SOME THEOREMS ON FOURIER SERIES

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1. Introduction. Suppose that a function $f(x)$ is of period 2π. Denoting by $s_m(x)$ the m-th partial sum of the Fourier series of $f(x)$, a classical problem of Zalcwasser reads as follows [5]:

Let $\{p_k\}$ be a strictly increasing sequence of positive integers. If a function $f(x)$ is squarely integrable over the period, then is it true that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |s_{p_k}(x) - f(x)| = 0 \tag{1}$$

for almost all x?

Concerning this problem various results were obtained under some additional conditions of the sequence $\{p_k\}$ (cf. [7]). On the other hand we proved the following theorem in which we imposed an additional condition on the function $f(x)$ but did not on the sequence $\{p_k\}$ [4].

THEOREM. *Let $f(x)$ be continuous and of modulus of continuity $o(1/\log\log(1/|h|))$ as $h\to 0$. If $\{p_k\}$ is a strictly increasing sequence of positive integers, then we have the conclusion* (1) *for almost all x.*

In this paper we shall improve the above theorem in two points, namely, we shall treat an integrable function instead of continuous function, and we shall add an exponent in the summand of the relation (1). Incidentally the similar method of the proof of our theorem will work to prove a theorem of R. Salem concerning a convergence criterion of Fourier series (Theorem 2 below).

In what follows we suppose that a function $f(x)$ is of period 2π and we denote the n-th partial sum of its Fourier series by the usual notation $s_n(x) = s_n(f; x)$ and its conjugate by $\bar{s}_n(x) = \bar{s}_n(f; x)$.

2. The improved form of the above theorem is this:

THEOREM 1. *Let $f(x)$ be an integrable function on $(0, 2\pi)$ and let $\{p_k\}$ be a strictly increasing sequence of positive integers. If*

$$\int_0^h \{f(x+t) - f(x-t)\}\, dt = o(|h|/\log\log(1/|h|)) \tag{2}$$

as $h \to 0$ *uniformly for* x, $0 \leqq a \leqq x \leqq b \leqq 2\pi$, *then for any positive number* α *we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |s_{p_k}(x) - f(x)|^{\alpha} = 0 \tag{3}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\bar{s}_{p_k}(x) - \bar{f}(x)|^{\alpha} = 0 \tag{4}$$

for almost all x *in* (a, b), *where* $\bar{f}(x)$ *is conjugate to* $f(x)$.

The proof will be done in the next sections.

THEOREM 2 (R. Salem [3]). *Let* $f(x)$ *be an integrable function on* $(0, 2\pi)$. *If*

$$\int_0^h \{f(x+t) - f(x-t)\}\, dt = o(|h|/\log(1/|h|)) \tag{5}$$

as $h \to 0$ *uniformly for* x, $0 \leqq a \leqq x \leqq b \leqq 2\pi$, *then the Fourier series of the function* $f(x)$ *converges almost everywhere in* (a, b).

R. Salem proved further the uniform convergence of the Fourier series in the case of continuous functions.

For the Fourier series of a function $L^p(0, 2\pi)$ $(p > 1)$, the sequence $\{s_{n_k}(x)\}$ of partial sums with the Hadamard gaps converges almost everywhere by the well known Littlewood and Paley theorem, but as Zygmund pointed out [8] this result does not remain true in the case $p = 1$. We shall furnish a sufficient condition for this conclusion:

THEOREM 3. *In the assumption of Theorem* 2, *if we replace the condition* (5) *by the weaker condition* (2) *uniformly for* x, $0 \leqq a \leqq x \leqq b \leqq 2\pi$, *then for any sequence of positive integers with the Hadamard gaps* $\{n_k\}$, $n_k/n_{k-1} > q > 1$ $(k = 1, 2, \ldots.)$, *the sequence* $\{s_{n_k}(x)\}$ *converges almost everywhere in* (a, b).

