RIEMANN-CESÀRO METHODS OF SUMMABILITY

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1. Introduction. In the theory of trigonometrical series, Riemann introduced the well known methods $(R, 1), (R_1), (R, 2)$ and (R_2) for evaluation of series. These methods have been generalized to the following forms. Throughout this paper, p denotes a positive integer. A series $\sum_{k=1}^{\infty} a_k$ is said to be evaluable (R, p) to s if the series in

$$f_p(t) = \sum_{k=1}^{\infty}\left(\frac{\sin kt}{kt}\right)^p a_k$$

converges in some interval $0 < t < t_0$ and $f_p(t) \to s$ as $t \to 0$. A series $\sum_{k=1}^{\infty} a_k$ is said to be evaluable (R_p) to s if the series in

$$F_p(t) = C_p^{-1} t \sum_{k=1}^{\infty}\left(\frac{\sin kt}{kt}\right)^p s_k,$$

where

$$C_p = \int_0^{\infty} u^{-p}(\sin u)^p \, du,$$

converges in some interval $0 < t < t_0$ and $F_p(t) \to s$ as $t \to 0$. It is well known that (R, p) and (R_p) are regular when $p \geqq 2$, while $(R, 1)$ and (R_1) are not regular.

It is the purpose of this paper to obtain information about these Riemann methods and generalizations of them by studying transformations which involve simultaneously the Riemann and Cesàro transformations of series. In terms of standard notation used by Zygmund [8, p. 42] and others, the Cesàro transform of order α of $\sum_{k=1}^{\infty} a_k$ is defined by

$$\sigma_n^\alpha = s_n^\alpha / A_n^\alpha, \tag{1.1}$$

where s_n^α and A_n^α being given by the relations

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1} \quad \text{and} \quad \sum_{n=0}^{\infty} s_n^\alpha x^n = \frac{\sum_{n=0}^{\infty} s_n x^n}{(1-x)^\alpha} = \frac{\sum_{n=0}^{\infty} a_n x^n}{(1-x)^{\alpha+1}}. \tag{1.2}$$

If we put $\alpha = -1$, for example, in (1.2), then we have $A_0^{-1} = 1$, $A_n^{-1} = 0\ (n = 1, 2, \dots)$ and $s_n^{-1} = a_n\ (n = 0, 1, 2, \dots)$: hence σ_n^{-1} has not meaning when $n = 1, 2, \dots$, while s_n^{-1} has meaning. It is well known that $A_n^\alpha \sim n^\alpha/\Gamma(\alpha+1)$, $\alpha \neq -1, -2, \dots$. A series is said to be evaluable (C, α) to s if $\sigma_n^\alpha \to s$ as $n \to \infty$, and to be evaluable $|C, \alpha|$ to s if $\Sigma|\sigma_n^\alpha - \sigma_{n+1}^\alpha|$ is convergent and $\sigma_n^\alpha \to s$ as $n \to \infty$. In the following, let α be a real number, not necessarily an integer, for which $\alpha \geqq -1$. In formulating our defini tion, we use the Cesàro sums s_n^α instead of the Cesàro means σ_n^α, which is meaningless when $\alpha = -1$.

A series $\sum_{k=1}^{\infty} a_k$ will be said to be evaluable to s by the Riemann-Cesàro method of order p and index α, or, briefly evaluable (R, p, α) to s, if the series in

$$\sigma(p, \alpha, t) = C_{p,\alpha}^{-1}\, t^{\alpha+1} \sum_{k=1}^{\infty} \left(\frac{\sin kt}{kt}\right)^p s_k^\alpha, \tag{1.3}$$

where

$$C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty u^{\alpha-p} (\sin u)^p\, du & (-1 < \alpha < p-1) \\ \pi/2 & (\alpha = 0, p = 1) \\ 1 & (\alpha = -1), \end{cases} \tag{1.4}$$

converges in some interval $0 < t < t_0$ and $\sigma(p, \alpha, t) \to s$ as $t \to 0$.

Under this definition, the $(R, p, -1)$-means $\sigma(p, -1, t)$ are identical with the (R, p)-means $f_p(t)$, by $s_n^{-1} = a_n$ and the $(R, p, 0)$-means $\sigma(p, 0, t)$ are identical with the (R_p)-means $F_p(t)$, by $s_n^0 = s_n$. Therefore, the $(R, p, -1)$ method and the $(R, p, 0)$ method are reduced to the (R, p) method and the (R_p) method, respectively. Our main results are the following.

THEOREM 1. *Let* $0 < \delta < 1$, *let* $\sum_{k=1}^{\infty} a_k$ *be evaluable* $(C, p-\delta)$ *to* s *and let*

$$\sum_{k=1}^{n} |\sigma_k^{p-\delta-1}| = O(n). \tag{1.5}$$

Then the series $\sum_{k=1}^{\infty} a_k$ *is evaluable* (R, p, α) *to* s *when* $-1 \leqq \alpha < p - \delta - 1$.

This result implies that if $0 < \delta < 1$ and $\sum_{k=1}^{\infty} a_k$ is evaluable $(C, p-1-\delta)$ to s, then $\sum_{k=1}^{\infty} a_k$ is evaluable (R, p, α) to s when $-1 \leqq \alpha < p - \delta - 1$. Since convergence implies Cesàro summability of each positive order, we obtain

COROLLARY 1. *The method* (R, p, α) *is regular when* $p \geqq 2$ *and* $-1 \leqq \alpha < p - 1$.

