# ON SOME RANDOM RIEMANN-SUMS

SHIGERU TAKAHASHI

(Received March 30, 1956)

**1.** In the present note $\{t_i(\omega)\}$, $i = 1, 2, \ldots,$ will denote a sequence of independent random variables defined on a probability space $(\Omega, \mathbf{B}, P)$ and each $t_i(\omega)$ have the uniform distribution on the interval $[0, 1]$, that is, for $0 \leqq x \leqq 1$

$$P[t_i(\omega) < x] = x.$$ [1)]

For each $\omega$ let $t_{i,n}(\omega)$ be the $i$-th value of $\{t_j(\omega)\}$ $(1 \leqq j \leqq n)$ arranged in the increasing order and let, for all $n$,

$$t_{0,n}(\omega) \equiv 0 \quad \text{and} \quad t_{n+1,n}(\omega) \equiv 1.$$

Further let $f(t)$, $0 \leqq t \leqq 1$, denote a Borel-measurable and integrable function.

It is an interesting problem, proposed by K. Ito, whether the following Riemann-sums

$$S_n(\omega) = \sum_{i=1}^{n} f(t_{i,n}(\omega))(t_{i+1,n}(\omega) - t_{i,n}(\omega)) \tag{1.1}$$

converge to $\int_0^1 f(t)\,dt$ or not, in any sense. In [2] we proved that under certain local conditions, we have

$$P\left[\lim_{n\to\infty} S_n(\omega) = \int_0^1 f(t)\,dt\right] = 1. \tag{1.2}$$

In this note we prove the following

THEOREM 1. *If* $f(t) \in L_p(0, 1)$ $p > 1$, *then* (1. 2) *holds.*

For $f(t) \in L(0, 1)$ we can not prove whether (1. 2) holds or not.

**2.** Let us put, for $1 \leqq i \leqq n$ and $n = 1, 2, \ldots,$

$$d_{i,n}(\omega) = t_{j+1,n}(\omega) - t_i(\omega), \text{ if } t_i(\omega) = t_{j,n}(\omega) \qquad (j = 1, 2, \ldots, n) \tag{2.1}$$

and

$$d'_{i,n}(\omega) = t_i(\omega) - t_{j-1,n}(\omega), \text{ if } t_i(\omega) = t_{j,n}(\omega) \qquad (j = 1, 2, \ldots, n). \tag{2.1'}$$

Then we can write

$$S_n(\omega) = \sum_{i=1}^{n} d_{i,n}(\omega) f(t_i(\omega)) \tag{2.2}$$

and

1) For the notations and definitions in the theory of probability see [1].

$$\int_0^1 f(t)\,dt = \sum_{i=1}^{n} \int_0^{d_{i,n}(\omega)} f(t_i(\omega)+u)\,du + \int_0^{t_{1,n}(\omega)} f(t)\,dt. \tag{2.2'}$$

LEMMA 1. *We have, for* $0 \leqq h \leqq 1$,

$$P[d_{i,n} < h] = 1-(1-h)^n \text{ [2)]}.$$

PROOF. By (2.1), we have

$$P[d_{i,n} < h] = P[(d_{i,n} < h) \cap (t_i \leqq 1-h)] + P[t_i > 1-h]$$
$$= \int_0^{1-h} P[d_{i,n} < h \,|\, t_i = x]\,dx + h.$$

where $P(E|F)$ denotes the conditional probability of $E$ under the hypothesis $F$.

From the independency of $\{t_i\}$, it is seen that

$$P[d_{i,n} < h \,|\, t_i = x] = P\Big[\bigcup_{\substack{j=1\\ j\neq i}}^{n} (t_j \in [x, x+h]) \,|\, t_i = x\Big]$$
$$= P\Big[\bigcup_{\substack{j=1\\ j\neq i}}^{n} (t_j \in [x, x+h])\Big]$$
$$= 1 - P\Big[\bigcap_{\substack{j=1\\ j\neq i}}^{n} (t_j \bar{\in} [x, x+h])\Big]$$
$$= 1-(1-h)^{n-1}.$$

Thus we obtain

$$P[d_{i,n} < h] = \int_0^{1-h} \{1-(1-h)^{n-1}\}dx + h$$
$$= 1-(1-h)^n.$$

From this lemma, it follows that

$$P\Big[\operatorname*{Max}_{1\leqq i\leqq n} d_{i,n} \geqq \frac{3\log n}{n}\Big] \leqq \sum_{i=1}^{n} P\Big[d_{i,n} \geqq \frac{3\log n}{n}\Big] \tag{2.3}$$
$$= n\Big(1-\frac{3\log n}{n}\Big)^n = O(1/n^2) \qquad (n \to +\infty).$$