3. For the proof of theorems we need some preliminary lemmas.

LEMMA 1. *Let* $f(x)$ *be an integrable function on* $(0, 2\pi)$ *and let* $F(x)$ *be the periodic part of the primitive of* $f(x)$. *Let* $\omega_1(\delta)$ *be an increasing function of* $\delta > 0$ *such that* $\delta\omega_1(\delta)$ *and* $\delta/\omega_1(\delta)$ *are also monotone increasing and that* $\delta\omega_1(\delta) \to 0$, $\omega_1(\sqrt{\delta}) = O(\omega_1(\delta))$ *as* $\delta \to 0$. *If*

$$\int_0^h \{f(x+t) - f(x-t)\}dt = o(|h|\,\omega_1(|h|))$$

uniformly for all x, *then there exists a trigonometric polynomial* $P_n(x)$ *of order* n *such that*

$$\left| P_n(x) - F(x) \right| = o\left(\frac{1}{n}\,\omega_1\left(\frac{1}{n}\right)\right)$$

uniformly for all x *and that*

$$\lim_{n\to\infty} \{P_n'(x) - f(x)\} = 0$$

for almost all x.

For example, we may take $\omega_1(\delta) = 1/\log(1/\delta)$ or $1/\log\log(1/\delta)$. This lemma is essentially known ([10], [3]). By the assumption we get

$$F(x+h) + F(x-h) - 2F(x) = o(|h|\,\omega_1(|h|))$$

uniformly. Using the kernel

$$\frac{1}{m^3}\left(\frac{\sin mt}{\sin t}\right)^4 \tag{6}$$

which is a trigonometric polynomial of order $2m-2$ in $2t$, we define

$$I_m(x) = \frac{1}{h_m}\int_{-\pi/2}^{\pi/2} F(x+2u)\,\frac{1}{m^3}\left(\frac{\sin mu}{\sin u}\right)^4 du$$

where

$$h_m = \frac{1}{m^3}\int_{-\pi/2}^{\pi/2}\left(\frac{\sin mt}{\sin t}\right)^4 dt$$

The function $I_m(x)$ is a trigonometric polynomial of order $2m-2$ and h_m is greater than a positive constant independent of m. We have

$$|I_m(x) - F(x)| = \frac{1}{h_m}\left|\int_0^{\pi/2}\{F(x+2u) + F(x-2u) - 2F(x)\}\,\frac{1}{m^3}\left(\frac{\sin mu}{\sin u}\right)^4 du\right|$$

$$= O(1)\left\{\left|\int_0^{\pi/m}\right| + \left|\int_{\pi/m}^{\pi/\sqrt{m}}\right| + \left|\int_{\pi/\sqrt{m}}^{\pi/2}\right|\right\}$$

$$= o\left(m\int_0^{\pi/m} u\,\omega_1(2u)\,du\right) + o\left(\frac{1}{m^3}\int_{\pi/m}^{\pi/\sqrt{m}} \frac{\omega_1(2u)}{u^3}\,du\right) + O\left(\frac{1}{m^3}\int_{\pi/\sqrt{m}}^{\pi/2}\frac{du}{u^4}\right)$$

$$= o\left(\frac{1}{m}\,\omega_1\left(\frac{1}{m}\right)\right) + o\left(\frac{1}{m}\,\omega_1\left(\frac{1}{m}\right)\right) + O\left(\frac{1}{m^{3/2}}\right),$$

that is, $|I_m(x) - F(x)| = o\left(\frac{1}{m}\,\omega_1\left(\frac{1}{m}\right)\right)$ uniformly.

On the other hand, since $f(x) = F'(x)$ almost everywhere, we get

$|I'_m(x) - f(x)|$

$$= \frac{1}{h_m}\left|\int_0^{\pi/m} \{f(x+2u) + f(x-2u) - 2f(x)\} \frac{1}{m^3}\left(\frac{\sin mu}{\sin u}\right)^4 du\right|$$

for almost all x, and the kernel (6) is a positive kernel we conclude by the well known theorem that

$$\lim_{m\to\infty} \{I_m'(x) - f(x)\} = 0 \tag{7}$$

if $\int_0^h |f(x+2u) + f(x-2u) - 2f(x)|\, du = o(|h|)$, that is, (7) subsists for almost all x.

To complete the proof for any n, we take m equal to $\frac{n}{2}+1$ or $\frac{1}{2}(n+1)$ according as n is even or odd, and put $P_n(x) = I_m(x)$.

Lemma 2. *If* $f(x) \in L^r(0,2\pi)$ $(r \geqq 2)$, *then*

$$\int_0^{2\pi} |s_n(x)|^r dx \leqq 3^r r^r \int_0^{2\pi} |f(x)|^r dx.$$

The factor 3^r on the right may be made smaller, but this is irrelevant to our purpose. This lemma is well known in a slightly different form ([9] p. 153).

Lemma 3. *If* $t_n(x)$ *is a trigonometric polynomial of order* n, *and* $r \geqq 1$, *then*

$$\int_0^{2\pi} |t_n'(x)|^r dx \leqq n^r \int_0^{2\pi} |t_n(x)|^r dx.$$

This is also known (cf. [9] p. 155).

Lemma 4. *If* $F(x)$ *is continuous and*

$$F(x+t) + F(x-t) - 2F(x) = o(|t|) \tag{8}$$

uniformly for x, *then the conjugate* $\overline{F}(x)$ *is continuous and satisfies the similar relation as* (8) *uniformly.*

If $F(x)$ *is continuous and*

$$F(x+t) + F(x-t) - 2F(x) = o(|t|/\log\log(1/|t|)) \tag{9}$$

uniformly, then $\overline{F}(x)$ *satisfies the similar relation as* (9) *uniformly.*

The first part of Lemma 4 is due to Zygmund ([10]) Theorem 7), and the second will be proved along the same line as the above Zygmund proof, and so we may omit the proof to avoid the repeated complication.

4. We are now in a position to prove Theorem 1. Without loss of generality we may suppose $(a, b) = (0, 2\pi)$. In fact, if $0 < a < b < 2\pi$, we put

$$f_1(x) = \begin{cases} f(x) & \text{for } a \leqq x \leqq b, \\ f\left(\dfrac{b-a}{a}x + a\right) & \text{for } 0 \leqq x < a, \\ f\left(\dfrac{b-a}{2\pi - b}x + \dfrac{2\pi a - b^2}{2\pi - b}\right) & \text{for } b < x \leqq 2\pi, \end{cases}$$

that is, in the intervals $(0, a)$ and $(b, 2\pi)$ the function $f_1(x)$ is similar to $f(x)$ in (a, b). As we see easily, $f_1(x)$ satisfies uniformly for x in $(0, 2\pi)$ the condition (2) replaced f by f_1. Evidently $f(x) - f_1(x) = 0$ for $a \leqq x \leqq b$, hence by the localization theorem $s_n(f - f_1; x)$ converges to zero uniformly in any closed interval contained in the open interval (a, b); and from the identity

$$s_n(f; x) = s_n(f_1; x) + s_n(f - f_1; x)$$

the theorem will be proved immdiately if the theorem is proved for $f_1(x)$ instead of $f(x)$.

Supposing that the equality (2) holds uniformly in $(0, 2\pi)$, let $P_n(x)$ be the polynomial of order n which is determined by Lemma 1 taking $\omega_1(\delta) = 1/\log\log(1/\delta)$, and we write for the simplicity

$$Q_k(x) = P_{p_k}(x), \qquad q_k(x) = Q'_k(x).$$

Hence $Q_k(x) - F(x) = o(1/(p_k \log\log p_k))$ uniformly, and $q_k(x) - f(x) = o(1)$ almost everywhere as $k \to \infty$. Since $q_k(x)$ is of order at most p_k we write

$$s_{p_k}(f; x) - f(x) = s_{p_k}(f - q_k; x) + \{q_k(x) - f(x)\}.$$

To prove (3) it is sufficient then to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} | s_{p_k}(f - q_k; x) |^{\alpha} = 0 \tag{10}$$

almost everywhere. If (10) is true for some $\alpha > 0$, it is also true for β, $0 < \beta < \alpha$; hence we may suppose that α is an integer not smaller than 2.

Now we shall show that, for any positive number c, if n is large enough, we have

$$\int_0^{2\pi} \exp\left\{ c(\log n)\left(\frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} | s_{p_k}(f - q_k; x) |^{\alpha}\right)^{1/\alpha}\right\} dx < M_{\alpha, c} \tag{11}$$

where $M_{\alpha, c}$ is a constant independent of n.

To prove this, the left hand side of (11) is equal to

$$\int_0^{2\pi} \sum_{r=0}^{\infty} \frac{(c\log n)^r}{r!} \left(\frac{1}{2^n} \sum_{k=2n+1}^{2^{n+1}} | s_{p_k}(f - q_k; x) |^{\alpha}\right)^{r/\alpha}\Bigg\} dx$$

$$=\sum_{r=0}^{\infty}\frac{(c\log n)^r}{r!\,2^{nr/\alpha}}\int_0^{2\pi}\Big(\sum_{k=2^n+1}^{2^{n+1}}|s'_{p_k}(F-Q_k;x)|^{\alpha}\Big)^{r/\alpha}dx$$

$$=\sum_{r=0}^{\alpha-1}+\sum_{r=\alpha}^{\infty}=A+B$$

say. The sum B is, by the Minkowski inequality, not greater than

$$\sum_{r=\alpha}^{\infty}\frac{(c\log n)^r}{r!\,2^{nr/\alpha}}\Big\{\sum_{k=2^n+1}^{2^{n+1}}\Big(\int_0^{2\pi}|s'_{p_k}(F-Q_k;x)|^r\,dx\Big)^{\alpha/r}\Big\}^{r/\alpha}.$$

In virtue of Lemma 3 this is not greater than

$$\sum_{r=\alpha}^{\infty}\frac{(c\log n)^r}{r!\,2^{nr/\alpha}}\Big\{\sum_{k=2^n+1}^{2^{n+1}}p_k^{\alpha}\Big(\int_0^{2\pi}|s_{p_k}(F-Q_k;x)|^r\,dx\Big)^{\alpha/r}\Big\}^{r/\alpha}.$$

By Lemma 1 we can take an integer $N=N(c)$ sufficiently large such that

(12) $\qquad |F(x)-Q_k(x)|<\delta/(p_k\log\log p_k) \qquad$ for all $k>N$,

where we put $\delta=1/(12ce)$. Hence if n is large enough

$$B\leqq\sum_{r=\alpha}^{\infty}\frac{(c\log n)^r}{r!\,2^{nr/\alpha}}\Big\{\sum_{k=2^n+1}^{2^{n+1}}\Big(\frac{\delta}{\log\log p_k}\Big)^{\alpha}\Big(\int_0^{2\pi}\Big|s_{p_k}\Big(\frac{(F-Q_k)p_k\log\log p_k}{\delta};x\Big)\Big|^r dx\Big)^{\alpha/r}\Big\}^{r/\alpha}$$

$$\leqq\sum_{r=\alpha}^{\infty}\frac{(c\log n)^r}{r!\,2^{nr/\alpha}}\Big\{\sum_{k=2^n+1}^{2^{n+1}}\Big(\frac{\delta}{\log\log p_k}\Big)^{\alpha}\Big(3^r r^r\int_0^{2\pi}dx\Big)^{\alpha/r}\Big\}^{r/\alpha}$$

by Lemma 2 and (12). Since $p_k\geqq k$ we get easily

$$B\leqq\sum_{r=\alpha}^{\infty}\frac{(c\log n)^r}{r!\,2^{nr/\alpha}}\Big\{\frac{\delta^{\alpha}3^{\alpha}r^{\alpha}(2\pi)^{\alpha/r}}{(\log\log 2^n)^{\alpha}}2^n\Big\}^{r/\alpha}$$

and using the Stirling formula $n!=n^ne^{-n}\sqrt{2\pi n}\,e^{\theta/(12n)}(0<\theta<1)$ we have

$$B=O\Big(\sum_{r=\alpha}^{\infty}\frac{(6ce\delta)^r}{\sqrt{2\pi r}}\Big)=O\Big(\sum_{r=\alpha}^{\infty}\frac{1}{2^r\sqrt{2\pi r}}\Big)=O(1),$$

if n is large enough.

On the other hand, by the Hölder inequality,

$$A\leqq\sum_{r=0}^{\alpha-1}\frac{(c\log n)^r}{r!\,2^{nr/\alpha}}\int_0^{2\pi}\Big(\sum_{k=2^n+1}^{2^{n+1}}|s'_{p_k}(F-Q_k;x)|^{\alpha}\Big)^{r/\alpha}dx$$

$$\leqq\sum_{r=0}^{\alpha-1}\frac{(c\log n)^r}{r!\,2^{nr/\alpha}}\Big(\int_0^{2\pi}\sum_{k=2^n+1}^{2^{n+1}}|s'_{p_k}(F-Q_k;x|^{\alpha}dx\Big)^{r/\alpha}(2\pi)^{1-r/\alpha}.$$

Applying Lemmas 2 and 3 again we get

$$A \leqq \sum_{r=0}^{\alpha-1} \frac{(c \log n)^r}{r!\, 2^{nr/\alpha}} \Big(\sum_{k=2^n+1}^{2^{n+1}} p_k^\alpha \int_0^{2\pi} | s_{p_k}(F - Q_k ; x) |^\alpha dx \Big)^{r/\alpha} (2\pi)^{1-r/\alpha}$$

$$= \sum_{r=0}^{\alpha-1} \frac{(c \log n)^r (2\pi)^{1-r/\alpha}}{r!\, 2^{nr/\alpha}} \Big(\sum_{k=2^n+1}^{2^{n+1}} \Big(\frac{\delta}{\log \log p_k}\Big)^\alpha \int_0^{2\pi} \Big| s_{p_k}\Big(\frac{(F - Q_k) p_k \log \log p_k}{\delta} ; x\Big) \Big|^\alpha dx \Big)^{r/\alpha}$$

$$\leqq \sum_{r=0}^{\alpha-1} \frac{(c \log n)^r (2\pi)^{1-r/\alpha}}{r!\, 2^{nr/\alpha}} \Big(\sum_{k=2^n+1}^{2^{n+1}} \frac{\delta^\alpha 3^\alpha \alpha^\alpha (2\pi)^\alpha}{(\log \log 2^n)^\alpha} \Big)^{r/\alpha}$$

$$\leqq \sum_{r=0}^{\alpha-1} \frac{(12\pi c \delta \alpha)^r (2\pi)^{1-r/\alpha}}{r!} < C_{\alpha, c}$$

where $C_{\alpha, c}$ is independent of n, if n is large enough. Thus we obtained the inequality (11).

Now we shall be able to complete the proof quite easily. For any positive ε we denote by E_n the set of all points $x \in (0, 2\pi)$ such that

$$\Big(\frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} | s_{p_k}(f - q_k ; x) |^\alpha \Big)^{1/\alpha} > \varepsilon.$$

Then we get from (11), taking $c = 2/\varepsilon$,

$$M_{\alpha, c} > \int_0^{2\pi} \exp \Big\{ \frac{2}{\varepsilon} (\log n) \Big(\frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} | s_{p_k}(f - q_k ; x) |^\alpha \Big)^{1/\alpha} \Big\} dx$$

$$\geqq \int_{E_n} \exp \Big\{ \frac{2}{\varepsilon} \log n \cdot \varepsilon \Big\} dx$$

$$= n^2 | E_n |,$$

if n is large enough. Therefore $\sum_n | E_n | < \infty$, that is

$$\limsup_{n \to \infty} \Big(\frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} | s_{p_k}(f - q_k ; x) |^\alpha \Big)^{1/\alpha} \leqq \varepsilon$$

almost everywhere. Since $\varepsilon > 0$ is arbitrary we conclude that

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} | s_{p_k}(f - q_k ; x) |^\alpha = 0$$

almost everywhere. For arbitrary large N, we take n such that $2^n \leqq N < 2^{n+1}$, then

$$\frac{1}{N} \sum_{k=1}^{N} | s_{p_k}(f - q_k ; x) |^\alpha < \frac{1}{2^n} \sum_{k=1}^{2^{n+1}} = o(1) + \frac{1}{2^n} \sum_{l=0}^{n} \sum_{k=2^l+1}^{2^{l+1}}$$

$$= o(1) + \frac{1}{2^n}\sum_{l=0}^{n} 2^l \cdot o(1) = o(1),$$

which is the same as (10), and the first part of Theorem 1 was proved.

The second part of Theorem 1 is an immediate consequence of the first part and Lemma 4.

Theorem 1 was proved completely.

THEOREM 4. *If $f(x)$ is a bounded function, there exists a constant $c > 0$ such that, for any sequence of positive integers $\{p_k\}$ and positive number α,*

$$\int_0^{2\pi} \exp\left\{c\left(\frac{1}{n}\sum_{k=1}^{n} |s_{p_k}(x)|^{\alpha}\right)^{1/\alpha}\right\} dx < M_{\alpha,c}, \tag{13}$$

$$\int_0^{2\pi} \exp\left\{c\left(\frac{1}{n}\sum_{k=1}^{n} |\bar{s}_{p_k}(x)|^{\alpha}\right)^{1/\alpha}\right\} dx < M_{\alpha,c}, \tag{14}$$

where $M_{\alpha,c}$ is a constant independent of n.

In the case $\alpha = n = 1$, Theorem 4 is known ([6] Theorem 5).

Since the proof will be done by the similar way to the proof of the inequality (11), we shall give only a sketch of the proof. We may suppose that α is an integer greater than 2. The left hand side of (13) is equal to

$$\sum_{r=0}^{\infty} \frac{c^r}{r!\, n^{r/\alpha}} \int_0^{2\pi} \left(\sum_{k=1}^{n} |s_{p_k}(x)|^{\alpha}\right)^{r/\alpha} dx$$

$$= \sum_{r=0}^{\alpha-1} + \sum_{r=\alpha}^{\infty} = A + B$$

say. Suppose that $|f(x)| < K$ for all x. By the Hölder inequality and by Lemma 2 we have

$$A \leqq \sum_{r=0}^{\alpha-1} \frac{c^r}{r!\, n^{r/\alpha}} \left(\int_0^{2\pi} \sum_{k=1}^{n} |s_{p_k}(x)|^{\alpha} dx\right)^{r/\alpha} (2\pi)^{1-r/\alpha}$$

$$\leqq \sum_{r=0}^{\alpha-1} \frac{c^r}{r!\, n^{r/\alpha}} \left(\sum_{k=1}^{n} 3^{\alpha}\alpha^{\alpha} \int_0^{2\pi} K^{\alpha} dx\right)^{r/\alpha} (2\pi)^{1-r/\alpha}$$

$$= 2\pi \sum_{r=0}^{\alpha-1} \frac{(3Kc\alpha)^r}{r!} < M'$$

where M' does not depend on n. By the Minkowski inequality and by Lemma 2 we get

$$B \leqq \sum_{r=\alpha}^{\infty} \frac{c^r}{r!\, n^{r/\alpha}} \left\{\sum_{k=1}^{n} \left(\int_0^{2\pi} |s_{p_k}(x)|^{r} dx\right)^{\alpha/r}\right\}^{r/\alpha}$$

$$\leqq \sum_{r=\alpha}^{\infty} \frac{c^r}{r!\, n^{r/\alpha}} \left\{ \sum_{k=0}^{n} \left(3^r r^r \int_0^{2\pi} K^r dx \right)^{\alpha/r} \right\}^{r/\alpha}$$

$$\leqq 2\pi \sum_{r=\alpha}^{\infty} \frac{(3Kec)^r}{\sqrt{2\pi r}},$$

in virtue of the Stirling formula. The last series is convergent if $c < (3Ke)^{-1}$, with a bound independent of n. We get therefore the inequality (13).

To prove (14) we need only to repeat the same argument, but we must use the following inequality in place of Lemma 2.

$$\int_0^{2\pi} | \bar{s}_n(f;x) |^r dx \leqq 10^r r^r \int_0^{2\pi} | f(x) |^r dx \qquad \text{for } r \geqq 2. \tag{15}$$

This being known essentially, we give a sketch of the proof. As we know,

$$\bar{s}_n(x) = -\frac{1}{\pi} \int_0^{2\pi} \frac{f(x+t)}{2 \tan \frac{t}{2}} (1 - \cos nt) dt + \frac{1}{2\pi} \int_0^{2\pi} f(t) \sin n(x-t) dt$$

$$= \bar{f}(x) + \frac{\cos nx}{\pi} \int_0^{2\pi} \frac{f(x+t) \cos n(x+t)}{2 \tan \frac{t}{2}} dt$$

$$+ \frac{\sin nx}{\pi} \int_0^{2\pi} \frac{f(x+t) \sin n(x+t)}{2 \tan \frac{t}{2}} dt + \frac{1}{2\pi} \int_0^{2\pi} f(t) \sin n(x-t) dt.$$

Hence

$$| \bar{s}_n(x) | \leqq | \bar{f}(x) | + | g_1(x) | + | g_2(x) | + \frac{1}{2\pi} \int_0^{2\pi} | f(t) | dt$$

where $g_1(x)$ is conjugate to $f(x) \cos nx$ and $g_2(x)$ to $f(x) \sin nx$. Therefore by the Minkowski inequality, if $r \geqq 1$

$$\left(\int_0^{2\pi} | \bar{s}_n(x) |^r dx \right)^{1/r} \leqq 3 \left(\int_0^{2\pi} | \bar{f}(x) |^r dx \right)^{1/r} + \int_0^{2\pi} | f(x) | dx. \tag{16}$$

