-1), \omega^*(-2), \cdots, \omega^*(-n))$ in $W_{M^*}^{(n)}$ which is minimal with respect to the order $\succ$ among the elements $(\omega'(-1), \cdots, \omega'(-n))$ in $W_{M^*}^{(n)}$ satisfying the properties $\omega'(-n)=\omega(-n)$ and $(\omega'(-1), \cdots, \omega'(-n))\succ(\omega(-1), \cdots, \omega(-n))$. We put $\tau(\omega)=\infty$, if such an n does not exist. We see that the set $\{\omega\in Y_{M^*}^+\mid \tau(\omega)=\infty\}$ is μ_{M^*}-null and its image under the map ρ_{M^*} is a set of Lebesgue measure zero in [0, 1). We define a mapping B_{M^*} on $Y_{M^*}^+\cap\{\omega\mid\tau(\omega)<\infty\}$ into $Y_{M^*}^+$ by

$$B_{M^*}(\omega)=(\omega^*(-1),\cdots,\omega^*(-\tau(\omega))\cdot\sigma^{\tau(\omega)}(\omega))$$
$$=(\omega^*(-1),\cdots,\omega^*(-\tau(\omega)),\ \omega(-\tau(\omega)-1),\ \omega(-\tau(\omega)-2),\cdots).$$

On the other hand the Lebesgue measure of the Borel set $\rho_{M^*}([\omega(-1),\cdots,\omega(-n)])$ is equal to $y_{\omega(-n)}/\lambda^{n-1}$. So it follows that the mapping

$$B=\rho_{M^*}\circ B_{M^*}\circ\pi_{M^*}\colon [0,1)\longrightarrow[0,1)\quad(\mathrm{mod}\ 0)$$

is Lebesgue measure-preserving.

This automorphism B (defined on $[0,1)$ except on some Lebesgue-null set) can be considered to be a generalization of the so-called "adding-machine" transformation.

REMARK 3. It can be seen that the mapping B_{M^*} satisfies the following commutation relation:

$$B_{M^*}\sigma_{M^*}(\omega)=\sigma_{M^*}B_{M^*}(\omega)\quad\text{if}\quad 3\leqq\tau(\omega)<\infty .$$

We now construct a transversal flow for the Baker's transformation of Markov type $(D, f_{M^*}\times f_M, \nu_{M^*}\times\nu_M)$. Let $\varepsilon=\min_{0\leqq i,j\leqq s-1}\{x_i, y_j\}/2$. For $t\in[0,\varepsilon]$ and $(x,x')\in Q_j\times P_k\subset D$, we define $Z_t(x,x')$ by putting

$$Z_t(x,x')=\begin{cases}(x,\ x+t) & \text{if } (x,x'+t)\in D.\\ (x,\ x'+t+\sum_{k'>j>k}x_j) & \text{if } (x,x'+t)\notin D \text{ and if there exists } i \text{ such that } m(j,i)=1,\ i>k. \text{ Here } k' \text{ is the smallest } i>k \text{ such that } m(j,i)=1.\\ \left(Bx,\ x'+t-\sum_{i=0}^{k}x_i+\sum_{i=0}^{j'-1}x_i\right) & \text{if } (x,x'+t)\notin D, \text{ and if } m(j,i)=0 \text{ for all } i(s-1\geqq i>k). \text{ Here } j' \text{ is defined to be } j'=\min\{i\colon m(\pi_{M^*}(Bx)(0),i)=1\}.\end{cases}$$

For $t\in[-\varepsilon,0]$ and $(x,x')\in Q_j\times P_k\subset D$, we define $Z_t(x,x')$ as

$$Z_t(x,x')=\begin{cases}(x,\ x'+t) & \text{if } (x,x'+t)\in D.\\ (x,\ x'+t-\sum_{k'<i<k}x_i) & \text{if } (x,x'+t)\notin D, \text{ and if there exists } i \text{ such that } m(j,i)=1,\ i<k. \text{ Here } k' \text{ is the largest } i<k \text{ such that } m(j,i)=1.\\ \left(B^{-1}x,\ x'+t-\sum_{i=0}^{k-1}x_i+\sum_{i=0}^{j'}x_i\right) & \text{if } (x,x'+t)\notin D \text{ and if } m(j,i)=0 \text{ for all } i\ (0\leqq i<k). \text{ Here } j' \text{ is defined by } j'=\max\{i\colon m\ (\pi_{M^*}(B^{-1}x)(0),i)=1\}.\end{cases}$$