On the other hand, for any $\varepsilon\,(0 < \varepsilon < 1)$, we have

$$P\Big[\sum_{i=1}^{n} d_{i,n} \leqq 1-\varepsilon\Big] = P[t_{1,n} \geqq \varepsilon] \tag{2.4}$$
$$= P\Big[\bigcap_{i=1}^{n} (t_i \geqq \varepsilon)\Big]$$

2) We write simply $d_{i,n}$ for $d_{i,n}(\omega)$ and so on.

$$= (1-\varepsilon)^n.$$

Hence we have

$$\sum_{n=1}^{\infty} P\Big[\sum_{i=1}^{n} d_{i,n} \leqq 1-\varepsilon\Big] < +\infty,$$

and this implies

$$P\Big[\lim_{n\to\infty}\sum_{i=1}^{n} d_{i,n} = 1\Big] = 1. \tag{2.4'}$$

LEMMA 2. *Let $x$ and $y$ be any two numbers.*

1° *If $0 \leqq x < 1-y \leqq 1$, then we have*

$$P[(d_{i,n} < y) \cap (t_i < x)] = x\{1-(1-y)^{n-1}\}.$$

2° *If $0 \leqq y < x \leqq 1$, then we have*

$$P[(d'_{i,n} < y) \cap (t_i \geqq x)] = (1-x)\{1-(1-y)^{n-1}\}.$$

PROOF. By the independency of $\{t_i\}$, we have

$$\begin{aligned} P[(d_{i,n} < y) \cap (t_i < x)] &= \int_0^x P[d_{i,n} < y \mid t_i = z]\, dz \\ &= \int_0^x P\Big[\bigcup_{\substack{j=1\\ j\neq i}}^{n} (t_j \in [z, z+y) \mid t_i = z)\Big] dz \\ &= \int_0^x P\Big[\bigcup_{\substack{j=1\\ j\neq i}}^{n} (t_j \in [z, z+y))\Big] dz \\ &= \int_0^x \{1-(1-y)^{n-1}\}\, dz \\ &= x\{1-(1-y)^{n-1}\}. \end{aligned}$$

In the same way, we can prove 2°.

LEMMA 3. *Let $x_1$, $y_1$, $x_2$ and $y_2$ be non-negative numbers and satisfy either the condition*

$$x_2 > y_1 + y_2 \quad and \quad x_2 + x_1 < 1, \tag{2.5}$$

*or the condition*

$$y_2 > x_1 + x_2 \quad and \quad y_2 + y_1 < 1. \tag{2.5'}$$

*Then we have, for $i \neq j$,*

$$\begin{aligned} &P[(d_{i,n} < x_1) \cap (d_{j,n} < y_1) \mid t_i = x_2,\ t_j = y_2] \\ &= 1-(1-x_1)^{n-2}-(1-y_1)^{n-2}+(1-x_1-y_1)^{n-2}. \end{aligned}$$

PROOF. We prove the lemma under (2.5), for under (2.5′) the proof is analogous.

It is seen that

$$(2.6)\qquad P[(d_{i,n} < x_1)\cap(d_{j,n} < y_1)\,|\,t_i = x_2,\ t_j = y_2]$$
$$= P[d_{i,n} < x_1\,|\,t_i = x_2,\ t_j = y_2] + P[d_{j,n} < y_1\,|\,t_i = x_2, t_j = y_2]$$
$$- P[(d_{i,n} < x_1)\cup(d_{j,n} < y_1)\,|\,t_i = x_2,\ t_j = y_2].$$

By (2.5), we have

$$(2.7)\qquad P[d_{i,n} < x_1\,|\,t_i = x_2,\ t_j = y_2]$$
$$= P\Big[\bigcup_{\substack{k=1\\k\neq i,j}}^{n}(t_k \in [x_2, x_2 + x_1))\,|\,t_i = x_2,\ t_j = y_2\Big]$$
$$= P\Big[\bigcup_{\substack{k=1\\k\neq i,j}}^{n}(t_k \in [x_2, x_2 + x_1))\Big] = 1-(1-x_1)^{n-2}$$