By the well known Riesz inequality

$$\left(\int_0^{2\pi} | \bar{f}(x) |^r dx \right)^{1/r} \leqq 2r \left(\int_0^{2\pi} | f(x) |^r dx \right)^{1/r} \qquad \text{for } r \geqq 2, \tag{17}$$

we get, combining (16) and (17),

$$\left(\int_0^{2\pi} | \bar{s}_n(x) |^r dx \right)^{1/r} \leqq 6r \left(\int_0^{2\pi} | f(x) |^r dx \right)^{1/r} + (2\pi)^{1-1/r} \left(\int_0^{2\pi} | f(x) |^r dx \right)^{1/r}$$

$$\leqq 10r \left(\int_0^{2\pi} | f(x) |^r dx \right)^{1/r} \qquad \text{for } r \geqq 2.$$

THEOREM 5. *Let $f(x)$ be an integrable function on $(0, 2\pi)$. If*

$$\int_0^h \{f(x+t) - f(x-t)\}\, dt = o(|h|)$$

as $h \to 0$, uniformly for x in (a, b), and if $\{p_k\}$ is any sequence of intergers, then we have

$$\frac{1}{n}\sum_{k=1}^{n} | s_{p_k}(x) - f(x) |^{\alpha} = o((\log\log n)^{\alpha}),$$

$$\frac{1}{n}\sum_{k=1}^{n} | \bar{s}_{p_k}(x) - \bar{f}(x) |^{\alpha} = o((\log\log n)^{\alpha})$$

almost everywhere.

Since the proof is also done, mutatis mutandis, along the line of the proof of Theorem 1, we shall only mention some points. Let $(a, b) = (0, 2\pi)$. The periodic part $F(x)$ of the primitive of $f(x)$ satisfies that

$$F(x+t) + F(x-t) - 2F(x) = o(|t|)$$

uniformly, and so $\bar{F}(x)$ satisfies the similar condition by Lemma 4. There exists a trigonometric polynomial $P_n(x)$ of order n such that

$$F(x) - P_n(x) = o(1/n)$$

uniformly for x (cf. [10] Theorem 8), and $\lim_{n\to\infty} \{f(x) - P_n'(x)\} = 0$ almost everywhere.

We remark that obviously we can remove $-\bar{f}(x)$ and $-f(x)$ in the summands of the conclusion.

5. We shall now prove Theorem 2. We may suppose again that $(a, b) = (0, 2\pi)$. Let $F(x)$ be the periodic part of the primitive of $f(x)$, then

$$F(x+t) + F(x-t) - 2F(x) = o(|t| / \log(1/|t|))$$

as $t \to 0$ uniformly for x. By Lemma 1 we can find a trigonometric polynomial $P_n(x)$ of order n such that

$$F(x) - P_n(x) = o(1/(n\log n))$$

uniformly for x, and

$$\lim_{n\to\infty} \{f(x) - P_n'(x)\} = 0$$

almost everywhere. Hence, by the relation

$$\begin{aligned} s_n(f; x) - f(x) &= s_n(f - P_n'; x) + \{P_n'(x) - f(x)\} \\ &= s_n'(F - P_n; x) + \{P_n'(x) - f(x)\}, \end{aligned}$$

it is sufficient to prove that

$$\lim_{n\to\infty} s_n'(F - P_n; x) = 0 \tag{18}$$

almost everywhere. We shall evaluate the integral

$$I=\int_0^{2\pi}\exp\Big(c\log n\,|\,s_n'(F-P_n;x)\,|\Big)dx,$$

where c is any positive constant. By the above approximation, we have

(19) $$|\,F(x)-P_n(x)\,|<\delta/(n\log n)$$

for large n, where $\delta=1/(6ce)$. As $e^a<e^a+e^{-a}=2\cosh a$, we get

$$I\leqq 2\int_0^{2\pi}\cosh\Big(c\,(\log n)\,|\,s_n'(F-P_n;x)\,|\Big)dx$$

$$=2+2\sum_{r=1}^{\infty}\frac{(c\log n)^{2r}}{(2r)!}\int_0^{2\pi}|\,s_n'(F-P_n;x)\,|^{2r}dx$$

$$\leqq 2+2\sum_{r=1}^{\infty}\frac{(c\log n)^{2r}}{(2r)!}\,n^{2r}\int_0^{2\pi}|\,s_n(F-P_n;x)\,|^{2r}dx$$

(by Lemma 3)

$$\leqq 2+2\sum_{r=1}^{\infty}\frac{(c\delta)^{2r}}{(2r)!}\int_0^{2\pi}\Big|\,s_n\Big(\frac{(F-P_n)\,n\log n}{\delta};x\Big)\Big|^{2r}dx$$

$$\leqq 2+2\sum_{r=1}^{\infty}\frac{(c\delta)^{2r}}{(2r)!}\,(6r)^{2r}(2\pi)$$

(by Lemma 2 and (19))

$$\leqq 2+2\sqrt{\pi}\sum_{r=1}^{\infty}\frac{1}{2^{2r}\sqrt{r}} \qquad \text{(by the Stirling formula)}$$

which is an absolute constant, say M. Therefore $I<M$ if n is large enough. Let E_n be the set of all $x\in(0,2\pi)$ for which

$$|\,s_n'(F-Q_n;x)\,|>\varepsilon.$$

Taking $c=2/\varepsilon$ we have

$$M>I>\int_{E_n}\exp(\varepsilon c\log n)dx=n^2\,|\,E_n\,|,$$

hence $|\,E_n\,|<M/n^2$ for large n, or $\sum_n|\,E_n\,|<\infty$. We obtain then

$$\limsup_{n\to\infty}|\,s_n'(F-P_n;x)\,|\leqq\varepsilon$$

almost everywhere. Since $\varepsilon>0$ is arbitrary we get (18) almost everywhere. q.e.d.

6. For the proof of Theorem 3, it is enough to use the same pattern as before. And then we may leave it to the reader with the proof of the following

THEOREM 6. *Let $f(x)$ be an integrable function on $(0,2\pi)$ and let $\{p_k\}$*

be a sequence of positive integers. If

$$\int_0^h \{f(x+t) - f(x-t)\}\, dt = o(|h|) \tag{20}$$

as $h \to 0$ *uniformly for* $x \in (a, b)$, *then*

$$s_{p_k}(x) = o(\log k) \quad \text{and} \quad \bar{s}_{p_k}(x) = o(\log k) \tag{21}$$

almost everywhere in (a, b).

We can replace o's in (20) and (21) by O's simultaneously.

7. Our argument just employed will be available to some problems of interpolation.

Suppose that $f(x)$ is a continuous function. Let $U_n(f; x)$ be a trigonometric polynomial of order n defined by

$$U_n(f; x_i) = f(x_i)$$

where $x_i = i\dfrac{2\pi}{2n+1}$ $(i = 0, 1, 2, \dots, 2n)$. Then, as we know, putting

$$\varphi_n(t) = i\frac{2\pi}{2n+1} \quad \text{for} \quad i\frac{2\pi}{2n+1} \leqq t < (i+1)\frac{2\pi}{2n+1} \quad (i = 0, 1, \dots, 2n),$$

we have

$$U_n(f; x) = \frac{1}{\pi}\int_0^{2\pi} f(t) D_n(x-t)\, d\varphi_n(t)$$

where $D_n(t)$ is the Dirichlet kernel.

By the Marcinkiewicz theorem ([2] proof of Theorem 11)

$$\int_0^{2\pi} |U_n(f; x)|^r dx \leqq (6\pi A r)^r \max|f(x)|^r \qquad (r \geqq 2). \tag{22}$$

From (22) and a suitable approximation of the function as in Lemma 1, we can prove the following theorems by the argument in the preceding sections. The details may be left to the reader.

THEOREM 7. *If* $f(x)$ *is a continuous functions, and if* $\{p_k\}$ *is a sequence of positive integers, then for any* $\alpha > 0$,

$$\frac{1}{n}\sum_{k=1}^{n} |U_{p_k}(f; x)|^\alpha = o\big((\log\log n)^\alpha\big)$$

almost everywhere.

THEOREM 8. *If* $f(x)$ *is a continuous function of modulus of continuity* $o(1/\log\log(1/|h|))$ *as* $h \to 0$, *and if* $\{p_k\}$ *is strictly increasing sequence of positive integers, then for any* $\alpha > 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} | U_{p_k}(f; x) - f(x) |^{\alpha} = 0$$

for almost all x.

These theorems are generalizations of the theorems of Erdös [1] in which $\alpha = 1$ or 2 and $p_k = k$ $(k = 1, 2, \dots .)$ (cf. also [4]).

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