Tohoku Math. J.
70 (2018), 339–352

ON A CLASS OF SINGULAR SUPERLINEAR ELLIPTIC SYSTEMS IN A BALL

DANG DINH HAI

(Received September 3, 2015, revised April 5, 2016)

Abstract. We establish the existence of large positive radial solutions for the elliptic system

$$\begin{cases} -\Delta u = \lambda f(v) \text{ in } B\,, \\ -\Delta v = \lambda g(u) \text{ in } B\,, \\ u = v = 0 \text{ on } \partial B\,, \end{cases}$$

when the parameter $\lambda > 0$ is small, where B is the open unit ball $\mathbb{R}^N$, $N > 2$, $f, g : (0, \infty) \to \mathbb{R}$ are possibly singular at 0 and $f(u) \sim u^p, g(v) \sim v^q$ at ∞ for some $p, q > 0$ with $pq > 1$. Our approach is based on fixed point theory in a cone.

1. Introduction. In this paper, we investigate the existence of positive solutions for the superlinear elliptic system

$$\begin{cases} -\Delta u = \lambda f(v) \text{ in } B\,, \\ -\Delta v = \lambda g(u) \text{ in } B\,, \\ u = v = 0 \text{ on } \partial B\,, \end{cases} \tag{1.1}$$

where B is the open unit ball $\mathbb{R}^N$, $N > 2$, $f, g : (0, \infty) \to \mathbb{R}$, and λ is a positive parameter.

Systems described by (1.1) arise in the study of steady states reaction-diffusion and hydrodynamical problems (see e.g. [1] and the references therein). Let us briefly look at the literature on the superlinear system (1.1) when f, g are nonsingular. In [20, Theorem 3], Peletier and Vorst established the existence and nonexistence of positive solutions to (1.1) for $\lambda > 0$, $N \geq 4$ and superlinear f, g satisfying $f(0) = g(0) = 0$ and $f(t), g(t) > 0$ for $t > 0$. In particular, when $f(t) = t^p$ and $g(t) = t^q$, where $p, q \geq 1$, [20, Theorem 4] gave the existence of a unique radial positive radial solution to (1.1) for

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}\,, \tag{1.2}$$

and the nonexistence of positive solutions to (1.1) for

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}\,. \tag{1.3}$$

Similar existence results under the assumption (1.2) on a bounded convex domain in $\mathbb{R}^N$, $N \geq 3$, were obtained by Clement, de Figueiredo, and Mitidieri [3, Theorem 3.1], which improves a previous result by Cosner [5, Theorem 2]. In [8, Theorem 1.2 (i)] Dalmasso showed the

2010 *Mathematics Subject Classification*. Primary 35J57; Secondary 35J75.
Key words and phrases. Singular elliptic systems, positive radial solutions.

existence of a positive solution to (1.1) under condition (1.2) with $p > 1, q \in (0,1)$, and $pq > 1$, thus complementing the results in [5, 18, 20]. The nonexistence of positive to (1.1) in a bounded domain was obtained in [18, Proposition 3.1] when f, g are pure powers satisfying (1.3). The case when $f(0)$ and $g(0)$ are negative was discussed in [13, Theorem 2.1], where the existence of a large positive radial solution to (1.1) was obtained for $\lambda > 0$ small when f, g satisfy conditions similar to the ones in [20] at ∞. In this paper, we are interested in studying positive radial solutions to (1.1) in the case when f, g are allowed to have a combined superlinear at ∞, singular at 0, and change sign, which has not been considered in the literature to our knowledge. In particular, our result when applied to the model case

$$\begin{cases} -\Delta u = \lambda\left(av^{-\alpha} + v^p\right) \text{ in } B\,, \\ -\Delta v = \lambda\left(bu^{-\beta} + u^q\right) \text{ in } B\,, \\ \quad u = v = 0 \text{ on } \partial B, \end{cases} \tag{1.4}$$

where $\alpha, \beta \in (0,1), a, b \in \mathbb{R}$, $p, q > 0$ with $pq > 1$ and satisfying (1.2), gives the existence of a positive radial solution to (1.4) when $N \geq 2 + \frac{4}{\min(p,q)}$ and $\lambda > 0$ is sufficiently small.

We refer to [2, 4, 6, 7, 9, 10, 12, 14-18] for related results in the single equation case. Our approach is based on fixed point theory in a cone.

We shall make the following assumptions:

(A1) $f, g : (0,\infty) \to \mathbb{R}$ are continuous and there exist positive constants $l_0, l_1, p, q > 0$ with $pq > 1$ such that

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}\,,$$

$$N \geq 2 + \frac{4}{\min(p,q)}\,, \tag{1.5}$$

and

$$\lim_{t\to\infty} \frac{f(t)}{t^p} = l_0\,,\ \lim_{t\to\infty} \frac{g(t)}{t^q} = l_1\,. \tag{1.6}$$

(A2) There exists a constant $\gamma \in (0,1)$ such that

$$\limsup_{t\to 0^+} t^{\gamma}(|f(t)| + |g(t)|) < \infty\,.$$

Our main result is

THEOREM 1.1. *Let* (A1)–(A2) *hold. Then there exists a positive constant* $\lambda_0 < 1$ *such that for* $\lambda < \lambda_0$, *problem* (1.1) *has a positive radial solution* (u_λ, v_λ) *with*

$$(1-r)^{-1}\min(u_\lambda(r), v_\lambda(r)) \to \infty$$

uniformly in $r \in [0,1)$ *as* $\lambda \to 0$.

REMARK 1.1. Note that (1.5) is satisfied if $p, q \geq 1$ and $N \geq 6$.