and

$$(2.7')\qquad P[d_{j,n} < y_1\,|\,t_i = x_2,\ t_j = y_2] = 1-(1-y_1)^{n-2}.$$

And in the same way, it follows that

$$(2.8)\qquad P[(d_{i,n} < x_1)\cup(d_{j,n} < y_1)\,|\,t_i = x_2,\ t_i = y_2]$$
$$= P\Big[\bigcup_{\substack{k=1\\k\neq i,j}}^{n}(t_k \in [x_2, x_2 + x_1)) \cup \bigcup_{\substack{m=1\\m\neq i,j}}^{n}(t_m\in[y_2, y_2 + y_1))\,|\,t_i = x_2,\ t_j = y_2\Big]$$
$$= P\Big[\bigcup_{\substack{k=1\\k\neq i,j}}^{n}(t_k \in [x_2, x_2 + x_1)) \cup \bigcup_{\substack{m=1\\m\neq i,j}}^{n}(t_m \in [y_2, y_2 + y_1))\Big]$$
$$= P\Big[\bigcup_{\substack{k=1\\k\neq i,j}}^{n}(t_k \in [x_2, x_2 + x_1) \text{ or } (t_k \in [y_2,\ y_2 + y_1))\Big]$$
$$= 1 - P\Big[\bigcap_{\substack{k=1\\k\neq i,j}}^{n}(t_k \,\overline{\in}\, [x_2,\ x_2 + x_1) \text{ and } (t_k \,\overline{\in}\,[y_2, y_2 + y_1))\Big]$$
$$= 1-(1-x_1-y_1)^{n-2}.$$

By (2.6)(2.7)(2.7′) and (2.8), we can prove the lemma.

**3.** Let us put

$$(3.1)\qquad \Omega_n \equiv \Big[\omega\,;\ \operatorname*{Max}_{1\leq i\leq n} d_{i,n} < \frac{3\log n}{n}\Big].$$

LEMMA 4. *If $f(t)$ satisfy the following conditions*

$$(3.2)\qquad \int_0^1 f(t)\,dt = 0$$

*and*

$$(3.2')\qquad \int_0^1 f^2(t)\,dt < +\infty,$$

***then we have, for $n \geqq 6$,***

$$\int_{\Omega_n} S_n^2 dP \leqq A \frac{(\log n)^{5/2}}{n^{1/2}} \int_0^1 f^2(t)\, dt,$$

*where A is a constant independent of f(t) and n.*

PROOF. We divide the proof in several steps.

1°. We have, by (3. 1),

$$\int_{\Omega_n} d_{i,n} f^2(t_i) dP \leqq \frac{(3\log n)^2}{n^2} \int_{\Omega} f^2(t_i)\, dP = \frac{(3\log n)^2}{n^2} \int_0^1 f^2(t)\, dt. \tag{3.3}$$

We have, for $i \neq j$,

$$\int_{\Omega_n} d_{i,n} d_{j,n} f(t_i) f(t_j) dP$$

$$= \Big( \int_{\substack{\Omega_n \\ t_j \leqq t_i \leqq t_j + d_{j,n}}} + \int_{\substack{\Omega_n \\ t_j > t_i + d_{i,n}}} + \int_{\substack{\Omega_n \\ t_i > t_j + d_{j,n}}} + \int_{\substack{\Omega_n \\ t_i < t_j \leqq t_i + d_{i,n}}} \Big) d_{i,n} d_{j,n} f(t_i) f(t_j)\, dP.$$

From the definitions of $d_{i,n}$ and $d_{j,n}$, it is seen that

$$[\omega\,;\, t_j \leqq t_i \leqq t_j + d_{j,n}] = [\omega\,;\, t_i = t_j + d_{j,n}]$$

and

$$[\omega\,;\, t_i < t_j \leqq t_i + d_{i,n}] \simeq [\omega\,;\, t_j = t_i + d_{i,n}]^{3)}.$$

Thus we have

$$\int_{\Omega_n} d_{i,n} d_{j,n} f(t_i) f(t_j)\, dP$$

$$= \Big( \int_{\substack{\Omega_n \\ t_i = t_j + d_{i,n}}} + \int_{\substack{\Omega_n \\ t_j > t_i + d_{i,n}}} + \int_{\substack{\Omega_n \\ t_i > t_j + d_{j,n}}} + \int_{\substack{\Omega_n \\ t_j = t_i + d_{i,n}}} \Big) d_{i,n} d_{j,n} f(t_i) f(t_j)\, dP.$$

By (3. 1) and the independency of $\{t_i\}$, we have

$$\Big( \int_{\substack{\Omega_n \\ t_i = t_j + d_{j,n}}} + \int_{\substack{\Omega_n \\ t_j = t_i + d_{i,n}}} \Big) |d_{i,n} d_{j,n} f(t_i) f(t_j)|\, dP$$

$$\leqq \Big( \frac{3\log n}{n} \Big)^2 \int_{|t_i - t_j| < \frac{3\log n}{n}} |f(t_i) f(t_j)|\, dP$$

3) If $P[(E - E\cap F) \cup (F - F\cap E)] = 0$, then we write $E \simeq F$.