Let us now define Z_t for other values of t in order to define a one-parameter group $\{Z_t: -\infty < t < \infty\}$ on D: For a positive number t, we write

$$t = n\varepsilon + \delta \quad \text{where} \quad n = \left[\frac{t}{\varepsilon}\right]$$

and let

$$Z_t(x, x') = Z_\delta(Z_\varepsilon^{(n)}(x, x')) ,$$

where $Z_\varepsilon^{(n)}$ denotes the n-th iterate of Z_ε, i.e., $Z_\varepsilon^{(1)} = Z_\varepsilon$ and $Z_\varepsilon^{(n)} = Z_\varepsilon(Z_\varepsilon^{(n-1)})$ $n = 1, 2, 3, \cdots$. We define Z_t for negative t in a similar manner.

The following three types may occur by considering the behavior of $Z_{-s}(x, x')$ as $s \downarrow 0$. If (x, x') is in the interior of D, then $Z_{-s}(x, x')$ is continuous at $s=0$. If (x, x') lies on the boundary of D, then $Z_{-s}(x, x')$ may be discontinuous at $s=0$. A point (x, x') is called a point of discontinuity of the first type if $(x, x') \in Q_j \times P_k$, $x' = \sum_{i=0}^{k-1} x_i$ and if there exists an $i < k$ such that $m(j, i) = 1$. In this case we put $k' = \max\{i: m(j, i) = 1, i < k\}$ and we will identify the point (x, x') with the limit

$$\lim_{s \downarrow 0} Z_{-s}(x, x') = \lim_{s \downarrow 0} (x, x' - \sum_{k' < i < k} x_i - s) ,$$

which is a point of $\bar{D}$. A point (x, x') is called a point of discontinuity of the second type if

$$(x, x') \in Q_j \times P_k , \quad x' = \sum_{i=0}^{k-1} x_i$$

and if there exists no $i < k$ such that $m(j, i) = 1$. In this case we put

$$k_0 = \max\{i: m(\pi_{M^*}(B^{-1}x)(-1), i) = 1\}$$

and we will identify the point (x, x') with the limit

$$\lim_{s \downarrow 0} Z_{-s}(x, x') = \lim_{s \downarrow 0} \left(B^{-1}x, \sum_{i=0}^{k_0} x_i - s\right) ,$$

which again lies in $\bar{D}$.

For any two-sided sequence $\omega \in Y_{M^*}^+ \otimes Y_M^+$, we will write $(\omega(n), n \geqq -k)$ for the one-sided sequence $(\omega(-k), \omega(-k+1), \cdots, \omega(0), \omega(1), \cdots)$ belonging to Y_M^+.

LEMMA 3.1. (1) *If (x, x') is a point of discontinuity of the first type, then*

$$\lim_{s\downarrow 0} \rho_M(\pi_{M^*}\times\pi_M(Z_{-s}(x, x'))(n)\ ,\ n\geqq -1)$$
$$=\rho_M(\pi_{M^*}\times\pi_M(x, x'))(n)\ ,\ n\geqq -1)\ .$$

(2) *If (x, x') is a point of discontinuity of the second type, then*

$$\lim_{s\downarrow 0} \rho_M(\pi_{M^*}\times\pi_M(Z_{-s}(x, x'))(m);\ m\geqq -n)$$
$$=\rho_M(\pi_{M^*}\times\pi_M(x, x'))(m);\ m\geqq -n)\ ,$$

where the integer n is given by $n=n(x)=\tau(\pi_{M^*}(B^{-1}x))$, not smaller than 2.