By (A1), there exist constants $t_0, t_1 > 0$ such that $f(t) \geq f(t_0)$ for $t \geq t_0$ and $g(t) \geq g(t_1)$ for $t \geq t_1$. Define

$$h_0(t) = \begin{cases} f(t) & \text{if } 0 < t \leq t_0 \\ f(t_0) & \text{if } t > t_0 \end{cases}, \quad f_0(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_0 \\ f(t) - f(t_0) & \text{if } t > t_0 \end{cases},$$

$$k_0(t) = \begin{cases} g(t) & \text{if } 0 < t \leq t_1 \\ g(t_1) & \text{if } t > t_1 \end{cases}, \quad g_0(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_1 \\ g(t) - g(t_1) & \text{if } t > t_1 \end{cases}.$$

Then $f = f_0 + h_0,\ g = g_0 + k_0$ on $(0, \infty)$. Note that f_0, g_0 are nonnegative, continuous on $[0, \infty)$, and $\lim_{t\to\infty} \frac{f_0(t)}{t^p} = l_0$, $\lim_{t\to\infty} \frac{g_0(t)}{t^q} = l_1$. By (A2), there exists a constant $k > 0$ such that

$$|h_0(t)| + |k_0(t)| \leq kt^{-\gamma} \tag{1.7}$$

for all $t > 0$. Hence, radial solutions to (1.1) are solutions of the ODE system

$$\begin{cases} -(r^{N-1}u')' = \lambda r^{N-1}(h_0(v) + f_0(v)),\ 0 < r < 1\,, \\ -(r^{N-1}v')' = \lambda r^{N-1}(k_0(u) + g_0(u)),\ 0 < r < 1\,, \\ \quad u'(0) = v'(0) = u(1) = v(1) = 0\,. \end{cases} \tag{1.8}$$

2. Preliminary results. Let $E = C[0, 1] \times C[0, 1]$ be equipped with norm $||(u, v)|| = \max(||u||_\infty, ||v||_\infty)$ and let $\mathbf{K}$ be the nonnegative cone in E.

We first recall the following fixed point theorem for cone expansion, which is a special case of [11, Theorem 2.5].

THEOREM A. *Let $T : E \to E$ be a completely continuous operator such that $T(\mathbf{K}) \subset \mathbf{K}$ and satisfying*

(a) *There exists $r > 0$ such that all solutions $(u, v) \in \mathbf{K}$ of*

$$(u, v) = \theta T(u, v)\,,\ \theta \in (0, 1)$$

satisfy $||(u, v)|| \neq r$.

(b) *There exists $R > r$ such that all solutions $(u, v) \in \mathbf{K}$ of*

$$(u, v) = T(u, v) + (t, t)\,,\ t \geq 0$$

satisfy $||(u, v)|| \neq R$.

Then T has a fixed point $(u, v) \in \mathbf{K}$ with $r \leq ||(u, v)|| \leq R$.

Let $\psi(r) = 1 - r$, $\lambda \in (0, 1)$, and $M > 0$. For $(\tilde{u}, \tilde{v}) \in E$, define $T_{\lambda,M}(\tilde{u}, \tilde{v}) = (u, v)$, where u, v satisfy

$$\begin{cases} -(r^{N-1}u')' = \lambda r^{N-1}(h_0(\tilde{v}_M) + f_0(\tilde{v}_M)),\ 0 < r < 1\,, \\ -(r^{N-1}v')' = \lambda r^{N-1}(k_0(\tilde{u}_M) + g_0(\tilde{u}_M)),\ 0 < r < 1\,, \\ \quad u'(0) = v'(0) = u(1) = v(1) = 0\,, \end{cases} \tag{2.0}$$

where $\tilde{z}_M \equiv \max(\tilde{z}, M\psi)$. By (1.7),

$$|h_0(\tilde{v}_M)|,\ \ |k_0(\tilde{u}_M)| \leq k(M\psi)^{-\gamma}\,. \tag{2.1}$$

Since $\psi^{-\gamma} \in L^q(0, 1)$ for $1 < q < 1/\gamma$, it follows from [15, Lemma 3.1] that (2.0) has a unique solution $(u, v) \in C^{1,\nu}[0, 1] \times C^{1,\nu}[0, 1]$ for some $\nu \in (0, 1)$, and $T_{\lambda,M} : E \to E$ is completely continuous. We shall show next that $T_{\lambda,M} : \mathbf{K} \to \mathbf{K}$ if M is large enough.

LEMMA 2.1. *There exists a constant $M > 1$ such that $T_{\lambda,M} : E \to \mathbf{K}$. Furthermore, if $(u, v) \in T_{\lambda,M}(\mathbf{K})$ then u, v are decreasing on* $[0, 1]$.

PROOF. In view of (1.6), there exist constants $c_0, c_1 > 0$ such that

$$f_0(t) \geq c_0 t^p - c_1 \quad \text{and} \quad g_0(t) \geq c_0 t^q - c_1 \tag{2.2}$$

for $t \geq 0$. Since

$$\lim_{s \to 0^+} s^{-N} \int_0^s \tau^{N-1} \psi^{-\gamma} d\tau = \lim_{s \to 0^+} s^{-N} \int_0^s \tau^{N-1} \psi^l d\tau = 1/N,$$

where $l \in \{p, q\}$, and $s^{-N} \int_0^s \tau^{N-1} \psi^{-\gamma} d\tau,\ s^{-N} \int_0^s \tau^{N-1} \psi^l d\tau$ are positive and continuous on $(0, 1]$, there exist constants $\tilde{c}_0, \tilde{c}_1 > 0$ such that

$$\int_0^s \tau^{N-1} \psi^l d\tau \geq \tilde{c}_0 s^N \ \text{ and } \int_0^s \tau^{N-1} \psi^{-\gamma} d\tau \leq \tilde{c}_1 s^N$$

for $s > 0, l \in \{p, q\}$. Hence it follows that

$$\int_0^s \tau^{N-1} \left(-k(M\psi)^{-\gamma} + c_0 (M\psi)^l - c_1\right) d\tau \geq \left(-k\tilde{c}_1 M^{-\gamma} + c_0 \tilde{c}_0 M^l - \frac{c_1}{N}\right) s^N > 0 \tag{2.3}$$

for $s > 0$ if $M > 1$ is large enough, which we assume. We claim that $T_{\lambda,M} : \mathbf{K} \to \mathbf{K}$. Let $(u, v) = T_{\lambda,M}(\tilde{u}, \tilde{v})$ where $(\tilde{u}, \tilde{v}) \in \mathbf{K}$. Using (2.1)-(2.3), we obtain

$$-u'(r) = \lambda r^{1-N} \int_0^r s^{N-1} \left(h_0(\tilde{v}_M) + f_0(\tilde{v}_M)\right) ds$$

$$\geq \lambda r^{1-N} \int_0^r s^{N-1} \left(-k(M\psi)^{-\gamma} + c_0 (M\psi)^p - c_1\right) ds > 0$$

for $r \in (0, 1]$ i.e. $u' < 0$ on $(0, 1]$. Similarly, $v' < 0$ on $(0, 1]$. Since $u(1) = v(1) = 0$, this completes the proof of Lemma 2.1. □

Let M be the constant given by Lemma 2.1. To avoid cumbersome notation we shall write $\tilde{z}$ for $\tilde{z}_M$ and T_λ for $T_{\lambda,M}$ for the rest of the paper.