$$= \left(\frac{3\log n}{n}\right)^2 \int_0^1 |f(x)|\, dx \int_{x-\frac{3\log n}{n}}^{x+\frac{3\log n}{n}} |f(y)|\,dy$$

$$\leqq \sqrt{2}\left(\frac{3\log n}{n}\right)^{5/2} \left(\int_0^1 f^2(t)\,dt\right).$$

Thus we obtain

$$(3.4) \qquad \left|\int_{\Omega_n} d_{i,n}d_{j,n}f(t_i)f(t_j)\,dP\right|$$

$$\leqq \sqrt{2}\left(\frac{3\log n}{n}\right)^{5/2}\int_0^1 f^2(t)\,dt + \left|\left(\int_{\substack{\Omega_n \\ t_i > t_j + d_{j,n}}} + \int_{\substack{\Omega_n \\ t_j > t_i + d_{i,n}}}\right) d_{i,n}d_{j,n}f(t_i)f(t_j)\,dP\right|.$$

2°. *We have, by* (3.1) *and the independency of* $\{t_i\}$,

$$\left(\int_{\substack{\Omega_n \\ t_i > t_j + d_{j,n} \\ t_i + d_{i,n} = 1}} + \int_{\substack{\Omega_n \\ t_j > t_i + d_{i,n} \\ t_j + d_{j,n} = 1}}\right) |d_{i,n}d_{j,n}f(t_i)f(t_j)|\,dP$$

$$\leqq \left(\frac{3\log n}{n}\right)^2 \left(\int_{1 \geqq t_i > 1 - \frac{3\log n}{n}} + \int_{1 \geqq t_j \geqq 1 - \frac{3\log n}{n}}\right) |f(t_i)f(t_j)|\,dP$$

$$\leqq \left(\frac{3\log n}{n}\right)^2 \left(\int_{\Omega} |f(t_j)|\,dP \int_{1 \geqq t_i > 1 - \frac{3\log n}{n}} |f(t_i)|\,dP + \int_{\Omega} |f(t_i)|\,dP \int_{1 \geqq t_j > 1 - \frac{3\log n}{n}} |f(t_j)|\,dP\right)$$

$$\leqq 2\left(\frac{3\log n}{n}\right)^{5/2} \int_0^1 f^2(t)\,dt.$$

Let us put

$$E_i \equiv \left[\omega;\ (t_i > d_{j,n} + t_j) \cap (t_i + d_{i,n} < 1) \cap \left(d_{i,n} < \frac{3\log n}{n}\right) \cap \left(d_{j,n} < \frac{3\log n}{n}\right)\right]$$

and

$$E_j \equiv \left[\omega;\ (t_j > d_{i,n} + t_i) \cap (t_j + d_{j,n} < 1) \cap \left(d_{i,n} < \frac{3\log n}{n}\right) \cap \left(d_{j,n} < \frac{3\log n}{n}\right)\right],$$

then we have, by (3.1),

$$\left|\left(\int_{E_i} - \int_{E_i \cap \Omega_n}\right) d_{i,n}d_{j,n}f(t_i)f(t_j)\,dP\right|$$

$$\leqq \int_{(\Omega - \Omega_n) \cap E_i} |d_{i,n}d_{j,n}f(t_i)f(t_j)|\,dP$$

$$\leqq \left(\frac{3\log n}{n}\right)^2 \sum_{\substack{k=1\\k\neq i,j}}^{n} \int_{d_{k,n}\geqq\frac{3\log n}{n}} |f(t_i)f(t_j)|\,dP.$$

From the definition of $d_{k,n}$, we have for $k \neq i$ and $k \neq j$,

$$\int_{d_{k,n}\geqq\frac{3\log n}{n}} |f(t_i)f(t_j)|\,dP \leqq \int_{\substack{\mathrm{Min}(t_m-t_k)\geqq\frac{3\log n}{n}\\1\leqq m\leqq n\\m\neq i,j,k\\t_m\geqq t_k}} |f(t_i)f(t_j)|\,dP$$

$$= P\left[\underset{\substack{1\leqq m\leqq n\\m\neq i,j\\t_m\geqq t_k}}{\mathrm{Min}} (t_m - t_k) > \frac{3\log n}{n}\right] \int_{\Omega} |f(t_i)f(t_j)|\,dP$$

$$\leqq \left(1 - \frac{3\log n}{n}\right)^{n-3} \int_0^1 f^2(t)\,dt \leqq K\frac{1}{n^3}\int_0^1 f^2(t)\,dt,$$

where $K$ is a constant independent of $n$ and $f(t)$. Hence we have

$$\left|\left(\int_{E_i} - \int_{E_i\cap\Omega_n}\right) d_{i,n}d_{j,n}f(t_i)f(t_j)\,dP\right| \leqq K\frac{(3\log n)^2}{n^4}\int_0^1 f^2(t)\,dt,$$

and

$$\left|\left(\int_{E_j} - \int_{E_j\cap\Omega_n}\right) d_{i,n}d_{j,n}f(t_i)f(t_j)\,dP\right| \leqq K\frac{(3\log n)^2}{n^4}\int_0^1 f^2(t)\,dt.$$