PROOF. Let (x, x') be a point of discontinuity of the first type and suppose $(x, x')\in Q_j\times P_k$, $x'=\sum_{j=0}^{k-1} x_j$ and $k'=\max\{i: m(j, i)=1, i<k\}$. Then, for any $s>0$, the x-coodinate of $Z_{-s}(x, x')$ equals x, and therefore,

$$\pi_{M^*}\times\pi_M(Z_{-s}(x, x'))(-1)=\pi_{M^*}\times\pi_M(x, x')(-1)=j\ .$$

Since $x'=\sum_{j=0}^{k-1} x_j$, the one-sided sequence $(\pi_{M^*}\times\pi_M(x, x')(n);\ n\geqq -1)$ is minimal with respect to the order in Y_M^+ among the elements satisfying $\omega(-1)=j$, $\omega(0)=k$. On the other hand, for any $\omega'\in Y_M^+$ with $\omega'(-1)=j$ and $\omega'(0)=k'$, there exists an $s_0>0$ such that

$$\omega'<(\pi_{M^*}\times\pi_M(Z_{-s}(x, x'))(n);\ n\geqq -1)$$

for every s, $0<s<s_0$. If we set

$$\bar{\omega}=\sup\{(\pi_{M^*}\times\pi_M(Z_{-s}(x, x'))(n);\ n\geqq -1\,|\,0<s<s_0\}\ ,$$

where the supremum is taken with respect to the order in Y_M^+, then it follows from Proposition 2.2 that

$$\bar{\omega}\in X_M^+ - Y_M$$

and that $\rho_M(\bar{\omega})=\rho_M(\pi_{M^*}\times\pi_M(x, x')(n); n\geqq -1)$, which proves the assertion(1).

Next, let (x, x') be a point of discontinuity of the second type. From the definition of the mapping B, it follows that there exists the smallest integer $n=n(x)=\tau(\pi_{M^*}(B^{-1}x))(\geqq 2)$ such that $(\pi_{M^*}\times\pi_M(x, x')(-n+k);\ 0\leqq k\leqq n)$ is minimal with respect to the order in $W_M^{(n)}$ among such elements $(\omega(-n), \cdots, \omega(-1))\in W_M^{(n)}$ that $(\omega(-n), \cdots, \omega(-1))>(\pi_{M^*}\times\pi_M(Z_{-s}(x, x'))(-n+k);\ 0\leqq k\leqq n)$ and that $\omega(-n)=\pi_{M^*}\times\pi_M(Z_{-s}(x, x'))(-n)$ for each s with $\varepsilon>s>0$. Since (x, x') is a point of discontinuity of the second type, the one-sided sequence $(\pi_{M^*}\times\pi_M(x, x')(m);\ m\geqq -n)$ is minimal with respect to the order of Y_M^+ among the elements of Y_M^+ whose first n entries coincide with $(\pi_{M^*}\times\pi_M(x, x')(-n+k);\ 0\leqq k\leqq n-1)$. Therefore, for any $\omega'\in Y_M^+$ with $\omega'(0)=\pi_{M^*}\times\pi_M(x, x')(-n)$ and

$$\omega' < (\pi_{M^*} \times \pi_M(x, x')(m);\ m \geqq -n)\ ,$$

there exists an $s_0 > 0$ such that $\omega' < (\pi_{M^*} \times \pi_M(Z_{-s}(x, x'))(m);\ m \geqq -n)$ for every s with $0 < s < s_0$. So, we obtain

$$\begin{aligned}\lim_{s \downarrow 0} \rho_M(\pi_{M^*} \times \pi_M(Z_{-s}(x, x'))(m)\ ;\ m \geqq -n) \\ = \rho_M(\pi_{M^*} \times \pi_M(x, x')(m)\ ;\ m \geqq -n)\ ,\end{aligned}$$

proving the assertion (2).

