NOTE ON A THEOREM OF HILTON

TÔRU WADA

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1. Introduction. P. J. Hilton [4] showed as a generalization of Chang-Whitehead theorem the following theorem: Let X be a connected CW-complex which is the union of CW-complexes A, B. Let A be $(p-1)$—connected $(p \geqq 3)$, B be $(q-1)$ connected $(q \geqq 3)$, and let $A \cap B = C$ be contractible over itself. Then

$$\pi_n(X) = i'_{3*}\pi_n(A) + i'_{4*}\pi_n(B), \qquad n < p+q-1$$

$$\pi_{p+q-1}(X) = i'_{3*}\pi_{p+q-1}(A) + i'_{4*}\pi_{p+q-1}(B) + P'(\pi_p(A) \otimes \pi_q(B)).$$

The purpose of this paper is to generalize the latter as follows:

THEOREM. *Let* $(X: A, B)$ *be an excisive triad* (see [3] p. 335), (A, C) *be* $(p-1)$-*connected* $(p \geqq 3)$ (see [1] p. 389), (B, C) *be* $(q-1)$-*connected* $(q \geqq 3)$, *and let* C *be* l-*connected* $(l \geqq 1)$. *Then*

$$\pi_n(X, C) = i_{3*}\,\pi_n(A, C) + i_{4*}\,\pi_n(B, C) \qquad n < l + \min(p, q) - 1 = n_0$$

$$\pi_{n_0}(X, C) = i_{3*}\pi_{n_0}(A, C) + i_{4*}\pi_{n_0}(B, C) + P(\pi_p(A) \otimes \pi_q(B)),$$

where P *is a univalent homomorphism and is given by*

$$P(\alpha \otimes \beta) = i_*[\alpha, \beta] \qquad \alpha \in \pi_p(A), \ \beta \in \pi_q(B),$$

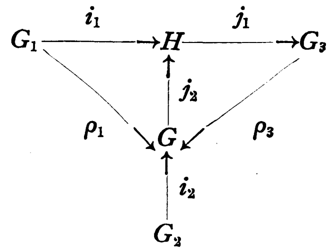
[,] *being the Whitehead product, and* i_* *is a univalent homomorphism to appear in Prop.* 1.

This follows readily from Prop. 1, Prop. 2, in the next section, and Blakers-Massey's triad theorem [2].

We use in the following the same notations, of Blakers-Massey [1], but our fundamental tool is Lemma 2.

2. Statements and Proofs.

LEMMA 1. *In the diagram*



of groups and homomorphisms, assume that $i_2\rho_1 = i_1$, $j_1 j_2 \rho_3 = 1$, *image* $i_* =$

kernel j_α ($\alpha = 1, 2$), i_1, j_2, ρ_1, ρ_2 *are univalent homomorphisms, and* j_1, j_2 *are onto homomorphisms. Then G decomposes into the direct sum*

$$G = \rho_1 G_1 + i_2 G_2 + \rho_3 G_3,$$

if i_α *has a left inverse* i_α^*, $(\alpha = 1, 2)$.

PROOF. Consider $x \in \rho_1 G_1 \cap \rho_3 G_3$, since $x \in \rho_1 G_1$, it follows that $j_3(x)$ belongs to $i_1 G_1$, hence $j_1 j_2(x) = 0$. Since $x \in \rho_3 G_3$, it follows that there is a $y \in G_3$, such that $x = \rho_3(y)$. Then

$$y = j_1 j_2 \rho_3(y) = j_1 j_2(x) = 0,$$

and this implies $x = \rho_3(y) = 0$, hence $\rho_1 G_1 \cap \rho_3 G_3 = 0$.

Next, if $g_1 \in G_1$, $g_3 \in G_3$, and $x \in i_2 G_2$, let $x = \rho_1(g_1) + \rho_3(g_3)$. Then

$$j_2 \rho_1(g_1) + j_2 \rho_3(g_3) = j_2(x) = 0$$

by virture of our assumption image i_2 = kernel j_2. Let j_1^* be a univalent homomorphism such that $j_2 \rho_3 = j_1^*$, while $j_2 \rho_1 = i_1$. Then

$$0 = j_1(i_1(g_1) + j_1^*(g_3)) = 0 + g_3.$$

Since $g_3 = 0$ and $g_1 = 0$, it follows that

$$(\rho_1 G_1 + \rho_3 G_3) \cap i_2 G_2 = 0.$$

Finally, let h be an element of the group H such that $j_2(g) = h$, for any $g \in G$, as j_2 is a homomorphism of G onto H. Since the group H decomposes into the direct sum $i_1 G_1 + j_1^* G_3$, it follows that there are $g_1 \in G_1$, $g_3 \in G_3$ such that