By (3.4) and the reasons in 2°, we have, for $i \neq j$,

(3.5)
$$\left|\int_{\Omega_n} d_{i,n}d_{j,n}f(t_i)f(t_j)\,dP\right|$$

$$\leqq K'\left(\frac{\log n}{n}\right)^{5/2}\int_0^1 f^2(t)dt + \left|\left(\int_{E_i} + \int_{E_j}\right) d_{i,n}d_{j,n}f(t_i)f(t_j)\,dP\right|,$$

where $K'$ is a constant independent of $f(t)$ and $n$.

3°. We define four dimensional sets whose points are $(x_1, y_1, x_2, y_2)$ as follows:

$$D \equiv \left[0 \leqq x_1 < \frac{3\log n}{n},\ 0 \leqq y_1 < \frac{3\log n}{n},\ 0 \leqq x_2 < 1 - x_1,\ 0 \leqq y_2 < 1 - y_1\right]$$

$$D_1 \equiv [y_1 + y_2 < x_2] \cap D,$$

$$D_1' \equiv [x_1 + x_2 < y_2] \cap D,$$

and

$$D_2 \equiv [x_2 - y_1 \leqq y_2 \leqq x_2 + x_1] \cap D.$$

Then any two of $D_1$, $D_1'$ and $D_2$ are disjoint and

$$D = D_1 \cup D_1' \cup D_2.$$

On the other hand by Lemma 3, we have

$$\Big(\int_{E_i} + \int_{E_j}\Big) d_{i,n} d_{j,n} f(t_i) f(t_j)\, dP = \Big(\iiiint_{D_1} + \iiiint_{D_1'}\Big) x_1 y_1 f(x_2) f(y_2)$$

$$\cdot P\Big[\,(d_{i,n} < x_1) \cap (d_{j,n} < y_1)\,\Big|\, t_i = x_2, t_j = y_2\,\Big] dx_1 dy_1 dx_2 dy_2$$

$$= \Big(\iiiint_{D_1} + \iiiint_{D_1'}\Big)(n-2)(n-3) x_1 y_1 (1 - x_1 - y_1)^{n-4} f(x_2) f(y_2) dx_1 dy_1 dx_2 dy_2$$

$$= \Big(\iiiint_{D} - \iiiint_{D_2}\Big)(n-2)(n-3) x_1 y_1 (1 - x_1 - y_1)^{n-4} f(x_2) f(y_2) dx_1 dy_1 dx_2 dy_2.$$

By Fubini's theorem, we have

$$\Big|\iiiint_{D} (n-2)(n-3) x_1 y_1 (1 - x_1 - y_1)^{n-4} f(x_2) f(y_2) dx_1 dy_1 dx_2 dy_2\Big|$$

$$= \Big|\int_0^{\frac{3\log n}{n}} y_1\, dy_1 \int_0^{\frac{3\log n}{n}} (n-2)(n-3) x_1 (1 - x_1 - y_1)^{n-4}\, dx_1 \int_0^{1-x_1} f(x_2) dx_2 \int_0^{1-y_1} f(y_2) dy_2\Big|.$$

Since $\frac{3\log n}{n} \leqq 1$ for $n \geqq 6$, we have, by (3.2),

$$\Big|\iiiint_{D} (n-2)(n-3) x_1 y_1 (1 - x_1 - y_1)^{n-4} f(x_2) f(y_2) dx_1 dy_1 dx_2 dy_2\Big|$$

$$\leqq \Big|\int_0^{\frac{3\log n}{n}} y_1\, dy_1 \int_0^{\frac{3\log n}{n}} (n-2)(n-3) x_1 (1 - x_1 - y_1)^{n-4} dx_1 \int_0^{1-x_1} f(x_2)\, dx_2 \int_{1-y_1}^{1} f(y_2) dy_2\Big|$$

$$\leqq \int_0^{\frac{3\log n}{n}} y_1^{3/2}\, dy_1 \int_0^{\frac{3\log n}{n}} (n-2)(n-3) x_1 (1 - x_1 - y_1)^{n-4}\, dx_1 \Big(\int_0^1 f^2(t)\, dt\Big)$$

$$\leqq \Big(\frac{3\log n}{n}\Big)^{5/2} \int_0^1 f^2(t) dt.$$

We divide $D_2$ in two disjoint subsets such that

$$D_2' \equiv (x_2 - y_1 \leqq y_2 \leqq 1 - y_1,\ 1 - y_1 \leqq x_1 + x_2) \cap D$$

and

$$D_2'' \equiv (x_2 - y_1 \leqq y_2 \leqq x_1 + x_2,\ 1 - y_1 > x_1 + x_2) \cap D.$$