LEMMA 3.2. (1) *If* (x, x') *is a continuity point,* i.e., *if* $\lim_{s \downarrow 0} Z_{-s}(x, x') = (x, x')$, *or if it is a point of discontinuity of the first type, then*

$$\lim_{s \downarrow 0} Z_{-s}(f_{M^*} \times f_M)^{-1}(x, x') = \lim_{s \downarrow 0} (f_{M^*} \times f_M)^{-1} Z_{-s}(x, x') = (f_{M^*} \times f_M)^{-1}(x, x')\ .$$

(2) *If* (x, x') *is a point of discontinuity of the second type and if* $n = n(x) = \tau(\pi_{M^*}(B^{-1}x)) = 2$, *then the point* $(f_{M^*} \times f_M)^{-1}(x, x')$ *is a point of discontinuity of the first type, and we have*

$$\begin{aligned}&\lim_{s \downarrow 0} \rho_M(\pi_{M^*} \times \pi_M(Z_{-s} \circ (f_{M^*} \times f_M)^{-1}(x, x'))(m);\ m \geqq -1) \\ &\quad = \lim_{s \downarrow 0} \rho_M(\pi_{M^*} \times \pi_M(f_{M^*} \times f_M)^{-1} Z_{-s}(x, x')(m);\ m \geqq -1) \\ &\quad = \rho_M(\pi_{M^*} \times \pi_M(f_{M^*} \times f_M)^{-1}(x, x')(m);\ m \geqq -1)\ .\end{aligned}$$

(3) *If* (x, x') *is a point of discontinuity of the second type and if* $n = n(x) = \tau(\pi_{M^*}(B^{-1}x)) > 2$, *then the point* $(f_{M_*} \times f_M)^{-1}(x, x')$ *is a point of discontinuity of the second type and we get*

$$\begin{aligned}&\lim_{s \downarrow 0} \rho_M(\pi_{M^*} \times \pi_M(Z_{-s}(f_{M^*} \times f_M)^{-1}(x, x'))(m);\ m \geqq -n+1) \\ &\quad = \lim_{s \downarrow 0} \rho_M(\pi_{M^*} \times \pi_M(f_{M^*} \times f_M)^{-1} Z_{-s}(x, x')(m);\ m \geqq -n+1) \\ &\quad = \rho_M(\pi_{M^*} \times \pi_M(f_{M^*} \times f_M)^{-1}(x, x')(m);\ m \geqq -n+1)\ .\end{aligned}$$

REMARK 3.2. The above lemma implies that almost every orbit of the flow $\{Z_t;\ -\infty < t < \infty\}$ is mapped onto another orbit; namely, for almost all $(x, x') \in D$

$$\begin{aligned}(f_{M^*} \times f_M)^{-1}\{Z_t(x, x');\ -\infty < t < \infty\} \\ = \{Z_t(f_{M^*} \times f_M)^{-1}(x, x')\ ;\ -\infty < t < \infty\}\ .\end{aligned}$$

PROOF. From Proposition 2.3 (2), it follows that if (x, x') is either a continuity point or a point of discontinuity of the first type, then $(f_{M^*} \times f_M)^{-1}(x, x')$ is a continuity point. If (x, x') is a point of discontinuity of the second type and if $\tau(\pi_{M^*}(B^{-1}x)) = 2$, then $(f_{M^*} \times f_M)^{-1}(x, x')$ is a point of discontinuity of the first type. It follows also that if (x, x') is a point of discontinuity of the second type and if $\tau(\pi_{M^*}(B^{-1}x))>2$, then $(f_{M^*}\times f_M)^{-1}(x, x')$ is a point of discontinuity of the second type. In this case we have $\tau(\pi_{M^*}(B^{-1}y))=\tau(\pi_{M^*}(B^{-1}x))-1$, where y denotes the x-coordinate of the point $(f_{M^*}\times f_M)^{-1}(x, x')$. Therefore, from Lemma 3.1 we have the following three identities