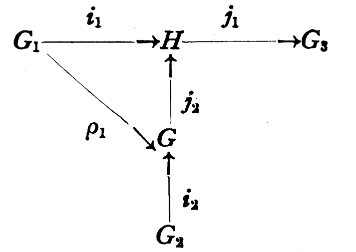
$$h = i_1(g_1) + j_1^*(g_3) \qquad \text{for all } h \in H.$$

The assumption implies $i_1(g_1) = j_2 \rho_1(g_1), j_1^*(g_3) = j_2 \rho_3(g_3)$, hence

$$j_2(g) = j_2 \rho_1(g_1) + j_2 \rho_3(g_3).$$

This completes the proof of the lemma.

LEMMA 2. *In the diagram*



of groups and homomorphisms, assume that the commutativity holds in the triangle, image i_α = *kernel* j_α $(\alpha = 1, 2)$, i_1, i_2 *are univalent homomorphisms, and* j_1, j_2 *are onto homomorphisms. Then there is a univalent homomorphism* ρ_3; $G_3 \to G$ *such that* $j_1 j_2 \rho_3 = 1$, *and G decomposes into the direct sum*

$$G = \rho_1 G_1 + i_2 G_2 + \rho_3 G_3$$

if and only if i_α *has a left inverse* i^*_α $(\alpha = 1, 2)$.

PROOF. Let there exists a univalent homomorphism ρ_3: $G_3 \rightarrow G$ such that $j_1 j_2 \rho_3 = 1$, and let the direct sum decomposition hold. Now, we define j_1^*: $G_3 \rightarrow H$ by $j_1^* = j_2\rho_3$, then $j_1 j_1^* = j_1 j_2 \rho_3 = 1$. The existence of a homomorphism i_1^*: $H \rightarrow G_1$ such that $i_1^*\, i_1 = 1$ follows from the fact that the group H decomposes into the direct sum

$$H = i_1 G_1 + j_1^* G_3.$$

Next, we define j_2^*: $H \rightarrow G$, by $j_2^* = \rho_1 i_1^{-1}$ on $i_1\, G_1$, and by $j_2^* = \rho_3 j_1^{*-1}$ on $j_1^*\, G_3$. Then

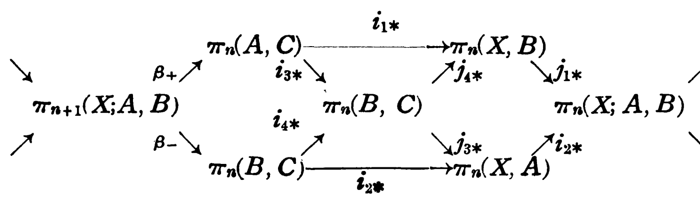
$$\begin{aligned} G &= \rho_1 G_1 + i_2 G_2 + \rho_3 G_3 \\ &= i_2 G_2 + j_2^* i_1 G_1 + j_2^* j_1^* G_3 \\ &= i_2 G_2 + j_2^*(i_1 G_1 + j_1^* G_3) = i_2 G_2 + j_2^* H, \end{aligned}$$

and follows the existence of a homomorphism i_2^*: $G \rightarrow G_2$ such that $i_2^* i_2 = 1$.

Conversely assume that i_α has a left inverse i^*_α $(\alpha = 1.2)$. Then there are subgroups X of the group G such that $G = i_2 G_2 + X$. Let X_0 denote one of such groups. Then $j_2 | X_0$ maps X_0 onto H, for $j_2 G = j_2 i_2 G_2 + j_2 X_0 = j_2 X_0$.