Then we have, for $n \geqq 6$,

$$\left|\iiiint\limits_{D_2'}(n-2)(n-3)x_1y_1(1-x_1-y_1)^{n-4}f(x_2)f(y_2)\,dx_1\,dy_1\,dx_2\,dy_2\right|$$

$$=\left|\int_0^{\frac{3\log n}{n}}y_1\,dy_1\int_0^{\frac{3\log n}{n}}(n-2)(n-3)x_1(1-x_1-y_1)^{n-4}\,dx_1\int_{1-y_1-x_1}^{1-x_1}f(x_2)\,dx_2\int_{x_2-y_1}^{1-y_1}f(y_2)\,dy_2\right|$$

$$\leqq\left|\int_0^{\frac{3\log n}{n}}y_1\,dy_1\int_0^{\frac{3\log n}{n}}(n-2)(n-3)x_1(1-x_1-y_1)^{n-4}dx_1\int_{1-y_1-x_1}^{1-x_1}f(x_2)(1-x_2)^{1/2}dx_2\right|\left(\int_0^1 f^2(t)dt\right)^{1}$$

$$\leqq\left(\int_0^{\frac{3\log n}{n}}y_1\,dy_1\int_0^{\frac{3\log n}{n}}(x_1+y_1)^{1/2}(n-2)(n-3)x_1(1-x_1-y_1)^{n-4}dx_1\right)\int_0^1 f^2(t)\,dt$$

$$\leqq\sqrt{2}\left(\frac{3\log n}{n}\right)^{5/2}\int_0^1 f^2(t)\,dt.$$

and

$$\left|\iiiint\limits_{D_2''}(n-2)(n-3)x_1y_1(1-x_1-y_1)^{n-4}f(x_2)f(y_2)dx_1dy_1dx_2dy_2\right|$$

$$=\left|\int_0^{\frac{3\log n}{n}}y_1\,dy_1\int_0^{\frac{3\log n}{n}}(n-2)(n-3)x_1(1-x_1-y_1)^{n-4}\,dx_1\int_0^{1-y_1-x_1}f(x_2)\,dx_2\int_{x_2-y_1}^{x_2+x_1}f(y_2)\,dy_2\right|$$

$$\leqq\left(\int_0^{\frac{3\log n}{n}}y_1\,dy_1\int_0^{\frac{3\log n}{n}}(n-2)(n-3)x_1(x_1+y_1)^{1/2}(1-x_1-y_1)^{n-4}dx_1\right)\left(\int_0^1 f^2(t)\,dt\right)$$

$$\leqq\sqrt{2}\left(\frac{3\log n}{n}\right)^{5/2}\int_0^1 f^2(t)dt.$$

Hence we have, for $n\geqq 6$,

$$\left|\left(\int\limits_{E_i}+\int\limits_{E_j}\right)d_{i,n}d_{j,n}f(t_i)f(t_j)\,dP\right|\leqq\left(\frac{3\log n}{n}\right)^{5/2}\int_0^1 f^2(t)\,dt. \tag{3.6}$$

By (3.3), (3.5) and (2.2), we can prove the lemma.

**4.** For the proof of Theorem 1 stated in § 1, it is sufficient to prove the following Theorem 1′. Because if $\int_0^1 f(t)\,dt\neq 0$, then by (2.4′), instead of $f(t)$, we may take the function $f'(t)$ such that

$$f'(t)=f(t)-\int_0^1 f(t)\,dt.$$

THEOREM 1'. *Let $f(t)$ be a function such that*

$$\int_0^1 f(t)\,dt = 0 \tag{4.1}$$

*and for some $p$, $1 < p \leqq 2$,*

$$\int_0^1 |f(t)|^p dt < +\infty. \tag{4.1'}$$

*Then we have*

$$P[\lim_{n\to\infty} S_n = 0] = 1.$$

PROOF. Let us define the functions $f_k(t)$, $k = 1, 2, \ldots,$ as follows:

$$f_k(t) = \begin{cases} f(t) - \alpha_k, & \text{if} \quad |f(t)| < k^{1/4}, \\ -\alpha_k, & \text{if} \quad |f(t)| \geqq k^{1/4}, \end{cases} \tag{4.2}$$