$$\lim_{s\downarrow 0} Z_s(f_{M^*}\times f_M)^{-1}(x, x')=(f_{M^*}\times f_M)^{-1}(x, x')\ ,$$
$$\lim_{s\downarrow 0} \rho_M(\pi_{M^*}\times\pi_M(Z_{-s}\circ(f_{M^*}\times f_M)^{-1}(x, x'))(m);\ m\geqq -1)$$
$$=\rho_M(\pi_{M^*}\times\pi_M(f_{M^*}\times f_M)^{-1}(x, x')(m);\ m\geqq -1)$$

and

$$\lim_{s\downarrow 0} \rho_M(\pi_{M^*}\times\pi_M(Z_{-s}\circ(f_{M^*}\times f_M)^{-1}(x, x'))(m);\ m\geqq -n+1)$$
$$=\rho_M(\pi_{M^*}\times\pi_M(f_{M^*}\times f_M)^{-1}(x, x')(m);\ m\geqq -n+1)\ .$$

The remaining identities in the assertion (1) and (2) are easy to verify directly, while the remaining identity in the assertion (3) follows from Remark 3.1.

The following lemma can be easily obtained:

LEMMA 3.3. *For almost all* $(x, x')\in D$,

$$(f_{M^*}\times f_M)^{-1}\{Z_s(x, x')\quad 0\leqq s\leqq t\}=\{Z_s(f_{M^*}\times f_M)^{-1}(x, x')\quad 0\leqq s\leqq t/\lambda\}\ .$$

PROPOSITION 3.3. *The one-parameter group* $\{Z_t;\ -\infty<t<\infty\}$ *constructed above satisfies the following properties*:

(1) *Each* Z_t *is a measure-preserving automorphism of* $(D, \nu_{M^*}\times\nu_M)$.

(2) $(f_{M^*}\times f_M)^{-1}Z_t=Z_{t/\lambda}(f_{M^*}\times f_M)^{-1}$ *holds* $\nu_{M^*}\times\nu_M$*-almost everywhere.*

PROOF. (1) follows easily from the fact that the measure $\nu_{M^*}\times\nu_M$ is a constant multiple of the restriction to D of the Lebesgue measure on $[0, 1)\times[0, 1)$. The assertion (2) follows from Lemmas 3.2 and 3.3.

By starting with Y_M^+ in place of $Y_{M^*}^+$ and going through the same procedure as above, we can construct another base transformation on $[0, 1)$ and a corresponding one-parameter flow $\{Z_{t'};\ -\infty<t<\infty\}$ on D. The orbits of $\{Z_t';\ -\infty<t<\infty\}$ are transversal to the orbits of $\{Z_t;\ -\infty<t<\infty\}$. Therefore, we obtain the following main theorem of this paper, using the fact that the mapping $\pi_{M^*}\times\pi_M$: $D\to X_{M^*}^+\otimes X_M^+(=X_M)$ is injective.

THEOREM 3.4. *Suppose that the maximal Markov automorphism* (X_M, σ, μ_M) *associated with a structure matrix* M *is non-atomic and ergodic. Then there exist two measure-preserving flows* $\{Z_t^{(1)};\ -\infty<t<\infty\}$ *and* $\{Z_t^{(2)};\ -\infty<t<\infty\}$ *satisfying the following properties*:

$$\sigma Z_t^{(1)} = Z_{\lambda t}^{(1)} \sigma \quad \text{for every } t \in R \text{ for almost all } \omega \in X_M$$

and

$$\sigma Z_t^{(2)} = Z_{t/\lambda}^{(2)} \sigma \quad \text{for every } t \in R \text{ for almost all } \omega \in X_M .$$

§ 4. Ergodic properties of a transversal flow for the maximal Markov automorphism.

In this section, we study some ergodic properties of the transversal flow for the maximal Markov automorphism constructed in § 3. For this purpose, we first investigate the properties of the base transformation B.

Let $(\varepsilon(-1), \varepsilon(-2), \cdots, \varepsilon(-n))$ be an element of $W_{M^*}^{(n)}$, and let us denote by $\Delta(\varepsilon(-1), \varepsilon(-2), \cdots, \varepsilon(-n))$ the image of the cylinder set $[\varepsilon(-1), \varepsilon(-2), \cdots, \varepsilon(-n)] \subset Y_{M^*}^+$ by the mapping ρ_{M^*}. Then, the subset $\Delta(\varepsilon(-1), \varepsilon(-2), \cdots, \varepsilon(-n))$ of the unit interval is also an interval and its length equals $y_{\varepsilon(-n)}/\lambda^{n-1}$. We will denote by $\{\xi_n: n \geqq 2\}$ the sequence of partitions of the unit interval defined by

$$\xi_n = \{\Delta(\varepsilon(-1), \varepsilon(-2), \cdots, \varepsilon(-n)) | (\varepsilon(-1), \cdots, \varepsilon(-n)) \in W_{M^*}^{(n)}\} .$$

For each k with $0 \leqq k \leqq s-1$, and $n \geqq 2$, let us define ε^* by

$$\varepsilon^*(-n+1) = \min\{i: m^*(i, k) = 1\} ,$$

and define inductively for $-1 \leqq j \leqq -n+2$ by

$$\varepsilon^*(j) = \min\{i: m^*(i, \varepsilon^*(j-1)) = 1\} .$$

Denoting by $C_{k,0}^{(n)}$ the sub-interval $\Delta(\varepsilon^*(-1), \varepsilon^*(-2), \cdots, \varepsilon^*(-n+1), k)$ of the unit interval ($0 \leqq k \leqq s-1$ and $n \geqq 2$), we define $C_{k,j}^{(n)}$ ($0 \leqq j \leqq r_k - 1$, $r_k = \sum_{j=0}^{s-1} m^{*(n-1)}(j, k)$) by

$$C_{k,j}^{(n)} = B^j C_{k,0}^{(n)} .$$

Note that the number r_k represents the number of M^*-admissible words of length n having the last entry k. It is easy to verify that for each $n \geqq 2$

$$\xi_n = \{C_{k,j}^{(n)} \mid 0 \leqq k \leqq s-1 , \quad 0 \leqq j \leqq r_k - 1\} .$$

LEMMA 4.1. *The number of ergodic components for the mapping B is less than the number* $1/\min_{0 \leqq i \leqq s-1}\{x_i y_i\}$.

PROOF. For Borel measurable subsets A and B of $[0, 1)$, we will denote by $A \triangle B$ the symmetric difference $A \cup B - A \cap B$ and by $|A|$ the

Lebesgue measure of A. Now suppose that F is a set of positive measure in $[0, 1)$ invariant under B. The sequence of partitions $\{\xi_n; n\geqq 2\}$ increases to the partition ε of $[0, 1)$ into individual points. Hence, for any $\delta>0$, we can find an integer n_0 such that for $n\geqq n_0$ there exists a ξ_n-measurable subset $F^{(n)}$ of $[0, 1)$ satisfyng $|F\triangle F^{(n)}|/|F^{(n)}|<\delta$. Thus, there exists a set $C^{(n)}_{k,j_0}\in\xi_n$ such that $|C^{(n)}_{k,j_0}\cap F|>(1-\delta)y_k/\lambda^{n-1}$. Since the set F is B-invariant, it follows that

$$|C^{(n)}_{k,j}\cap F|>(1-\delta)\frac{y_k}{\lambda^{n-1}} \text{ holds for every } j(0\leqq j\leqq r_k-1)\ ,$$

from which we have

$$|F|\geqq\sum_{j=0}^{r_k-1}|C^{(n)}_{k,j}\cap F|\geqq(1-\delta)\frac{y_k r_k}{\lambda^{n-1}}\ ;$$

δ being arbitrary, one has

$$|F|\geqq\frac{y_k r_k}{\lambda^{n-1}} \text{ for some } k(0\leqq k\leqq s-1)\ .$$

From the fact that $\sum_{j=0}^{s-1}x_j m^{*(n-1)}(j, k)=\lambda^{n-1}x_k$ and that $\sum_{j=0}^{s-1}x_j=1$, it is seen that $|F|\geqq x_k y_k$. This implies the assertion of the lemma.

PROPOSITION 4.1. *If a structure matrix M is aperiodic, then the base transformation B associated with M is ergodic.*

PROOF. Suppose that the number of ergodic components for the transformation B is $k>1$. Then, the transversal flow $\{Z^{(1)}_t: -\infty<t<\infty\}$ for (X_M, σ, μ_M) constructed by the transformation B has k ergodic components $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_k$, where we assume that $\mu_M(\bar{F}_i)\leqq\mu_M(\bar{F}_{i+1})$, $1\leqq i\leqq k-1$. From the relation $Z^{(1)}_{\lambda t}\sigma=\sigma Z^{(1)}_t$ it follows that the equalities

$$\sigma^p\bar{F}_i=\sigma^p Z^{(1)}_t\bar{F}_i=Z^{(1)}_{\lambda^p t}\sigma^p\bar{F}_i$$

holds for every $p>0$ and t. This implies that the set $\sigma^p\bar{F}_i$ is $Z^{(1)}_t$-invariant for each i $(1\leqq i\leqq k)$ and thus $\sigma^p\bar{F}_i\in\{\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_k\}$. This means that we can find an integer p_0 and a suffix i_0 such that

$$\sigma^{p_0}\bar{F}_{i_0}=\bar{F}_{i_0}\quad\text{(a.e.)}\ .$$

This contradicts to the fact that (X_M, σ, μ_M) is mixing.

PROPOSITION 4.2. *If a structure matrix M is irreducible and periodic with period $d>1$, then the transformation B associated with*

M is not ergodic and the number of the ergodic components of B is at least d.

PROOF. Let $\{D_\alpha: \alpha=1, 2, \cdots, d\}$ be the periodic decomposition of the state space $S=\{0, 1, \cdots, s-1\}$ with respect to M and let us denote by $\bar{D}_\alpha$ the subset of $[0, 1)$ defined by

$$\bar{D}_\alpha=\{\Delta(\omega(-1)) \mid \omega(-1) \in D_\alpha\} \quad (1 \leqq \alpha \leqq d) .$$

From the definition of the transformation B it follows that, for each

$$\omega \in Y_{M^*}^+ , \quad (B_{M^*}\omega)(-1) \in D_\alpha \quad \text{if} \quad \omega(-1) \in D_\alpha \quad \text{and if} \quad \tau(\omega)<\infty .$$

Therefore, we have $B\bar{D}_\alpha \subset \bar{D}_\alpha$ for each α $(1 \leqq \alpha \leqq d)$.

Note that the transformation B associated with M gives a concrete example of transformations admitting simple approximation multiplicity s in the sense of R. V. Chacon [8], where s is the number of states in S. The spectral multiplicity for the unitary operator induced by the transformation B is at most s; so the metric entropy of B must be 0. We summarize these results in the following:

THEOREM 4.1. *Let (X_M, σ, μ_M) be the maximal Markov automorphism which is non-atomic and ergodic, and let $\{Z_t^{(i)}: -\infty<t<\infty\}(i=1, 2)$ be the transversal flows for (X_M, σ, μ_M) constructed in § 3. Then,*

(1) *the flows $\{Z_t^{(i)}: -\infty<t<\infty\}$ are ergodic if the structure matrix M is aperiodic. They are non-ergodic if M is irreducible but not aperiodic and*

(2) *the entropy of the transversal flow $\{Z_t^{(i)}: -\infty<t<\infty\}$ is equal to 0 for $i=1, 2$.*

It is an interesting question to determine the spectral types of the transversal flows for a maximal Markov automorphism. The maximal Markov automorphism associated with the structure matrix $M=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ is nothing but the simple β-automorphism discussed in [6]. In this case it is known that the transversal flows have the discrete spectrum, (See [9]). However, for more general structure matrices, the question seems to be quite difficult to settle.

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