LEMMA 2.2. *There exist constants $\tilde{\lambda}_0 \in (0, 1)$ and $r_\lambda > 0$ with $r_\lambda \to \infty$ as $\lambda \to 0$ such that for $\lambda < \tilde{\lambda}_0$, all solutions $(u, v) \in \mathbf{K}$ of*

$$(u, v) = \theta T_\lambda(u, v),\ \theta \in (0, 1)$$

satisfy $||(u, v)|| \neq r_\lambda$.

PROOF. Let $(u, v) \in \mathbf{K}$ satisfy

$$(u, v) = \theta T_\lambda(u, v) \quad \text{for some } \theta \in (0, 1)\,.$$

Then $u, v \geq 0$ and

$$u(r) = \lambda\theta \int_r^1 s^{1-N} \left(\int_0^s \tau^{N-1} \left(h_0(\tilde{v}) + f_0(\tilde{v})\right) d\tau \right) ds.$$

In view of (1.6), there exist constant $d_0, d_1 > 0$ such that

$$f_0(t) \leq d_0 t^p + d_1 \quad \text{and} \quad g_0(t) \leq d_0 t^q + d_1 \tag{2.4}$$

for $t \geq 0$. Let $\nu = \max\{p, q\}$. Since $\psi \leq \tilde{v} \leq v + M$, it follows from (2.1) and (2.4) that

$$u(r) \leq \lambda \int_r^1 s^{1-N} \left(\int_0^s \tau^{N-1} (k\psi^{-\gamma} + d_0(v + M)^p + d_1) d\tau \right) ds$$

$$\leq \lambda d_2 (1 + ||v||_\infty^\nu) \quad \text{for } r \in (0, 1), \tag{2.5}$$

where $d_2 = k(1-\gamma)^{-1} + 2^{\nu-1} d_0 (1 + M^\nu) + d_1$. Here we have used the inequality $(x + y)^\nu \leq 2^{\nu-1}(x^\nu + y^\nu)$ for $x, y \geq 0$, $\nu > 1$ and the fact that

$$s^{1-N} \int_0^s \tau^{N-1} \psi^{-\gamma} d\tau \leq \int_0^s \psi^{-\gamma} d\tau \leq (1 - \gamma)^{-1} \quad \text{for } s > 0\,.$$

Similarly,

$$v(r) \leq \lambda d_2 (1 + ||u||_\infty^\nu) \tag{2.6}$$

for $r \in (0, 1)$. Combining (2.5) and (2.6), we get

$$||(u, v)|| \leq \lambda d_2 (1 + ||(u, v)||^\nu)\,. \tag{2.7}$$

Suppose $\lambda < (4d_2)^{-1}$ and let $r_\lambda = (4\lambda d_2)^{-1/(\nu-1)}$. Then $r_\lambda \to \infty$ as $\lambda \to 0$.

We claim that $||(u, v)|| \neq r_\lambda$. Indeed, suppose $||(u, v)|| = r_\lambda$. Since $r_\lambda > 1$, it follows from (2.7) that

$$r_\lambda \leq 2\lambda d_2 r_\lambda^\nu\,,$$

which implies $r_\lambda \geq (2\lambda d_2)^{-1/(\nu-1)}$, a contradiction which proves the claim. □

For the rest of the paper, we assume $\lambda < \tilde{\lambda}_0$.

LEMMA 2.3. *(i) Let* $(u, v) \in \mathbf{K}$ *be a solution of*

$$(u, v) = T_\lambda(u, v) + (t, t),\ t \geq 0\,. \tag{2.8}$$

Then there exist positive constants δ_0, δ_1 *independent of* u, v, λ, *such that*

$$u(r) \geq \lambda(\delta_0 v^p(r) - \delta_1)\,, \quad v(r) \geq \lambda(\delta_0 u^q(r) - \delta_1)$$

for $r \in [1/2, 3/4]$.

(ii) There exists a constant $t_\lambda > 0$ *such that if the equation* (2.8) *has a solution* $(u, v) \in \mathbf{K}$ *then*

$$u(1/2)\,, \quad v(1/2) \leq t_\lambda\,.$$

In particular, if (2.8) *has a solution in* $\mathbf{K}$ *then* $t \leq t_\lambda$.

PROOF. Let $(u, v) \in \mathbf{K}$ be a solution of (2.8) for some $t \geq 0$. Then $(u - t, v - t) = T_\lambda(u, v)$ and hence by Lemma 2.1, u, v are decreasing on [0, 1] and satisfy

$$\begin{cases} -(r^{N-1}u')' = \lambda r^{N-1}(h_0(\tilde{v}) + f_0(\tilde{v})),\ 0 < r < 1, \\ -(r^{N-1}v')' = \lambda r^{N-1}(k_0(\tilde{u}) + g_0(\tilde{u})),\ 0 < r < 1, \\ \quad u'(0) = v'(0) = 0,\ u(1) = v(1) = t. \end{cases} \tag{2.9}$$

Note that

$$u(r) = t + \lambda \int_r^1 \frac{1}{s^{N-1}} \left(\int_0^s \tau^{N-1}\, (h_0(\tilde{v}) + f_0(\tilde{v}))d\tau \right) ds.$$

Let $r \in [1/2, 3/4]$. Using (2.1)-(2.2), it follows that for $s \geq r$,

$$\int_0^s \tau^{N-1}\, (h_0(\tilde{v}) + f_0(\tilde{v}))d\tau \geq \int_0^r \tau^{N-1}(-k\psi^{-\gamma} + c_0 v^p - c_1)d\tau - c_2$$

$$\geq c_3 v^p(r) - c_4\,,$$

where $c_2, c_3,$ and c_4 are positive constants independent of u, v, λ. Hence

$$u(r) \geq \lambda \int_r^1 s^{1-N}(c_3 v^p(r) - c_4)ds \geq \lambda(\delta_0 v^p(r) - \delta_1)\,, \tag{2.10}$$

where $\delta_0 = c_3 \int_{3/4}^1 s^{1-N} ds$, $\delta_1 = c_4 \int_{1/2}^1 s^{1-N} ds$. Similarly,

$$v(r) \geq \lambda(\delta_0 u^q(r) - \delta_1)\,, \tag{2.11}$$

and (i) follows. Suppose $u(1/2) > \bar{t}_\lambda$, where $\bar{t}_\lambda > 0$ is large enough so that $\delta_0 \bar{t}_\lambda^q \geq 2\delta_1$, $\lambda^p(\delta_0/2)^{1+p}\ \bar{t}_\lambda^{pq} > 2\delta_1$, and $\bar{t}_\lambda^{pq-1} > (\lambda\delta_0/2)^{-(1+p)}$. Then it follows from (2.10) and (2.11) that

$$v(1/2) \geq \lambda(\delta_0/2)u^q(1/2)\,,$$

and

$$u(1/2) \geq \lambda(\delta_0/2)v^p(1/2)\,,$$

which implies

$$\bar{t}_\lambda^{pq-1} < u^{pq-1}(1/2) \le (\lambda\delta_0/2)^{-(1+p)},$$

a contradiction. Hence $u(1/2) \le \bar{t}_\lambda$. Similarly, there exists $\hat{t}_\lambda > 0$ such that $v(1/2) \le \hat{t}_\lambda$. Hence $u(1/2), v(1/2) \le t_\lambda = \max(\bar{t}_\lambda, \tilde{t}_\lambda)$, and $t = u(1) \le u(1/2) \le t_\lambda$, which completes the proof. □

LEMMA 2.4. *Let $(u, v) \in \mathbf{K}$ be a solution of* (2.8) *for some $t \ge 0$. Then*
(i)

$$\lambda(u^q(1/2) + v^p(1/2)) \to \infty \text{ as } ||(u, v)|| \to \infty.$$

(ii) *There exists a constant $R_\lambda > r_\lambda$ such that all solutions $(u, v) \in \mathbf{K}$ of* (2.8) *satisfy $||(u, v)|| < R_\lambda$, where r_λ is given by Lemma* 2.2.

PROOF. Define $\bar{f}_0(t) = \inf_{s\ge t} f_0(s)$, $\tilde{f}_0(t) = \sup_{0\le s\le t} f_0(s)$, $\bar{g}_0(t) = \inf_{s\ge t} g_0(s)$, $\tilde{g}_0(t) = \sup_{0\le s\le t} g_0(s)$, $\bar{F}_0(t) = \int_0^t \bar{f}_0(s)ds$, and $\bar{G}_0(t) = \int_0^t \bar{g}_0(s)ds$. Let

$$\xi(r) = r^N u'v' + \lambda r^N \Big[-k_0(u^{1-\gamma} + v^{1-\gamma}) + \bar{F}_0(v) + \bar{G}_0(u)\Big] + \alpha r^{N-1}u'v + \beta r^{N-1}uv',$$

where $\alpha, \beta > 0$ are such that $\alpha + \beta = N - 2$ and

$$\frac{N}{p+1} > \alpha, \quad \frac{N}{q+1} > \beta.$$

Let $||u|| = D_0$, $||v|| = D_1$ and without loss of generality suppose $D_0 \ge D_1$. Note that u, v are positive and decreasing on [0, 1]. We shall break down the proof of (i) in four steps. In Step 1, we establish a lower bound estimate for $\xi'(r)$. In Step 2, we show that $\lambda D_0^q,\ \lambda D_1^p \to \infty$ as $D_0 \to \infty$. In Step 3, we establish a lower bound estimate for $\xi(r), r \ge r_2$, where $r_2 = \max(r_0, r_1)$ and $u(r_0) = D_0/2,\ v(r_1) = D_1/2$. In Step 4, we establish (i) by considering the two cases $r_2 \ge 1/2$ and $r_2 < 1/2$. Since we want to establish (i), we shall assume that $D_0 >> 1$ in Steps 2-4.

Step 1. Establish a lower bound estimate for $\xi'(r)$.

By (1.7),

$$|h_0(\tilde{v})| \le k\tilde{v}^{-\gamma} \le kv^{-\gamma} \text{ and } |k_0(\tilde{u})| \le k\tilde{u}^{-\gamma} \le ku^{-\gamma}.$$

Hence, by multiplying the first equation in (2.9) by rv', the second by ru', and adding we get

$$-(r^N u'v')' + (2 - N)r^{N-1}u'v' = \lambda r^N \left[(h_0(\tilde{v}) + f_0(\tilde{v}))v' + (k_0(\tilde{u}) + g_0(\tilde{u}))u'\right]$$

$$\le \lambda r^N \left[(-kv^{-\gamma} + \bar{f}_0(v))v' + (-ku^{-\gamma} + \bar{g}_0(u))u'\right] \tag{2.12}$$

$$= \Big[\lambda r^N(-k_0v^{1-\gamma} + \bar{F}_0(v) - k_0u^{1-\gamma} + \bar{G}_0(u))\Big]'$$

$$-\lambda N r^{N-1}\Big[-k_0v^{1-\gamma} + \bar{F}_0(v) - k_0u^{1-\gamma} + \bar{G}_0(u)\Big],$$

where $k_0 = k(1-\gamma)^{-1}$.

Next, multiplying the first equation in (2.9) by αv, the second by βu, and adding, we get

$$-(\alpha r^{N-1}u'v + \beta r^{N-1}uv')' + (N-2)r^{N-1}u'v'$$

$$= \lambda r^{N-1}\left[\alpha(h_0(\tilde{v}) + f_0(\tilde{v}))v + \beta(k_0(\tilde{u}) + g_0(\tilde{u}))u)\right]$$

$$\leq \lambda r^{N-1}\left[\alpha(kv^{1-\gamma} + \tilde{f}_0(v+M)v) + \beta(ku^{1-\gamma} + \tilde{g}_0(u+M)u)\right]. \tag{2.13}$$

Adding (2.12) and (2.13), we obtain

$$\xi'(r) \geq \lambda r^{N-1}\left[N(-k_0v^{1-\gamma} + \bar{F}_0(v)) - \alpha(kv^{1-\gamma} + \tilde{f}_0(v+M)v)\right]$$

$$+\lambda r^{N-1}\left[N(-k_0u^{1-\gamma} + \bar{G}_0(u)) - \beta(ku^{1-\gamma} + \tilde{g}_0(u+M)u\right].$$

Since

$$\lim_{t\to\infty} \frac{N(-k_0t^{1-\gamma} + \bar{F}_0(t)) - \alpha(kt^{1-\gamma} + \tilde{f}_0(t+M)t)}{t^{p+1}} = \left(\frac{N}{p+1} - \alpha\right) l_0 > 0,$$

and

$$\lim_{t\to\infty} \frac{N(-k_0t^{1-\gamma} + \bar{G}_0(t)) - \beta(kt^{1-\gamma} + \tilde{g}_0(t+M)t)}{t^{q+1}} = \left(\frac{N}{q+1} - \beta\right) l_1 > 0,$$

there exist positive constants a and m independent of $u, v, \lambda,$ such that

$$\xi'(r) \geq \lambda r^{N-1}(a(u^{q+1} + v^{p+1}) - m) \tag{2.14}$$

for $r \in [0,1]$.

Step 2. Show $\lambda D_0^q,\ \lambda D_1^p \to \infty$ as $D_0 \to \infty$.

Note that λ is dependent on D_0 and it is not trivial that $\lambda D_0^q \to \infty$ as $D_0 \to \infty$. Our strategy here is to first use the equation for u and the fact that $t \leq t_\lambda$ to show that $\lambda D_1^p \to \infty$ as $D_0 \to \infty$, and then use the equation for v to show that $\lambda D_0^q \to \infty$ as $D_0 \to \infty$.

By Lemma 2.3 (ii), $t \leq t_\lambda$, which, together with (2.1) and (2.4), implies

$$u(r) \leq t_\lambda + \lambda \int_r^1 \frac{1}{s^{N-1}}\left(\int_0^s \tau^{N-1}(k\psi^{-\gamma} + d_0(v+M)^p + d_1)d\tau\right)ds$$

$$\leq t_\lambda + \lambda m_1(1 + D_1^p), \tag{2.15}$$

for $r \in [0,1]$, where $m_1 = k(1-\gamma)^{-1} + d_0 2^{\nu-1}(1+M^p) + d_1,\ \nu = \max(p,q)$. Similarly,

$$v(r) \leq t_\lambda + \lambda m_1(1 + D_0^q) \tag{2.16}$$

for $r \in [0,1]$. Suppose $D_0 > 4\tilde{t}_\lambda,\ (D_0/2m_1)^{1/p} > 4\tilde{t}_\lambda$, where $\tilde{t}_\lambda = \max(t_\lambda, m_1)$.

Since $\lambda < 1$,

$$t_\lambda + \lambda m_1 < t_\lambda + m_1 \leq 2\tilde{t}_\lambda < D_0/2,$$

from which (2.15) implies

$$\lambda D_1^p \geq (1/m_1)(D_0 - t_\lambda - \lambda m_1) \geq D_0/2m_1. \tag{2.17}$$

Consequently,

$$D_1 \geq (D_0/2m_1)^{1/p} > 4\tilde{t}_\lambda\,.$$

Hence it follows from (2.16) that

$$\lambda D_0^q \geq (1/m_1)(D_1 - t_\lambda - \lambda m_1) \geq D_1/2m_1 \geq D_0^{1/p} m_2\,, \tag{2.18}$$

where $m_2 = (2m_1)^{-(1/p+1)}$.

Step 3. Establish a lower bound estimate for $\xi(r)$, $r \geq r_2$.

Let us recall that $r_2 = \max(r_0, r_1)$ where $u(r_0) = D_0/2$, $v(r_1) = D_1/2$. Note that r_0, r_1 exist since $u(1) \leq t_\lambda < D_0/2$, $v(1) \leq t_\lambda < D_1/2$, and $u(0) > D_0/2$, $v(0) > D_1/2$.

It follows from (2.14) that for $r \geq r_2$,

$$\begin{aligned} \xi(r) &\geq \lambda \left(a \int_0^{r_0} s^{N-1} u^{q+1} ds + a \int_0^{r_1} s^{N-1} v^{p+1} ds - m \right) \\ &\geq \lambda \left(b r_0^N ({D_0}^{q+1} + b r_1^N {D_1}^{p+1} - m) \right)\,, \end{aligned} \tag{2.19}$$

where $b = (a/N)(1/2)^{\max(p,q)+1}$.

Next, we need estimates for r_0, r_1. Since there exists a positive constant m_3 depending only on $k, \gamma, d_0, d_1, p, m_1, M$ such that

$$\begin{aligned} \int_0^r s^{N-1}\,(h_0(\tilde{v}) + f_0(\tilde{v}))ds &\leq \int_0^r s^{N-1}(k\psi^{-\gamma} + d_0(v+M)^p + d_1)ds \\ &\leq m_3 D_1^p r^N\,, \end{aligned}$$

it follows that

$$-u'(r) = \lambda r^{1-N} \int_0^r s^{N-1}\,(h_0(\tilde{v}) + f_0(\tilde{v}))ds \leq \lambda m_3 D_1^p r\,. \tag{2.20}$$

Integrating (2.20) on $(0, r_0)$ gives

$$D_0/2 \leq \lambda m_3 D_1^p (r_0^2/2)\,. \tag{2.21}$$

By taking m_3 larger if necessary, we obtain in a similar fashion that

$$D_1/2 \leq \lambda m_3 D_0^q (r_1^2/2)\,. \tag{2.22}$$

From (2.21) and (2.22), we deduce that

$$r_0 \geq m_4 \sqrt{\frac{D_0}{\lambda D_1^p}} \text{ and } r_1 \geq m_4 \sqrt{\frac{D_1}{\lambda D_0^q}}\,, \tag{2.23}$$

where $m_4 = \sqrt{1/m_3}$. Using (2.23) in (2.19), we get

$$\xi(r) \geq \lambda^{1-N/2} b m_4^N \left(\frac{D_0^{q+1+N/2}}{D_1^{Np/2}} + \frac{D_1^{p+1+N/2}}{D_0^{Nq/2}} \right) - \lambda m\,. \tag{2.24}$$

Let $\delta = 1 + \frac{N}{2(q+1)} - \frac{Np}{2(p+1)}$. Then $\delta > 0$, by (A1). Since

$$\frac{D_0^{q+1+N/2}}{D_1^{Np/2}} = \frac{D_0^{(q+1)\left(\frac{q+1+N/2}{q+1}\right)}}{D_1^{(p+1)\left(\frac{Np}{2(p+1)}\right)}} \geq D_0^{(q+1)\delta}$$

if $D_0^{q+1} > D_1^{p+1}$, and

$$\frac{D_1^{p+1+N/2}}{D_0^{Nq/2}} = \frac{D_1^{(p+1)\left(\frac{p+1+N/2}{p+1}\right)}}{D_0^{(q+1)\left(\frac{Nq}{2(q+1)}\right)}} \geq D_0^{(q+1)\delta}$$

if $D_0^{q+1} \leq D_1^{p+1}$, it follows from (2.24) and $\lambda < 1$ that

$$\xi(r) \geq \lambda^{1-N/2} b m_4^N D_0^{(q+1)\delta} - \lambda m \geq \lambda^{1-N/2}(b m_4^N D_0^{(q+1)\delta} - m)$$

$$\geq m_5 \lambda^{1-N/2} D_0^{(q+1)\delta} \quad \text{for } r \geq r_2\,, \tag{2.25}$$

where $m_5 = b m_4^N/2$, provided that $D_0^{(q+1)\delta} > m/m_5$, which we assume.

Step 4. Proof of (i).

Case 1: $r_2 \geq 1/2$. If $r_2 = r_0$ then $u(1/2) \geq u(r_0) = D_0/2$, which, together with (2.18), implies

$$\lambda u^q(1/2) \geq \lambda (D_0/2)^q \geq m_2 D_0^{1/p}/2^q\,, \tag{2.26}$$

while if $r_2 = r_1$ then $v(1/2) \geq v(r_1) = D_1/2$, which together with (2.17), implies

$$\lambda v^p(1/2) \geq \lambda (D_1/2)^p \geq D_0/(2^{p+1} m_1)\,. \tag{2.27}$$

Case 2: $r_2 < 1/2$. Then, by (2.25),

$$\xi_0(r) \geq \xi(r) \geq m_5 \lambda^{1-N/2} D_0^{(q+1)\delta} \quad \text{for } r \geq 1/2\,,$$

where $\xi_0(r) = r^N u'v' + \lambda r^N(\bar{F}_0(v) + \bar{G}_0(u))$.

Since $\lim_{t\to\infty} t^{-(p+1)} \bar{F}_0(t) = l_1$ and $\lim_{t\to\infty} t^{-(q+1)} \bar{G}_0(t) = l_2$, there exist constants $l, m_6 > 0$ such that

$$u'v' + \lambda l(v^{p+1} + u^{q+1}) \geq m_5 \lambda^{1-N/2} D_0^{(q+1)\delta} - m_6 \geq m_7 \lambda^{1-N/2} D_0^{(q+1)\delta} \tag{2.28}$$

on $[1/2, 1]$, provided that $D_0^{(q+1)\delta} > 2m_6/m_5$, where $m_7 = m_5/2$.

Since $\lambda < 1$, it follows from Lemma 2.3 (i) that

$$\lambda v^p(r) \leq \delta_0^{-1}(u(r) + \delta_1) \quad \text{and} \quad \lambda u^q(r) \leq {\delta_0}^{-1}(v(r) + \delta_1) \tag{2.29}$$

for $r \in [1/2, 3/4]$. Multiplying the first inequality in (2.29) by lv, the second by lu, and adding to get

$$\lambda l(v^{p+1}(r) + u^{q+1}(r)) \leq m_8(uv + u + v)\,, \tag{2.30}$$

where m_8 is a positive constant depending on δ_0, δ_1, and l.

Combining (2.28) and (2.29), we obtain

$$u'v' + m_8(uv + u + v) \geq m_7\lambda^{1-N/2} D_0^{(q+1)\delta},$$

from which it follows that

$$u'v' + uv + u + v \geq m_9\lambda^{1-N/2} D_0^{(q+1)\delta},$$

where $m_9 = \frac{m_7}{\max(1, m_8)}$. Since $u', v' < 0$ on $(0, 1]$, this implies

$$(-u' - v' + u + v + 1)^2 \geq u'v' + uv + u + v \geq m_9\lambda^{1-N/2} D_0^{(q+1)\delta} \tag{2.31}$$

on $[1/2, 3/4]$. Let $w = u + v$. Then it follows from (2.31) and $\lambda < 1$ that

$$-w' + w \geq \sqrt{m_9}\lambda^{1/2-N/4} D_0^{(q+1)\delta/2} - 1 \geq m_{10}\lambda^{1/2-N/4} D_0^{(q+1)\delta/2},$$

on $[1/2, 3/4]$, provided that $D_0^{(q+1)\delta/2} \geq 2m_9^{-1/2}$, where $m_{10} = \sqrt{m_9}/2$.

Solving this differential inequality gives

$$w(1/2) \geq m_{11}\lambda^{1/2-N/4} D_0^{(q+1)\delta/2},$$

where $m_{11} = m_{10}(1 - e^{-1/4})$. Hence

$$u(1/2) \geq (m_{11}/2)\lambda^{1/2-N/4} D_0^{(q+1)\delta/2},$$

or

$$v(1/2) \geq (m_{11}/2)\lambda^{1/2-N/4} D_0^{(q+1)\delta/2}.$$

If $u(1/2) \geq (m_{11}/2)\lambda^{1/2-N/4} D_0^{(q+1)\delta/2}$ then

$$\lambda u^q(1/2) \geq m_{12}\lambda^{1+(1/2-N/4)q} D_0^{q(q+1)\delta/2} \geq m_{12} D_0^{q(q+1)\delta/2} \tag{2.32}$$

since $1 + (1/2 - N/4)q \leq 0$, where $m_{12} = (m_{11}/2)^q$.

On the other hand, if $v(1/2) \geq (m_{11}/2)\lambda^{1/2-N/4} D_0^{(q+1)\delta/2}$ then

$$\lambda v^p(1/2) \geq m_{13}\lambda^{1+(1/2-N/4)p} D_0^{p(q+1)\delta/2} \geq m_{13} D_0^{p(q+1)\delta/2}, \tag{2.33}$$

since $1 + (1/2 - N/4)p \leq 0$, where $m_{13} = (m_{11}/2)^p$.

Combining (2.26), (2.27), (2.32), and (2.33), it follows that

$$\lambda(u^q(1/2) + v^p(1/2)) \to \infty \text{ as } D_0 \to \infty,$$

i.e. (i) holds. In particular, there exists a constant $R_\lambda > r_\lambda$ such that $u^q(1/2) + v^p(1/2) > t_\lambda^q + t_\lambda^p$ for $||(u, v)|| \geq R_\lambda$. This implies $u(1/2) > t_\lambda$ or $v(1/2) > t_\lambda$ for $||(u, v)|| > R_\lambda$, which contradicts Lemma 2.3(ii). Hence (2.8) has no solution $(u, v) \in \mathbf{K}$ with $||(u, v)|| \geq R_\lambda$, which completes the proof of Lemma 2.4. □

LEMMA 2.5. *Let $z \in C^1[0,1]$ satisfy*

$$\begin{cases} -(r^{N-1}z')' \geq -\lambda k r^{N-1}\psi^{-\gamma} \text{ in } (0,1)\,, \\ z(1/2) \geq L,\ z(1) = 0\,, \end{cases} \tag{2.34}$$

where $\gamma \in (0,1)$, $k, L > 0$. Then

$$z(r) \geq L_0(1-r)$$

for $r \in [1/2, 1]$, where $L_0 = 2^{2-N}L - 2^{N-1}k(1-\gamma)^{-1}\lambda$.

PROOF. Let $z_0(r) = z(r) - z(1/2)\left(\int_r^1 s^{1-N}ds\right)\left(\int_{1/2}^1 s^{1-N}ds\right)^{-1}$, $r \in [0,1]$. Then $z_0(1/2) = z_0(1) = 0$ and z_0 satisfies the diffferential inequality in (2.34). Hence

$$z_0(r) \geq -\lambda k \int_{1/2}^1 K(r,s)s^{N-1}\psi^{-\gamma}ds, \tag{2.35}$$

where $K(r,s)$ is the Green's function of $-(r^{N-1}u')'$ with zero boundary condition on $(1/2, 1)$. Note that

$$K(r,s) = \begin{cases} \rho\left(\int_{1/2}^s \tau^{1-N}d\tau\right)\left(\int_r^1 \tau^{1-N}d\tau\right) \text{if } s \leq r\,, \\ \rho\left(\int_{1/2}^r \tau^{1-N}d\tau\right)\left(\int_s^1 \tau^{1-N}d\tau\right) \text{if } s > r\,, \end{cases}$$

where $\rho = \left(\int_{1/2}^1 \tau^{1-N}d\tau\right)^{-1}$. Since

$$K(r,s) \leq \int_r^1 \tau^{1-N}d\tau \leq 2^{N-1}(1-r)$$

for $1/2 \leq r, s \leq 1$, it follows from (2.35) that

$$z_0(r) \geq -2^{N-1}k\lambda \int_0^1 s^{N-1}\psi^{-\gamma}ds \geq -2^{N-1}k(1-\gamma)^{-1}\lambda(1-r)\,.$$

Hence

$$\begin{aligned} z(r) &= z(1/2)\left(\int_r^1 s^{1-N}ds\right)\left(\int_{1/2}^1 s^{1-N}ds\right)^{-1} + z_0(r) \\ &\geq (2^{2-N}L - 2^{N-1}k(1-\gamma)^{-1}\lambda)(1-r) \end{aligned}$$

for $r \in [1/2, 1]$, which completes the proof. □

3. Proof of the main result.

PROOF OF THEOREM 1.1. By Theorem A, Lemma 2.2, and Lemma 2.4 (ii), T_λ has a fixed point $(u_\lambda, v_\lambda) \in \mathbf{K}$ with $||(u_\lambda, v_\lambda)|| \geq r_\lambda$. Since $r_\lambda \to \infty$ as $\lambda \to 0$, it follows from Lemma 2.4(i) with $t = 0$ that

$$\lambda(u_\lambda^q(1/2) + v_\lambda^p(1/2)) \to \infty \tag{3.1}$$

as $\lambda \to 0$. By Lemma 2.3(i),

$$u_\lambda(1/2) \geq \lambda(\delta_0 v_\lambda^p(1/2) - \delta_1)\,, \tag{3.2}$$

and

$$v_\lambda(1/2) \geq \lambda(\delta_0 u_\lambda^q(1/2) - \delta_1)\,. \tag{3.3}$$

Let $M_0 > 0$. We shall show that

$$u_\lambda(r), v_\lambda(r) \geq M_0(1-r) \text{ on } (0,1)$$

if λ is sufficiently small. Let $K > 1$ be large enough so that

$$2^{2-N}\min(K^{1/\max(p,q)}, \delta_0 K - \delta_1) - 2^{N-1}k(1-\gamma)^{-1} > 2M_0\,. \tag{3.4}$$

In view of (3.1), there exists $\lambda_0 \in (0, \tilde{\lambda}_0)$ such that $\lambda u_\lambda^q(1/2) > K$ or $\lambda v_\lambda^p(1/2) > K$ for $\lambda \in (0, \lambda_0)$.

If $\lambda u_\lambda^q(1/2) > K$ then $u_\lambda(1/2) > K^{1/q}$ and it follows from (3.3) and $\lambda < 1$ that $v_\lambda(1/2) \geq \delta_0 K - \delta_1$. Since u_λ, v_λ satisfy (2.34) with $L = \min(K^{1/\max(p,q)}, \delta_0 K - \delta_1)$, (3.4) and Lemma 2.5 imply

$$u_\lambda(r), v_\lambda(r) \geq 2M_0(1-r) \tag{3.5}$$

for $r \in [1/2, 1]$. On the other hand, if $\lambda v_\lambda^p(1/2) > K$ then $v_\lambda(1/2) > K^{1/p}$ and it follows from (3.2) that $u_\lambda(1/2) \geq \delta_0 K - \delta_1$. Hence (3.5) follows from (3.4) and Lemma 2.5. Thus (3.5) holds in either case. Since u_λ, v_λ are decreasing, $u_\lambda(r) \geq u_\lambda(1/2) \geq M_0(1-r)$ and $v_\lambda(r) \geq v_\lambda(1/2) \geq M_0(1-r)$ for $r \in [0, 1/2)$. In particular, by taking $M_0 = M$, we see that (u_λ, v_λ) is a positive radial solution of (1.1) for $\lambda < \lambda_0$ with

$$(1-r)^{-1}\min(u_\lambda(r), v_\lambda(r)) \to \infty$$

uniformly in $r \in [0, 1)$ as $\lambda \to 0$, which completes the proof. □

Acknowledgment. The author wish to thank the referee for carefully reading the manuscript and providing helpful suggestions.

REFERENCES

[1] T. B. BENJAMIN, A unified theory of conjugate flows, Philos. Trans. Roy. Soc. A 269 (1971), 587–643.

[2] H. BREZIS AND R. L. TURNER, On a class of superlinear elliptic problems, Comm. Partial Differential Equations 2 (1977), 601–614.

[3] PH. CLEMENT, D. G. DE FIGUEREDO AND E. MITIDIERI, Positive solutions of semilinear elliptic systems, Comm. Partial Differential Equations 17 (1992), 923–940.

[4] PH. CLEMENT, R. MANASEVICH AND E. MITIDIERI, Positive solutions for a quasilinear elliptic system via blow up, Comm. Partial Differential Equations 18 (1993), 2071–2106.

[5] C. COSNER, Positive solutions for superlinear elliptic systems without variational structure, Nonlinear Anal. 8 (1984), 1427–1436.

[6] M. G. CRANDALL, P. H. RABINOWITZ AND L. TARTAR, On a Dirichlet problem with a singularity, Comm. Partial Differential Equations 12 (1977), 193–222.

[7] M. M. COCLITE AND G. PALMIERI, On a singular nonlinear Dirichlet problem, Comm. Partial Differential Equations 14 (1989), 1315–1327.

[8] R. DALMASSO, Existence and uniqueness of positive radial solutions for the Lane-Emden system, Nonlinear Anal. 57 (2004), 341–348.

[9] D. G. DE FIGUEREIDO, P. L. LIONS AND R. D. NUSSBAUM, A priori estimates and existence of positive solutions for semilinear elliptic equations, J. Math. Pures Appl. 61 (1982), 41–63.

[10] W. FULKS AND J. S. MAYBEE, A singular nonlinear equation, Osaka J. Math. 12 (1960), 1–19.

[11] G. B. GUSTAFSON AND K. SCHMITT, Nonzero solutions of boundary value problems for second order ordinary and delay-differential equations, J. Differential Equations 12 (1972), 129–147.

[12] B. GIDAS AND J. SPRUCK, A priori bounds for positive solutions of nonlinear elliptic equations, Comm. Partial Differential Equations 6 (1981), 883–901.

[13] D. D. HAI, On a class of semilinear elliptic systems, J. Math. Anal. Appl. 285 (2003), 477–486.

[14] D. D. HAI AND K. SCHMITT, On radial solutions of quasilinear boundary value problems, in Topics in nonlinear analysis, 349–361, Progr. Nonlinear Differential Equations Appl. 35, Birkhauser, Basel, 1999.

[15] D. D. HAI AND J. L. WILLIAMS, Positive radial solutions for a class of quasilinear boundary value problems in a ball, Nonlinear Anal. 75 (2012), no. 4, 1744–1750.

[16] D. D. HAI, Positive radial solutions for singular quasilinear elliptic equations in a ball, Publ. Res. Inst. Math. Sci. 50 (2014), 341–362.

[17] J. HERNANDEZ, J. KARATSON AND P. L. SIMON, Multiplicity for semilinear elliptic equations involving singularity, Nonlinear Anal. 65 (2006), 265–283.

[18] M. GARCIA-HUIDOBRO, R. MANASEVICH AND K. SCHMITT, Positive radial soltutions of quasilinear elliptic partial differential equations in a ball, Nonlinear Anal. 35 (1999), 175–190.

[19] E. MITIDIERI, A Rellich type identity and applications, Comm. Partial Differential Equations 18 (1993), 125–151.

[20] L. A. PELETIER AND R. C. A. M. VAN DER VORST, Existence and nonexistence of positive solutions of nonlinear elliptic systems and the biharmonic equation, Differential Integral Equations 5(1992), no. 4, 747–767.

DEPARTMENT OF MATHEMATICS AND STATISTICS
MISSISSIPPI STATE UNIVERSITY
MISSISSIPPI STATE, MS 39762
USA

E-mail address: dang@math.msstate.edu