Let $x_1, x_2 \in X_0$, and $j_2 x_1 = j_2 x_2$. Then $j_2 x_1 - j_2 x_2 = j_2(x_1 - x_2) = 0$, hence the element $(x_1 - x_2)$ of the group X_0 belongs to $i_2 G_2$. Since $i_2 G_2 + X_0$ is a direct sum, it follows that $x_1 = x_2$, therefore j_2 is an isomorphism of the group X_0 onto the group H. Let μ be an inverse isomorphism of $j_2 | X_0$, then $j_2\, \mu(h) = h$, for all $h \in H$. We define now the univalent homomorphism ρ_3: $G_3 \rightarrow G$ by $\rho_3 = \mu j_1^*$. Then $j_2 \rho_3(g_3) = j_2\, \mu\, j_1^*(g_3) = j_1^*(g_3)$ $g_3 \in G_3$, and Lemma 1 implies the conclusion.

Consider the various groups and homomorphisms indicated by the following diagram (see [1], Lemma 3.5.5)



Then the following commutativity relationship holds:

$$i_{1*} = j_{4*} i_{3*}, \quad j_{2*} = j_{3*} i_{4*}.$$

Moreover, we have the following result:

PROPOSITION 1. *Let* $(X; A, B)$ *be a triad, then there is a univalent homomorphism*

$$i_*\colon \pi_n(X\colon A, B) \rightarrow \pi_n(X, C) \qquad (3 \leqq n \leqq r)$$

such that

$$j_{1*} j_{4*} i_* = 1,$$

and the group $\pi_n(X, C)$ *decomposes into the direct sum*

$$\pi_n(X, C) = i_{3*}\pi_n(A, C) + i_{4*}\pi_n(B, C) + i_*\pi_n(X; A, B)$$

if injection homomorphisms

$$i_{1*}\colon \pi_n(A, C) \to \pi_n(X, B)$$
$$i_{2*}\colon \pi_n(B, C) \to \pi_n(X, C)$$

have left inverses for all n $(2 \leqq n \leqq r)$.

PROOF. In the above diagram, since i_{1*}, i_{2*}, i_{3*}, i_{4*} are univalent homomorphisms for $n-1$, then j_{1*}, j_{2*}, j_{3*}, j_{4*} are onto homomorphisms for n. The assumption for n implies that i_{3*}, i_{4*}, are univalent homomorphisms for n. Then, we obtain the following diagram

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow & & & \\
0 \longrightarrow & \pi_n(A, C) & \xrightarrow{i_{1*}} & \pi_n(X, B) & \xrightarrow{j_{1*}} & \pi_n(X; A, B) & \longrightarrow 0 \\
 & & \searrow^{i_{3*}} & \uparrow j_{4*} & & & \\
 & & & \pi_n(X, C) & & & \\
 & & & \uparrow i_{4*} & & & \\
 & & & \pi_n(B, C) & & & \\
 & & & \uparrow & & & \\
 & & & 0 & & &
\end{array}$$

and Proposition 1 now follows from Lemma 2.

PROPOSITION 2. *Let* $(X; A, B)$ *be an excisive triad,* (A, C) *be* $(p-1)$*-connected* $(p \geqq 3)$, (B, C) *be* $(q-1)$*-connected* $(q \geqq 3)$, *and let* C *be* l*-connected* $(l \geqq 1)$, *then injection homomorphisms*

$$i_{1*}\colon \pi_n(A, C) \to \pi_n(X, B)$$
$$i_{2*}\colon \pi_n(B, C) \to \pi_n(X, A)$$

have left inverses at least for $n \leqq l + \min(p, q) - 1$.

PROOF. Corollary 3.2 in [3] implies that

$$i_{1*}\colon \pi_n(A, C) \to \pi_n(X, B)$$

has a left inverse for $n \leqq l + p - 1$. Similarly

$$i_{2*}\colon \pi_n(B, C) \to \pi_n(X, A)$$

has a left inverse for $n \leqq l + q - 1$. Therefore the Proposition is true.

NOTE. If $(X; A, B)$ is an excisive triad, (A, C) is $(p-1)$-connected $(p \geqq 3)$, (B, C) is $(q-1)$-connected $(q \geqq 3)$, and C is $(p+q-1)$-connected, then

$$\pi_n(X) = i'_{3*}\pi_n(A) + i_{4*}\pi_n(B) \qquad n \leqq p + q - 2$$
$$\pi_{p+q-1}(X) = i'_{3*}\pi_{p+q-1}(A) + i'_{4*}\pi_{p+q-1}(B) + P'(\pi_p(A \otimes \pi_q(B)).$$

This follows immediately from the Theorem.

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY.