# ON THE MAXIMUM MODULUS AND THE COEFFICIENTS OF AN ENTIRE DIRICHLET SERIES

Qazi Ibadur Rahman

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I. Consider the Dirichlet series

$$f(s) = \sum_{1}^{\infty} a_n e^{s\lambda_n},$$

$$\lambda_{n+1} > \lambda_n,\ \ \lambda_1 \geqq 0,\ \ \lim_{n\to\infty} \lambda_n = \infty,\ s = \sigma + it,$$

where

(1.1) $$\overline{\lim_{n\to\infty}} \frac{\log n}{\lambda_n} = D < \infty.$$

It defines in its half-plane of convergence a holomorphic function. Let $\sigma_c$ and $\sigma_a$ be the abscissa of convergence and the abscissa of absolute convergence respectively of $f(s)$.

Let $\mu(\sigma)$ be the maximum of $|a_n|e^{\sigma\lambda_n}$ $(n = 1, 2, \ldots.)$, and $M(\sigma)$ the l. u. b. of $|f(\sigma + it)|$, $-\infty < t < \infty$, where $\sigma$ is a constant smaller than $\sigma_a$. If $\sigma_c = \infty$, $\sigma_a = \infty$, $f(s)$ defines an *entire function*. Let $\lambda_{\nu(\sigma)}$ be the $\lambda_n$ corresponding to the maximum term of the series for $re(s) = \sigma$. Then $\lambda_{\nu(\sigma)}$ is evidently a non-decreasing function of $\sigma$.

Since for $\sigma < \sigma_a$,

$$a_n e^{\sigma\lambda_n} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} e^{-it\lambda_n} f(\sigma + it)\, dt,$$

one has

$$|a_n| e^{\sigma\lambda_n} \leqq M(\sigma) \qquad (n = 1, 2, \ldots.),$$

and consequently

[A] $$\mu(\sigma) \leqq M(\sigma), \qquad \sigma < \sigma_a.$$

On the other hand, for every positive $\varepsilon$ one can choose a positive integer $N(\varepsilon)$ such that $\log n < \lambda_n(D + \varepsilon/2)$ for $n \geqq N(\varepsilon)$. Therefore

$$\begin{aligned} M(\sigma) &\leqq \sum |a_n| e^{\sigma\lambda_n} \qquad (n = 1, 2, \ldots.), \\ &= \sum_{1}^{N(\epsilon)-1} |a_n| e^{\sigma\lambda_n} + \sum_{N(\epsilon)}^{\infty} |a_n| e^{(\sigma+D+\epsilon)\lambda_n} e^{-(D+\epsilon)\lambda_n}, \\ &= N(\varepsilon)\mu(\sigma) + \mu(\sigma + D + \varepsilon) \sum_{N(\epsilon)}^{\infty} \frac{1}{n^{(D+\epsilon)/(D+\epsilon/2)}} \\ &< K'\mu(\sigma + D + \varepsilon), \end{aligned}$$

$K'$ being a constant depending on $f(s)$ and $\varepsilon$.

If $\sigma_a = \infty$, $\log \mu(\sigma)$ being convex and indefinitely increasing, one has

$$\log \mu(\sigma + \eta) = \log \mu(\sigma) + \eta p(\sigma), \qquad \eta > 0, \quad p(\sigma) \to \infty,$$

and it follows that

[B] $$M(\sigma) < \mu(\sigma + D + \varepsilon), \qquad \varepsilon > 0, \quad \sigma > \sigma(\varepsilon).$$

In this paper we prove several relations between these auxiliary functions $M(\sigma)$ etc., which are true whether $f(s)$ be of finite or infinite order. We shall use the following notations.