where

$$\alpha_k = \int_{|f(t)|<k^{1/4}} f(t)\,dt. \tag{4.2'}$$

Then we have, by (4.1)(4.1') and the definition of $f_k(t)$,

$$\int_0^1 f_k(t)\,dt = 0, \tag{4.3}$$

$$\int_0^1 f_k^2(t)\,dt = \int_{|f(t)|<k^{1/4}} f^2(t)\,dt - \alpha_k^2 \tag{4.3'}$$

$$\leqq \int_{|f(t)|<k^{1/4}} f^2(t)dt = O(k^{(2-p)/4}) \qquad (k\to+\infty),$$

and

$$|\alpha_k| = \int_{|f(t)|\geqq k^{1/4}} f(t)\,dt \leqq \Big(\int_{|f(t)|\geqq k^{1/4}} |f(t)|^p\,dt\Big)^{\frac{1}{p}} \Big(\int_{|f(t)|\geqq k^{1/4}} dt\Big)^{\frac{p-1}{p}} \tag{4.3''}$$

$$= O(k^{-1/4(p-1)}) \qquad (k\to+\infty).$$

By (2.2), we have

$$\int_{\Omega_k} |S_k|\,dP \leqq \sum_{i=1}^{k} \int_{\Omega_k} d_{i,k}|f(t_i) - f_k(t_i)|\,dP + \Big\{\int_{\Omega_k} \Big(\sum_{i=1}^{k} d_{i,k} f_k(t_i)\Big)^2 dP\Big\}^{1/2}.$$

By the definition of $f_k(t)$ and (4.3''), it follows that

$$\sum_{i=1}^{k} \int_{\Omega_k} d_{i,k}|f(t_i) - f_k(t_i)|\,dP$$

$$\leqq \sum_{i=1}^{k} \int_{\Omega_k} |\alpha_k| d_{i,k} dP + \sum_{i=1}^{k} \int_{\Omega_k} d_{i,k} |f(t_i)| dP$$

$$\leqq |\alpha_k| + O\left(\frac{\log k}{k^{(p-1)/4}}\right) = O\left(\frac{\log k}{k^{(p-1)/4}}\right) \qquad (k \to +\infty).$$

By Lemma 4 and (4. 3′), we have

$$\left\{\int_{\Omega_k} \left(\sum_{i=1}^{k} d_{i,k} f_k(t_i)\right)^2 dP\right\}^{1/2} \leqq \left(A \frac{(\log k)^{5/2}}{k^{1/2}} \int_0^1 f^2(t)\, dt\right)^{1/2}$$

$$= O\left(\frac{(\log k)^{5/4}}{k^{p/8}}\right) \qquad (k \to +\infty).$$

Since $0 < p-1 \leqq p/2$ for $1 < p \leqq 2$, we have

$$\int_{\Omega_k} |S_k| dP \leqq \sum_{i=1}^{k} \int_{\Omega_k} d_{i,k} |f(t_i) - f_k(t_i)| dP + \left\{\int_{\Omega_k} \left(\sum_{i=1}^{k} d_{i,k} f(t_i)\right)^2 dP\right\}^{1/2}$$

$$= O\left(\frac{(\log k)^{5/4}}{k^{(p-1)/4}}\right) \qquad (k \to +\infty)$$

Therefore if we take an integer $\alpha$ such that $\alpha(p-1)/4 > 1$, then

$$\sum_{k=1}^{\infty} \int_{\Omega_{k^\alpha}} |S_{k^\alpha}|\, dP = O\left(\sum_{k=1}^{\infty} \frac{(\log k)^{5/4}}{k^{\alpha(p-1)/4}}\right) = O(1). \tag{4. 4}$$

On the other hand, by (3. 1) and (2. 3), we have

$$\sum_{k=1}^{\infty} P[\Omega - \Omega_{k^\alpha}] = O\left(\sum \frac{1}{k^{2\alpha}}\right) = O(1). \tag{4. 4′}$$

By (4. 4) and (4. 5), we can prove that

$$P[\lim_{k\to\infty} S_{k^\alpha} = 0] = 1. \tag{4. 5}$$

Next let us put

$$\Omega_k' \equiv \bigcap_{n=k^\alpha}^{(k+1)^\alpha - 1} \Omega_n,$$

then we have

$$\sum_{k=1}^{\infty} P[\Omega - \Omega_k'] \leqq \sum_{k=1}^{\infty} \sum_{n=k^\alpha}^{(k+1)^\alpha - 1} O(1/n^2) = O(1). \tag{4. 6}$$

It is seen that

$$\max_{k^\alpha \leqq n < (k+1)^\alpha} |S_n - S_{k^\alpha}| \leqq \sum_{n=k^\alpha+1}^{(k+1)^\alpha - 1} |S_n - S_{n-1}|. \tag{4. 7}$$

On the other hand by (2. 2), we have

$$S_n - S_{n-1} = \begin{cases} d_{n,n}f(t_n), & \text{if } t_n = t_{1,n}, \\ d_{n,n}\{f(t_n) - f(t_n - d'_{n,n})\}, & \text{if } t_n \neq t_{1,n}. \end{cases}$$

By the definition of $t_{1,n}$, we can see that

$$[\omega;\ t_n \neq t_{1,n}] \simeq [\omega;\ d'_{n,n} < t_n],$$

Therefore we have, by Lemma 2,

$$\Big\{\int_{\substack{\Omega'_k \\ t_n \neq t_{1,n}}} |d_{n,n}f(t_n) - f(t_n - d'_{n,n})|^p\, dP\Big\}^{1/p} \leqq \Big(\frac{3\log n}{n}\Big)\Big(\int_{d'_{n,n}<t_n} |f(t_n) - f(t_n - d'_{n,n})|^p\, dP\Big)^{1/p}$$

$$\leqq \Big(\frac{3\log n}{n}\Big)\Big(\int_0^1 (n-1)(1-y)^{n-2}\,dy \int_y^1 |f(x) - f(x-y)|^p\, dx\Big)^{1/p}$$

$$\leqq 2\Big(\frac{3\log n}{n}\Big)\Big(\int_0^1 |f(t)|^p dt\Big)^{1/p}.$$

By (3.1), we have

$$\Big(\int_{\substack{\Omega'_k \\ t_n = t_{1,n}}} |d_{n,n}f(t_n)|^p\, dP\Big)^{1/p} \leqq \Big(\frac{3\log n}{n}\Big)\Big(\int_0^1 |f(t)|^p\, dP\Big)^{1/p}.$$

Hence we obtain that

$$\Big(\int_{\Omega'_k} |S_n - S_{n-1}|^p dP\Big)^{1/p} = O\Big(\frac{3\log n}{n}\Big) \qquad (n \to +\infty). \tag{4.8}$$

By (4.7) and (4.8), we have

$$\Big(\int_{\Omega'_k}\Big\{\operatorname*{Max}_{k^\alpha \leqq n < (k+1)^\alpha} |S_n - S_{k^\alpha}|\Big\}^p dP\Big)^{1/p} = O\Big(\sum_{n=k^\alpha+1}^{(k+1)^\alpha - 1} \frac{\log n}{n}\Big)$$

$$= O\Big(\frac{(\log k)\{(k+1)^\alpha - k^\alpha\}}{k^\alpha}\Big) = O\Big(\frac{\log k}{k}\Big) \qquad (k \to +\infty).$$

Thus we obtain that

$$\sum_{k=1}^\infty \int_{\Omega'_k}\Big(\operatorname*{Max}_{k^\alpha \leqq n < (k+1)^\alpha} |S_n - S_{k^\alpha}|\Big)^p dP < +\infty. \tag{4.9}$$

By (4.6) and (4.9), we can prove that

$$P\Big[\operatorname*{Max}_{k^\alpha \leqq n < (k+1)^\alpha} |S_n - S_{k^\alpha}| \to 0\Big] = 1. \tag{4.10}$$

By (4.5) and (4.10), we can prove theorem.

5. In this paragraph we prove the following.

THEOREM 2. *If $f(t) \in L(0,1)$, then we have, for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P\left[\left| S_n - \int_0^1 f(t)\,dt \right| > \varepsilon \right] = 0.$$

PROOF. It is sufficient to prove that

$$I_n = \int_\Omega \left| S_n - \int_0^1 f(t)\,dt \right| dP = o(1) \qquad (n \to +\infty). \tag{5.1}$$

By (2.2) and (2.2′), we have

$$I_n \leqq \sum_{i=1}^{n} \int_\Omega \left\{ \int_0^{d_{i,n}} |f(t_i) - f(t_i - u)|\,du \right\} dP + \int_\Omega \left| \int_0^{t_{1,n}} f(u)\,du \right| dP.$$

(5.1) can be shown easily, by the first two Lemmas in §2 and the fact

$$\int_0^1 |f(t+u) - f(t)|\,dt = o(1) \qquad (u \to 0).$$

## REFERENCES

[1] A. N. KOLMOGOROFF, Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin (1933).
[2] S. TAKAHASHI, Notes on the Riemann-Sum, Proc. Japan Acad., 31 (1955), 6-13.

INSTITUTE OF MATHEMATICS, KANAZAWA UNIVERSITY.