$$\begin{matrix} T_k \\ t_k \end{matrix} = \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k M(\sigma)}{l_{k-1}\sigma}; \qquad \begin{matrix} P_k \\ \rho_k \end{matrix} = \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k M(\sigma)}{l_{k-2}\sigma};$$

$$\begin{matrix} \Delta_k \\ \delta_k \end{matrix} = \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_{k-1}\lambda_{\nu(\sigma)}}{l_{k-1}\sigma}; \qquad \begin{matrix} N_k \\ \nu_k \end{matrix} = \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_{k-1}\lambda_{\nu(\sigma)}}{l_{k-2}\sigma};$$

$$\begin{matrix} S_k \\ s_k \end{matrix} = \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k \mu(\sigma)}{l_{k-1}\sigma}; \qquad \begin{matrix} F_k \\ \phi_k \end{matrix} = \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{l_k \mu(\sigma)}{l_{k-2}\sigma},$$

where $k$ is any fixed integer $\geqq 2$ and $l_k x$ denotes the $k$-th interate of $\log x$ [1, p. 16].

2.

THEOREM. *We have*

| | | |
|---|---|---|
| (2.1) | $t_k = s_k$ | $(k = 2, 3, \ldots.)$ |
| (2.2) | $T_k = S_k$ | ( 〃 ) |
| (2.3) | $t_2 = 1 + \delta_2$ | |
| (2.4) | $T_2 = 1 + \Delta_2$ | |
| (2.5) | $t_k = \max(1, \delta_k)$ | $(k = 3, 4, \ldots.)$ |
| (2.6) | $T_k = \max(1, \Delta_k)$ | ( 〃 ) |
| (2.7) | $\rho_k = \phi_k = \nu_k$ | $(k = 2, 3, \ldots.)$ |
| (2.8) | $P_k = F_k = N_k$ | ( 〃 ) |

3.

LEMMA. *Let*

(i) $\psi(x)$ *be a positive increasing function;*

(ii) $\liminf_{x\to\infty} \frac{\log \psi(x)}{x} = \alpha \qquad (0 \leqq \alpha < \infty).$

*Then corresponding to each pair of positive numbers* $\beta$, $\gamma$ *satisfying the inequalities*

(iii) $\alpha < \beta, \qquad \frac{\alpha}{\beta} < \gamma < 1,$

*there is a sequence* $x_1, x_2, \ldots.$ *tending to infinity such that*

(iv) $\psi(x) < e^{\beta x} \qquad (\gamma x_n \leqq x \leqq x_n).$

For let $x_1, x_2, \ldots.$ be a sequence such that

$$\frac{\log \psi(x_n)}{x_n} < \beta\gamma.$$

Then if $\gamma x_n \leqq x \leqq x_n$,

$$\log \psi(x) \leqq \log \psi(x_n) < \beta\gamma x_n \leqq \beta x, \text{ or } \qquad \psi(x) < e^{\beta x}.$$

**4.**

(i) Proof of $t_k = s_k$.

Since $M(\sigma) \geqq \mu(\sigma)$ we have

$$t_k \geqq s_k.$$

Moreover, $\log \mu(\sigma)$ is a convex function of $\sigma$ and therefore for any (arbitrarily large) positive constant $H$,

$$\log \mu(\sigma) > H\sigma \qquad \sigma > \sigma(H).$$

We get

$$s_k \geqq 1 \qquad (k \geqq 2),$$

and so

$$t_k \geqq s_k \geqq 1 \qquad (k \geqq 2). \tag{4.1}$$

To prove $t_k \leqq s_k$, we may suppose $s_k < \infty$. We have

$$l_{k-1}M(\sigma) < l_{k-1}\mu(\sigma + D + \mathcal{E}) < e^{(s_k+\epsilon')l_{k-1}(\sigma+D+\epsilon)}$$

for a sequence of values of $\sigma \to \infty$. Hence

$$t_k \leqq s_k$$

which holds when $s_k = \infty$.

(ii) Proof of $T_k = S_k$ is similar and is omitted.

(iii) Proof of $t_2 = 1 + \delta_2$.

We may suppose $\delta_2 < \infty$. We have [$B$ ; 2, p. 67]

$$\begin{aligned}\log M(\sigma) &< \log \mu(\sigma + D + \mathcal{E})\\ &= \log \mu(D + \mathcal{E}) + \int_{D+\epsilon}^{\sigma+D+\epsilon} \lambda_{\nu(x)}\, dx\\ &< \log \mu(D + \mathcal{E}) + \sigma\lambda_{\nu(\sigma+D+\epsilon)}\\ &< 2\,\sigma\lambda_{\nu(\sigma+D+\epsilon)}\end{aligned}$$

so that

$$\log\log M(\sigma) < \log 2 + \log \sigma + \log \lambda_{\nu(\sigma+D+\epsilon)}$$

and we get

$$t_2 \leqq 1 + \delta_2$$

which holds when $\delta_2 = \infty$.

To prove $t_2 \geqq 1 + \delta_2$, we may suppose $0 < \delta_2 < \infty$. We have

$$\log M(\sigma) \geqq \log \mu(\sigma) > \int_{\sigma/2}^{\sigma} \lambda_{\nu(x)}\, dx \;>\; \frac{\sigma}{2}\lambda_{\nu(\sigma/2)}$$

Hence

$$t_2 \geqq 1 + \delta_2.$$

which obviously holds when $\delta_2 = 0$. If $\delta_2 = \infty$, the above argument shows that $t_2 = \infty$.

(iv) Proof of $T_2 = 1 + \Delta_2$ is similar and is omitted.

(v) Proof of $s_k = \max(1, \delta_k)$, $(k \geqq 3)$.

Since

$$\int_{\sigma}^{\sigma+2} \lambda_{\nu(x)}\, dx < \log \mu(\sigma + 2) \text{ or } 2\lambda_{\nu(\sigma)} < \log \mu(\sigma + 2)$$

it follows that $\delta_k \leqq s_k$ and hence from (4.1), $\max(1, \delta_k) \leqq s_k$. If $\delta_k < 1$, let $\delta_k < \theta < 1$: then

$$l_{k-1}\lambda_{\nu(\sigma)} < \theta\, l_{k-1}\, \sigma, \tag{4.2}$$

for a sequence of values of $\sigma \to \infty$. Further

$$\begin{aligned} \log \mu(\sigma) &= \log \mu(a) + \int_a^{\sigma} \lambda_{\nu(x)}\, dx \\ &< \log \mu(a) + \sigma\lambda_{\nu(\sigma)} \\ &< 2\,\sigma\lambda_{\nu(\sigma)} \end{aligned}$$

$$\begin{aligned} \text{or} \quad \log\log \mu(\sigma) &< \log 2 + \log \sigma + \log \lambda_{\nu(\sigma)} \\ &< \log 2 + \log \sigma + \theta \log \sigma, \qquad \text{by (4.2)} \\ &< 2 \log \sigma + O(1) \end{aligned} \tag{4.3}$$

for a sequence of values of $\sigma \to \infty$. Hence $s_k \leqq 1$ and so $s_k = 1$. If $1 \leqq \delta_k < \infty$, let $\theta > \delta_k$. From (4.3)

$$\log\log \mu(\sigma) < 3\, e_{k-2}(\theta e_{k-1} \sigma)$$

for a sequence of values of $\sigma \to \infty$. Hence $s_k \leqq \delta_k$ which holds if $\delta_k = \infty$. Hence the result follows.

(vi) Proof of $S_k = \max(1, \Delta_k)$ is similar and is omitted.

(vii) Proof of $\rho_2 = \nu_2$.

We have

$$\log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(x)}\, dx. \tag{4.4}$$

[A] and (4.4) give

$$2\lambda_{\nu(\sigma)} \leqq \int_{\sigma}^{\sigma+2} \lambda_{\nu(x)}\, dx < \log \mu(\sigma + 2) \leqq \log M(\sigma + 2),$$

whence

$$\frac{\log 2}{\sigma} + \frac{\log \lambda_{\nu(\sigma)}}{\sigma} < \frac{\log\log M(\sigma + 2)}{\sigma}$$

$$\therefore \quad \nu_2 \leqq \rho_2 \qquad (0 \leqq \rho_2 \leqq \infty). \tag{4.5}$$

Now suppose that $\nu_2 < \infty$. $\lambda_{\nu(\sigma)}$ is a non-decreasing function and so by the lemma, if $\nu_2 < \beta$, $\nu_2/\beta < \gamma < 1$, there is a sequence $\sigma_1, \sigma_2, \ldots$ for which

(4.6) $$\frac{\log \lambda_{\nu(\sigma)}}{\sigma} < \beta \qquad (\gamma\sigma_n \leqq \sigma \leqq \sigma_n).$$

Take positive numbers $\delta$, $\varepsilon'$ such that $\gamma < \delta < 1$, $\gamma/\delta < \varepsilon' < 1$, and write $\xi_n = \delta\sigma_n$, so that

$$\gamma\sigma_n = \frac{\gamma}{\delta}\xi_n < \varepsilon'\xi_n < \xi_n.$$

By (4.4)

$$\log \mu(\xi_n) = \log \mu(\varepsilon'\xi_n) + \int_{\epsilon'\xi_n}^{\xi_n} \lambda_{\nu(x)}\, dx.$$

Since $f(s)$ defines an entire function

$$\lim_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty$$

and therefore for a sufficiently large $\xi_n$

$$\log \mu(\varepsilon'\xi_n) = \log|a_{\nu(\epsilon'\xi_n)}| + \varepsilon'\xi_n\lambda_{\nu(\epsilon'\xi_n)} < \varepsilon'\xi_n \lambda_{\nu(\epsilon'\xi_n)}$$

so that

$$\begin{aligned}
\log \mu(\xi_n) &\geqq \log \mu(\varepsilon'\xi_n) + \lambda_{\nu(\epsilon'\xi_n)} \int_{\epsilon'\xi_n}^{\xi_n} dx \\
&= \log \mu(\varepsilon'\xi_n) + (1-\varepsilon')\xi_n\lambda_{\nu(\epsilon'\xi_n)} \\
&> \log \mu(\varepsilon'\xi_n) + \frac{1-\varepsilon'}{\varepsilon'} \log \mu(\varepsilon'\xi_n) \\
&= \frac{1}{\varepsilon'} \log \mu(\varepsilon'\xi_n)
\end{aligned}$$

$$-\varepsilon' \log \mu(\xi_n) < -\log \mu(\varepsilon'\xi_n)$$

$$\begin{aligned}
\log\mu(\xi_n) - \varepsilon' \log \mu(\xi_n) &< \log \mu(\xi_n) - \log \mu(\varepsilon'\xi_n) \\
&= \int_{\epsilon'\xi_n}^{\xi_n} \lambda_{\nu(x)}\, dx.
\end{aligned}$$

Thus using (4.6),

(4.7) $$\begin{aligned}
(1-\varepsilon') \log \mu(\xi_n) &< \int_{\epsilon'\xi_n}^{\xi_n} e^{\beta\sigma}\, d\sigma \\
&= \frac{e^{\beta\xi_n} - e^{\beta\epsilon'\xi_n}}{\beta} = \frac{e^{\beta\xi_n}\{1 - e^{-(1-\epsilon')\beta\xi_n}\}}{\beta}
\end{aligned}$$

From [B]

$$\log M(\xi_n - D - \varepsilon) < \log \mu(\xi_n) < e^{\beta\xi_n} \frac{1 - e^{-(1-\epsilon')\beta\xi_n}}{(1-\varepsilon')\beta}$$

whence

(4.8) $$\rho_2 \leqq \nu_2$$

(4.5) and (4.8) are equivalent to $\rho_2 = \nu_2$.

Proof of $\rho_k = \nu_k$, for $k \geqq 3$, is similar to that for $k = 2$ and is omitted. Proof of $\rho_k = \phi_k$ is also omitted.

(viii) Proof of (2.8) is simple and is omitted.

## References

[1] G. H. Hardy, Orders of infinity, Cambridge, 1924.
[2] Y. C. Yung, Sur les droites de Borel de certaines fonctions entières, Ann. Ecole Normale, 68 (1951), 65-104.

Muslim University, Aligarh (India).