If $p \geqq 2$, then we can put $\alpha = 0$ in Theorem 1 and obtain

COROLLARY 2. *Under the conditions of Theorem* 1, *the series* $\sum_{k=1}^{\infty} a_k$ *is evaluable* (R_p) *to* s *when* $p \gneqq 2$.

This Corollary is due to Obreschkoff [4] who proved that the result holds when $p = 1$. (R, p) analogue is also due to Obreschkoff [4].

THEOREM 2. *The $(R, 1, \alpha)$ method is not regular when* $-1 \leqq \alpha \leqq 0$.

THEOREM 3. *If the series $\sum_{k=1}^{\infty} a_k$ is evaluable $|C, p|$ to s, it is evaluable (R, p, α) to s when $-1 \leqq \alpha < p-1$ and α is an integer. Further, if the series $\sum_{k=1}^{\infty} a_k$ is evaluable $|C, 1|$ to s, it is also evaluable $(R, 1, 0)$ to s.*

In this Theorem, if we put $\alpha = 0$, we have

COROLLARY 3. *If the series is evaluable $|C, p|$ to s, then it is evaluable (R_p) to s.*

(R, p) case of this Corollary is due to Obreschkoff [4].

THEOREM 4. *Suppose that*

$$\sum_{k=n}^{2n} (|a_k| - a_k) = O(1), \tag{1.6}$$

and the series $\sum_{k=1}^{\infty} a_k$ is evaluable to s by the Abel method. Then the series $\sum_{k=1}^{\infty} a_k$ is evaluable $(R, 1, \alpha)$ to s when $-1 \leqq \alpha \leqq 0$.

$(R, 1)$ and (R_1) cases are due to Szász [6, 7].

THEOREM 5. *Suppose that*

$$\sum_{k=n}^{2n} (|a_k| - a_k) = O(n^{1-r}), \qquad 0 < r < 1, \tag{1.7}$$

and

$$\sum_{k=1}^{n} |s_n - s| = o(n/\log n). \tag{1.8}$$

Then the series $\sum_{k=1}^{\infty} a_k$ is evaluable $(R, 1, \alpha)$ to s when $-1 \leqq \alpha \leqq 0$.

In this Theorem we can put $\alpha = 0$ and obtain

COROLLARY 4. *Under the conditions of Theorem 5, the series $\sum_{k=1}^{\infty} a_k$ is evaluable (R_1) to s.*

$(R, 1)$-analogue of this Corollary is due to Szász [5].

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2. Preliminary Lemmas.

LEMMA 1. *Let $\Delta^m\varphi(nt)$ denote the m-th difference of $\varphi(nt)$ with respect to n. Then we have*

$$\Delta^m\varphi(nt) = O(t^{m-p}/n^p), \tag{2.1}$$

when m is a non-negative integer and $\varphi(t) = (\sin t/t)^p$.

LEMMA 2. *Let $0 < \delta < 1$ and let $q_n \geqq 0$. Then*

$$\sum_{k=1}^{m} q_n = O(m) \tag{2.2}$$

implies

$$\sum_{n=m}^{\infty} q_n/n^{\delta+1} = O(m^{-\delta}) \tag{2.3}$$

and conversely.

These Lemmas are due to Obreschkoff [4].

LEMMA 3. *Let $A_n^\alpha = \binom{n+\alpha}{n}$ and let $-1 < \alpha < p-1$. Then*

$$\lim_{t\to 0} t^{\alpha+1}\sum_{n=1}^{\infty} A_{n-1}^{\alpha}\left(\frac{\sin nt}{nt}\right)^p = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty u^{\alpha-p}(\sin u)^p du = C_{p,\alpha}. \tag{2.4}$$

In particular, if $p=1$, $\alpha = 0$, we have

$$\lim_{t\to 0}\sum_{n=1}^{\infty}\frac{\sin nt}{n} = \frac{\pi}{2} = C_{1,0}. \tag{2.5}$$

This Lemma when $p \geqq 2$, $\alpha = 0$ is due to Obreschkoff [4].

PROOF. (2.5) is well known. For the proof of (2.4), firstly, we prove

$$\lim_{t\to 0} t^{\alpha+1}\sum_{n=1}^{\infty} n^\alpha\left(\frac{\sin nt}{nt}\right)^p = \Gamma(\alpha+1)\, C_{p,\alpha}.$$

Let a be an arbitrary positive number and let $m = [a/t]$. Then, we have, by the definition of definite integral,

$$\lim_{t\to 0}\ t^{\alpha+1}\sum_{n=1}^{m} n^\alpha\left(\frac{\sin nt}{nt}\right)^p = \int_0^a u^{\alpha-p}(\sin u)^p\, du. \tag{2.6}$$

Since $\alpha < p-1$, we have

$$\left|t^{\alpha+1}\sum_{n=m+1}^{\infty} n^\alpha\left(\frac{\sin nt}{nt}\right)^p\right| \leqq t^{\alpha-p+1}\sum_{n=m}^{\infty} n^{\alpha-p}$$
$$= O(t^{\alpha-p+1}m^{\alpha-p+1})$$
$$= O(a^{\alpha-p+1}).$$

Thus, we have

$$\lim_{t\to 0} t^{\alpha+1}\sum_{n=1}^{\infty} n^{\alpha}\left(\frac{\sin nt}{nt}\right)^{p} = \int_{0}^{a} u^{\alpha-p}(\sin u)^{p}\,du + O(a^{\alpha-p+1}). \tag{2.7}$$

Since a is arbitrary, if a tends to infinity, we have (2.6).

Next, we shall prove (2.4). Since $A_n^{\alpha} \sim n^{\alpha}/\Gamma(\alpha+1)$, we may write

$$A_{n-1}^{\alpha} = n^{\alpha}/\Gamma(\alpha+1) + \varepsilon_n n^{\alpha}, \tag{2.8}$$

where $\varepsilon_n \to 0$ as $n\to\infty$. Then, we shall prove

$$\lim_{t\to 0} t^{\alpha+1}\sum_{n=1}^{\infty} \varepsilon_n n^{\alpha}\left(\frac{\sin nt}{nt}\right)^{p} = 0 \tag{2.9}$$

For each positive number ε, there exists an N such that $|\varepsilon_n| < \varepsilon$ when $n \geqq N$. Since $\alpha > -1$, we have, for fixed N,

$$\lim_{t\to 0} t^{\alpha+1}\sum_{n=1}^{N-1} \varepsilon_n n^{\alpha}\left(\frac{\sin nt}{nt}\right)^{p} = 0.$$

Using the method analogous to the one which we obtained (2.6),

$$\begin{aligned}
&\limsup_{t\to 0}\left| t^{\alpha+1}\sum_{n=N}^{\infty} \varepsilon_n n^{\alpha}\left(\frac{\sin nt}{nt}\right)^{p}\right| \\
&\leqq \varepsilon \limsup_{t\to 0} t^{\alpha+1}\sum_{n=N}^{\infty} n^{\alpha}\left|\frac{\sin nt}{nt}\right|^{p} \\
&\leqq \varepsilon \limsup_{t\to 0} t^{\alpha+1}\sum_{n=1}^{\infty} n^{\alpha}\left|\frac{\sin nt}{nt}\right|^{p} \\
&= \varepsilon\int_{0}^{\infty} u^{\alpha-p}\,|\sin u|^{p}\,du = \varepsilon K,
\end{aligned}$$

say. Since ε is arbitrary, we obtain (2.9) and, using (2.6), (2.8) and (2.9), we obtain (2.4).

LEMMA 4. *If* $-1 \leqq \alpha < p-\delta-1$, $0<\delta<1$, *then we have*

$$G_k(t) = t^{\alpha+1}\sum_{n=k}^{\infty} A_{n-k}^{\alpha+\delta-p}\left(\frac{\sin nt}{nt}\right)^{p} = O(k^{-p}t^{-\delta}). \tag{2.10}$$

PROOF. Let us write

$$G_k(t) = t^{\alpha+1}\left(\sum_{n=k}^{k+\rho} + \sum_{n=k+\rho+1}^{\infty}\right) A_{n-k}^{\alpha+\delta-p}\left(\frac{\sin nt}{nt}\right)^{p} = U(t) + V(t),$$

where $\rho = [t^{-1}]$. Since $p-\alpha-\delta > 1$, using $(\sin nt/nt)^{p} = O(nt)^{-p}$, we have

$$\begin{aligned}
V(t) &= t^{\alpha+1}\sum_{n=k+\rho+1}^{\infty} A_{n-k}^{\alpha+\delta-p}\left(\frac{\sin nt}{nt}\right)^{p} \\
&= O\Big((k+\rho+1)^{-p}\,t^{\alpha-p+1}\sum_{n=k+\rho+1}^{\infty}(n-k)^{\alpha+\delta-p}\Big) \qquad (2.11) \\
&= O(k^{-p}\,t^{\alpha-p+1}\,\rho^{\alpha+\delta-p+1}) \\
&= O(k^{-p}t^{-\delta}).
\end{aligned}$$

Putting $\varphi(t) = (\sin t/t)^p$, by the repeated use of Abel's transformation p-times, we have

$$U(t) = t^{\alpha+1}\sum_{n=0}^{\rho-p} A_n^{\alpha+\delta}\Delta^p\varphi\,(\overline{n+k}\,t) + t^{\alpha+1}\sum_{i=0}^{p-1} A_{\rho-i}^{\alpha+\delta+i-p+1}\Delta^i\varphi(\overline{\rho+k-i}\,t)$$

$$= W(t) + \sum_{i=0}^{p-1} W_i(t),$$

say. Then, using Lemma 1, we have

$$\begin{aligned} W_i(t) &= t^{\alpha+1} A_{\rho-i}^{\alpha+\delta+i-p+1}\,\Delta^i\varphi(\overline{\rho+k-i}\,t) \\ &= O(t^{\alpha+1}\rho^{\alpha+\delta-i+p+1}\cdot t^{i-p}(\rho+k)^{-p}) \\ &= O(k^{-p}t^{-\delta}), \end{aligned} \tag{2.12}$$

when $i = 0, 1, 2, \ldots, p-1$. Again, using Lemma 1, we have

$$\begin{aligned} W(t) &= t^{\alpha+1}\sum_{n=0}^{\rho-p} A_n^{\alpha+\delta}\Delta^p\varphi(\overline{n+k}\,t) \\ &= O\Big(t^{\alpha+1}\sum_{n=0}^{\rho-p} n^{\alpha+\delta}(n+k)^{-p}\Big) \\ &= O(k^{-p}t^{\alpha+1}\rho^{\alpha+\delta+1}) \\ &= O(k^{-p}t^{-\delta}). \end{aligned} \tag{2.13}$$

Summing up (2.11), (2.12) and (2.13), we have (2.10).

LEMMA 5. *Under the conditions of Theorem* 4, *the series* $\sum_{k=1}^{\infty} a_k$ *is evaluable* $(C, 1)$ *to* s *and* $s_n = O(1)$ *and further*

$$\sum_{k=n}^{\infty} |a_k|/k = O(n^{-1}). \tag{2.14}$$

LEMMA 6. *If* (1.7) *and* (1.8) *hold, then we have*

$$s_n = O(n^{1-r}) \tag{2.15}$$

and

$$\sum_{k=n}^{\infty} |a_k|/k = O(n^{-r}). \tag{2.16}$$

Lemmas 5 and 6 are due to Szász [5].

LEMMA 7. *Let* $0 \leqq \delta \leqq 1$. *Then, we have*

$$H_k(t) = t^{-\delta}\sum_{n=k}^{\infty} A_{n-k}^{-\delta-1}\frac{\sin nt}{n} = O(k^{-1}). \tag{2.17}$$

PROOF. Since (2.17) when $\delta = 0, 1$ are obvious, we shall consider the case $0 < \delta < 1$. Let us write

$$t^{-\delta}\sum_{n=k}^{\infty} A_{n-k}^{-\delta-1}\frac{\sin nt}{n} = t^{-\delta}\Big(\sum_{n=k}^{\rho+k} + \sum_{n=\rho+k+1}^{\infty}\Big)A_{n-k}^{-\delta-1}\frac{\sin nt}{n} = U(t) + V(t),$$

where $\rho = [t^{-1}]$. Then, we have, by the Abel transformation,

$$U(t) = t^{-\delta} \sum_{n=k}^{k+\rho-1} A_{n-k}^{-\delta} \left(\frac{\sin nt}{n} - \frac{\sin (n+1)t}{n+1} \right) + t^{-\delta} A_\rho^{-\delta} \frac{\sin (k+\rho)t}{k+\rho}$$
$$= O\left(t^{-\delta} \cdot k^{-1} t \cdot \sum_{n=k}^{k+\rho-1} A_{n-k}^{-\delta} \right) + O\,(t^{-\delta} \rho^{-\delta} (k+\rho)^{-1})$$
$$= O(k^{-1} t^{-\delta+1} \rho^{-\delta+1}) + O(k^{-1})$$
$$= O(k^{-1}),$$

and

$$V(t) = t^{-\delta} \sum_{n=k+\rho+1}^{\infty} A_{n-k}^{-\delta-1} \frac{\sin nt}{n}$$
$$= O(t^{-\delta}(k+\rho)^{-1}(\rho+1)^{-\delta})$$
$$= O(k^{-1}).$$

Thus we have (2.17).

LEMMA 8. *Let* $0 \leqq \delta \leqq 1$. *Then we have*

$$\Delta^\gamma H_k(t) = O(k^{-1} t^\gamma), \tag{2.18}$$

where γ *is a nonnegative integer and* $\Delta^\gamma H_k(t)$ *is* γ*-th difference of* $H_k(t)$ *with respect to* k.

PROOF. Lemma when $\delta = 0, 1$ is due to Hirokawa and Sunouchi [3]. Hence, we shall restrict the case $0 < \delta < 1$. Let us write

$$\Delta^\gamma H_k(t) = \Delta^\gamma \left\{ t^{-\delta} \sum_{n=0}^{\infty} A_n^{-\delta-1} \frac{\sin(n+k)t}{n+k} \right\}$$
$$= t^{-\delta} \sum_{n=0}^{\infty} A_n^{-\delta-1} \Delta^\gamma \left(\frac{\sin (n+k)t}{n+k} \right)$$
$$= t^{-\delta} \left(\sum_{n=0}^{\rho} + \sum_{n=\rho+1}^{\infty} \right) A_n^{-\delta-1} \Delta^\gamma \left(\frac{\sin (n+k)t}{n+k} \right)$$
$$= U(t) + V(t),$$

where $\rho = [t^{-1}]$. Since $\Delta^\gamma (\sin nt/n) = O(n^{-1} t^\gamma)$, we have

$$U(t) = t^{-\delta} \sum_{n=0}^{\rho-1} A_n^{-\delta} \Delta^{\gamma+1} \left(\frac{\sin (n+k)t}{n+k} \right) + t^{-\delta} A_\rho^{-\delta} \Delta^\gamma \left(\frac{\sin (\rho+k)t}{\rho+k} \right)$$
$$= O\left(t^{-\delta} \cdot k^{-1} t^{\gamma+1} \sum_{n=0}^{\rho-1} A_n^{-\delta} \right) + O(t^{-\delta} \rho^{-\delta} (\rho+k)^{-1} t^\gamma)$$
$$= O(k^{-1} t^{\gamma-\delta+1} \rho^{1-\delta}) + O(k^{-1} t^\gamma)$$
$$= O(k^{-1} t^\gamma)$$

and

$$V(t) = O\left(t^{-\delta} \cdot k^{-1} t^\gamma \cdot \sum_{n=\rho+1}^{\infty} |A_n^{-\delta-1}| \right)$$
$$= O(k^{-1} t^{\gamma-\delta} \rho^{-\delta})$$
$$= O(k^{-1} t^\gamma).$$

Thus, we have the required.

LEMMA 9. *Let* $0 \leqq \delta \leqq 1$. *Then, we have*

$$\eta_m(t) = \sum_{k=m}^{\infty} H_k(t) = O(m^{-1}t^{-1}). \tag{2.19}$$

PROOF. It is known that

$$r_m = \sum_{n=m}^{\infty} \frac{\sin nt}{n} = O(m^{-1}t^{-1}).$$

See, for example, Hardy and Rogosinski [2, p. 29]. This proves Lemma when $\delta = 0$. When $\delta = 1$, Lemma is obvious. For the case $0 < \delta < 1$, we shall write

$$\begin{aligned}\sum_{k=m}^{\infty} H_k(t) &= t^{-\delta}\sum_{n=0}^{\infty} A_n^{-\delta-1} \sum_{k=m}^{\infty} \frac{\sin(n+k)t}{n+k} \\ &= t^{-\delta}\Big(\sum_{n=0}^{\rho} + \sum_{n=\rho+1}^{\infty}\Big) A_n^{-\delta-1} r_{m+k} \\ &= U(t) + V(t),\end{aligned}$$

say, where $\rho = [t^{-1}]$. Here, we may easily see that this rearrangement is permissable. We have, by the Abel transformation

$$\begin{aligned}U(t) &= t^{-\delta}\sum_{n=0}^{\rho-1} A_n^{-\delta} \frac{\sin(m+n)t}{m+n} + t^{-\delta} A_\rho^{-\delta} r_{m+\rho} \\ &= O(t^{-\delta}\cdot\rho^{1-\delta}\cdot m^{-1}) + O(t^{-\delta}\cdot\rho^{-\delta}\cdot(m+\rho)^{-1}t^{-1}) \\ &= O(m^{-1}t^{-1}),\end{aligned}$$

and

$$\begin{aligned}V(t) &= t^{-\delta} \sum_{n=\rho+1}^{\infty} A_n^{-\delta-1} r_{m+n} \\ &= O(t^{-\delta}(m+\rho)^{-1}t^{-1}\cdot\rho^{-\delta}) \\ &= O(m^{-1}t^{-1}).\end{aligned}$$

Thus, we have (2.19).

3. Proof of Theorem 1. We may suppose, without loss of generality by Lemma 3, that $s = 0$. It is known, from (1.2), that

$$s_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha+\delta-p}\, s_k^{p-\delta-1},$$

where A_n^α is Andersen's notation and $s_0 = 0$. Hence we have, putting $\varphi(t) = (\sin t/t)^p$,

$$\begin{aligned}t^{\alpha+1}\sum_{n=1}^{\infty} s_n^\alpha\, \varphi(nt) &= t^{\alpha+1} \sum_{n=1}^{\infty} \varphi(nt) \sum_{k=0}^{n} A_{n-k}^{\alpha+\delta-p}\, s_k^{p-\delta-1} \\ &= \sum_{k=0}^{\infty} s_k^{p-\delta-1}\, t^{\alpha+1} \sum_{n=k}^{\infty} A_{n-k}^{\alpha+\delta-p}\, \varphi(nt) \\ &= \sum_{k=0}^{\infty} s_k^{p-\delta-1}\, G_k(t),\end{aligned} \tag{3.1}$$

say. Here, we shall prove that this rearrangement is permissible. For this purpose, it is sufficient to show that, for fixed $t > 0$,

$$t^{\alpha+1}\sum_{k=0}^{N} s_k^{p-\delta-1}\sum_{n=N+1}^{\infty} A_{n-k}^{\alpha+\delta-p}\,\varphi(nt)=o(1)\quad\text{as } N\to\infty. \tag{3.2}$$

Since

$$\begin{aligned} t^{\alpha+1}\sum_{n=N+1}^{\infty} A_{n-k}^{\alpha+\delta-p-1}\,\varphi(nt) &= O\Big(\sum_{n=N+1}^{\infty}(n-k)^{\alpha+\delta-p-1}\cdot n^{-p}\Big)\\ &= O(N^{-p}(N-k+1)^{\alpha+\delta-p}),\end{aligned}$$

we have, using the Abel transformation,

$$\begin{aligned} & t^{\alpha+1}\sum_{k=0}^{N} s_k^{p-\delta-1}\sum_{n=N+1}^{\infty} A_{n-k}^{\alpha+\delta-p}\varphi(nt)\\ &= t^{\alpha+1}\sum_{k=0}^{N-1} s_k^{p-\delta}\sum_{n=N+1}^{\infty} A_{n-k-1}^{\alpha+\delta-p-1}\,\varphi(nt)+t^{\alpha+1}s_N^{p-\delta}\sum_{n=N+1}^{\infty}A_{n-N-1}^{\alpha+\delta-p}\,\varphi(nt)\\ &= o\Big(\sum_{k=0}^{N-1} k^{p-\delta}\cdot N^{-p}(N-k+1)^{\alpha+\delta-p}\Big)+o(N^{p-\delta}N^{-p})\\ &= o(N^{\alpha-p-1})+o(N^{-\delta})\\ &= o(1),\end{aligned}$$

when $\alpha<p-\delta-1$. Thus (3.2) is proved.

Therefore, for the proof, it is sufficient to prove that the series (3.1) converges in $0<t<t_0$ and its sum tends to zero as t tends to zero. Let us write

$$\sum_{k=0}^{\infty} s_k^{p-\delta-1}G_k(t)=\Big(\sum_{k=0}^{m}+\sum_{k=m+1}^{\infty}\Big)=U_1(t)+U_2(t),$$

where $m=[N/t]$ and N is an arbitrary fixed positive number. Using Lemma 2, we have, by (1.5),

$$\sum_{k=m+1}^{\infty}|\sigma_k^{p-\delta-1}|k^{-1-\delta}=O(m^{-\delta}) \tag{3.3}$$

and hence, using (1.1) and Lemma 4,

$$\begin{aligned} U_2(t) &= \sum_{k=m+1}^{\infty}\sigma_k^{p-\delta-1}A_k^{p-\delta-1}G_k(t)\\ &= O\Big(\sum_{k=m+1}^{\infty}|\sigma_k^{p-\delta-1}|k^{-1-\delta}t^{-\delta}\Big)\\ &= O(t^{-\delta}m^{-\delta})\\ &= O(N^{-\delta}).\end{aligned} \tag{3.4}$$

Thus, the series (3.1) converges for all t. By Abel's transformation, we have

$$\begin{aligned} U_1(t) &= \sum_{k=0}^{m} s_k^{p-\delta-1}G_k(t)\\ &= \sum_{k=0}^{m-1} s_k^{p-\delta}(G_k(t)-G_{k+1}(t))+s_m^{p-\delta}G_m(t),\end{aligned}$$

where
$$G_k(t)-G_{k+1}(t)=t^{\alpha+1}\sum_{n=0}^{\infty}A_n^{\alpha+\delta-p}\Delta\varphi(\overline{n+k}\,t)$$
$$=O(k^{-p}t^{1-\delta})$$

by the method analogous to the proof of Lemma 4. Hence we have, from Lemma 4 and summability $(C, p-\delta)$ of the series $\sum_{k=1}^{\infty}a_k$,

$$U_1(t)=o\left(\sum_{k=0}^{m-1}k^{p-\delta}k^{-p}t^{1-\delta}\right)+o(m^{p-\delta}m^{-p}t^{-\delta})$$
$$=o(m^{1-\delta}t^{1-\delta})+o(m^{-\delta}t^{-\delta})$$
$$=o(N^{1-\delta})+o(N^{-\delta})$$
$$=o(1),$$

for each arbitrary fixed N. Therefore, by (3. 3), (3. 4) and (3. 5), we have

$$\sum_{k=0}^{\infty}s_k^{p-\delta-1}G_k(t)=U_1(t)+U_2(t)$$
$$=o(1)+O(N^{-\delta}),$$

as t tends to zero, Since N is arbitrary, we have

$$\lim_{t\to 0}\sum_{k=0}^{\infty}s_k^{p-\delta-1}G_k(t)=0,$$

and the proof is complete.

4. Proof of Theorem 2. For the proof, we need the following:

LEMMA 10. *For each* δ, $0<\delta<1$, *there existsa series* $\sum_{n=1}^{\infty}a_n$ *such that* (i) *the series* $\sum_{n=1}^{\infty}a_n$ *converges to zero and* (ii) *the limit*

$$\lim_{t\to 0}t^{-\delta}\sum_{n=1}^{\infty}a_n\frac{\sin nt}{n}$$

does not exist.

This Lemma is due to Hardy and Littlewood [1] when $\delta=1$.

PROOF. We shall prove Lemma by Hardy and Littlewood's method. If we put $q_{\nu+1}=\exp(\nu q_\nu^\delta)$ and $q_0=1$, then we can see easily that

$$q_\nu\uparrow,\ q_{\nu+1}/q_\nu\to\infty\quad\text{and}\quad\log(q_{\nu+1}/q_\nu)/q_\nu^\delta\to\infty.$$

From this sequence $\{q_\nu\}$, we may construct a sequence of positive integers $\{n_\nu\}$ such that

$$n_\nu\uparrow,\ m_\nu=n_{\nu+1}/n_\nu\to\infty\quad\text{and}\quad(\log m_\nu)/n_\nu^\delta\to\infty,$$

where m's are integers. Further, we may find a sequence of positive integers $\{k_\nu\}$ such that

$$k_\nu\uparrow,\ k_{\nu+1}/k_\nu\to 1\quad\text{and}\quad(\log m_\nu)/k_\nu^{1-\delta}\,n_\nu^\delta\to\infty.$$

Then, if we put $t_\nu=2\pi k_\nu/n_\nu$, we have $\lim_{\nu\to\infty}t_\nu=0$. Now, we shall define our series $\sum_{\nu=1}^{\infty} a_\nu$ by

$$a_n = \frac{1}{n_\nu} \sin nt_\nu = \frac{1}{n_\nu} \sin \frac{2nk_\nu \pi}{n_\nu}$$

for $n_\nu \leqq n < n_{\nu+1}$. Then, we may prove Lemma by the method analogous to the proof of the Hardy and Littlewood lemma, so that we omit the proof in detail.

PROOF OF THEOREM 2. When $\alpha = -1, 0$, Theorem is well known. Hence we shall restict to the case $-1 < \alpha < 0$. When the series $\sum_{\nu=1}^{\infty} a_\nu$ is defined as in Lemma 10 for $\delta = -\alpha$, we find a series $\sum_{\nu=1}^{\infty} b_\nu$ such that $B_n^\alpha = a_n$, where B_n^α is (C, α) sums of the series $\sum_{\nu=0}^{\infty} b_\nu$ with $b_0 = 0$. Then, putting $s_n = \sum_{\nu=1}^{n} a_\nu$, we have

$$\begin{aligned}\sum_{n=1}^{n} b_n = B_n^0 &= \sum_{\nu=0}^{n} A_{n-\nu}^{-\alpha-1} B_\nu^\alpha \\ &= \sum_{\nu=0}^{n} A_{n-\nu}^{-\alpha-1} a_\nu \\ &= \sum_{\nu=0}^{n-1} A_{n-\nu}^{-\alpha-2} s_\nu + A_0^{-\alpha-2} a_n \\ &= o\left(\sum_{\nu=1}^{n} \nu^{-\alpha-2}\right) + o(1) \\ &= o(1).\end{aligned}$$

Hence, the series $\sum_{\nu=1}^{\infty} b_\nu$ converges to zero. But, by Lemma 10, we see that

$$\lim_{t\to 0} t^\alpha \sum_{n=1}^{\infty} a_n \frac{\sin nt}{n} = \lim_{t\to 0} t^{\alpha+1} \sum_{n=1}^{\infty} B_n^\alpha \frac{\sin nt}{nt}$$

does not exists. Therefore, the series $\sum_{\nu=1}^{\infty} b_\nu$ is not evaluable $(R, 1, \alpha)$ and the proof is complete.

5. **Proof of Theorem 3.** Putting $\varphi(t) = (\sin t/t)^p$, by the repeated use of Abel's transformation $(p-\alpha)$-times, we have

$$(5.1) \qquad t^{\alpha+1} \sum_{k=1}^{n} s_k^\alpha \varphi(kt) = t^{\alpha+1} \sum_{k=1}^{n-p+\alpha} s_k^p \Delta^{p-\alpha}\varphi(kt) + t^{\alpha-1} \sum_{i=1}^{p-\alpha} s_{n-i+1}^{\alpha+i} \Delta^{i-1}\varphi(\overline{n-i+1}\, t).$$

We may suppose, without loss of generality, that $s = 0$. Since summability

$|C,p|$ implies summability (C,p), we have $s_n^i = o(n^p)$ when $0 \leqq i \leqq p$. Then, using Lemma 1, we have

$$t^{\alpha+1}\sum_{i=1}^{p-\alpha} s_{n-i+1}^{\alpha+i}\,\Delta^{i-1}\,\varphi(\overline{n-i+1}\;t)$$
$$= o\left(t^{\alpha+1}\sum_{i=1}^{p-\alpha}(n-i+1)^p\cdot(n-i+1)^{-p}t^{i-p-1}\right)$$
$$= o(1),$$

as $n\to\infty$, for fixed $t>0$. Therefore, for the proof it is sufficient to prove that the series

$$t^{\alpha+1}\sum_{k=1}^{\infty} s_k^p\,\Delta^{p-\alpha}\varphi(kt) \tag{5.2}$$

converges in $0<t<t_0$, and its sum tends to zero as $t\to 0$. Using (1.1) and Abel's transformation, we have

$$t^{\alpha+1}\sum_{k=n}^{m} s_k^p\,\Delta^{p-\alpha}\varphi(kt) = t^{\alpha+1}\sum_{k=n}^{m}\sigma_k^p A_k^p\,\Delta^{p-\alpha}\varphi(kt)$$
$$= \sum_{k=n}^{m-1} U_k(t)\Delta\sigma_k^p + U_m(t)\,\sigma_m^p - U_{n-1}(t)\,\sigma_n^p, \tag{5.3}$$

where $U_n(t) = t^{\alpha+1}\sum_{k=1}^{n} A_k^p\;\;\Delta^{p-\alpha}\varphi(kt)$ and $\Delta\sigma_k^p = \sigma_k^p - \sigma_{k+1}^p$. Now, we shall show that $U_n(t)$ is uniformly bounded in $0<t<\pi$ and for all n. If $nt\leqq 1$, then, using Lemma 1, we have

$$U_n(t) = t^{\alpha+1}\sum_{k=1}^{n} A_k^p\,\Delta^{p-\alpha}\varphi(kt)$$
$$= O\left(t^{\alpha+1}\sum_{k=1}^{n} k^p\cdot k^{-p}\,t^{-\alpha}\right)$$
$$= O(nt)$$
$$= O(1).$$

On the other hand, following Obreschkoff, we shall consider the series $1 + 0 + 0 + \ldots\ldots$ Concerning this series, we have

$$s_n^\beta = A_n^\beta,\qquad \sigma_n^\beta = 1\qquad (n=1,2,\ldots\ldots;\ \beta>-1)$$

Hence, from (5.1), we have

$$t^{\alpha+1}\sum_{k=1}^{n} A_k^\alpha\,\varphi(kt) = t^{\alpha+1}\sum_{k=1}^{n-p+\alpha} A_k^\alpha\,\Delta^{p-\alpha}\varphi(kt)$$
$$+\,t^{\alpha+1}\sum_{i=1}^{p-\alpha} A_{n-i+1}\,\Delta^{i-1}\varphi(\overline{n-i+1}\;t),$$

that is,

$$t^{\alpha+1}\sum_{k=1}^{n} A_k^\alpha\varphi(kt) = U_{n-p+\alpha}(t) + t^{\alpha+1}\sum_{i=1}^{p-\alpha} A_{n-i+1}^{\alpha+i}\,\Delta^{i-1}\,\varphi(\overline{n-i+1}\;t),$$

where

$$\begin{aligned} & t^{\alpha+1}\sum_{i=1}^{p-\alpha} A_{n-i+1}^{\alpha+i}\Delta^{i-1}\varphi(\overline{n-i+1}\ t) \\ & = O\Big(t^{\alpha+1}\sum_{i=1}^{p-\alpha}(n-i+1)^{\alpha+i}\cdot(n-i+1)^{-p}t^{i-p-1}\Big) \\ & = \sum_{i=1}^{p-\alpha} O((nt)^{\alpha+i-p}) \\ & = O(1) \end{aligned}$$

when $nt > 1$. Putting $\rho = [t^{-1}]$, we have

$$\begin{aligned} t^{\alpha+1}\sum_{k=1}^{n} A_k^{\alpha}\varphi(kt) &= t^{\alpha+1}\Big(\sum_{k=1}^{\rho} + \sum_{k=\rho+1}^{n}\Big) A_k^{\alpha}\ \varphi(kt) \\ &= O\Big(t^{\alpha+1}\sum_{k=1}^{\rho} k^{\alpha}\Big) + O\Big(t^{\alpha+1}\sum_{k=\rho+1}^{\infty} t^{-p}\cdot k^{\alpha-p}\Big) \\ &= O((\rho t)^{\alpha+1}) + O(t^{\alpha-p+1}\rho^{\alpha-p+1}) \\ &= O(1) \end{aligned}$$

when $p-\alpha > 1$. And, when $p = 1$, $\alpha = 0$, this result is well known. See, for example, Hardy and Rogosinski [2, p. 29]. Thus, $U_n(t)$ is uniformly bounded in $0 < t < \pi$ and for all n. Then, using above results and (5. 3), we have

$$t^{\alpha+1}\sum_{k=n}^{m} s_k^p\,\Delta^{p-\alpha}\varphi(kt) = O\Big(\sum_{k=n}^{m}|\Delta\sigma_k^p|\Big) + O(|\sigma_m^p|) + O(|\sigma_n^p|) = o(1) \tag{5. 4}$$

as $m, n \to \infty$, by our assumption. Hence the series (5. 2) converges in $0 < t < \pi$. Since $o(1)$ in (5. 4) is uniform in $0 < t < \pi$, for an arbitrary small $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that

$$\Big|t^{\alpha+1}\sum_{k=N}^{\infty} s_k^p\,\Delta^{p-\alpha}\varphi(kt)\Big| < \varepsilon.$$

Further, we have

$$\lim_{t\to 0} t^{\alpha+1}\sum_{k=1}^{N-1} s_k^p\,\Delta^{p-\alpha}\varphi(kt) = 0,$$

when $\alpha \geqq -1$. Therefore we have

$$\limsup_{t\to 0}\Big|t^{\alpha+1}\sum_{k=1}^{\infty} s_k^p\Delta^{p-\alpha}\varphi(kt)\Big| \leqq \varepsilon.$$

Since ε is arbitrary, we have

$$\lim_{t\to 0} t^{\alpha+1}\sum_{k=1}^{\infty} s_k^p\,\Delta^{p-\alpha}\varphi(kt) = 0,$$

which is the required, and the proof is complete.

6. Proof of Theorem 4. We may suppose, without loss of generality, that $s=0$. By Lemma 5, the series $\sum_{k=1}^{\infty} a_k$ is evaluable $(C, 1)$ to zero: hence

$$s_n^1 = \sum_{k=1}^{n} s_n = o(n). \tag{6.1}$$

Since we have, from (1.2),

$$s_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k,$$

we have

$$t^\alpha \sum_{n=1}^{\infty} s_n^\alpha \frac{\sin nt}{n} = t^\alpha \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k \right) \frac{\sin nt}{n}$$

$$= \sum_{k=0}^{\infty} s_k \left(t^{-\delta} \sum_{n=k}^{\infty} A_{n-k}^{\alpha-1} \frac{\sin nt}{n} \right) \tag{6.2}$$

$$= \sum_{k=0}^{\infty} s_k H_k(t), \tag{6.3}$$

say. Here, we shall prove that this rearrangement is permissble. For this purpose, it is sufficient to prove that, for fixed $t > 0$,

$$t^\alpha \sum_{k=0}^{N} s_k \sum_{n=N+1}^{\infty} A_{n-k}^{\alpha-1} \frac{\sin nt}{n} = o(1) \tag{6.4}$$

as $N \to \infty$. Since

$$\sum_{n=N+1}^{\infty} A_{n-k}^{\alpha-2} \frac{\sin nt}{n} = O\left(\sum_{n=N+1}^{\infty} (n-k)^{\alpha-2} \cdot n^{-1} \right)$$
$$= O(N^{-1}(N-k+1)^{\alpha-1}),$$

we have, using Abel's transformation and (6.1),

$$t^\alpha \sum_{k=0}^{N} s_k \sum_{k=N+1}^{\infty} A_{n-k}^{\alpha-1} \frac{\sin nt}{n}$$
$$= t^\alpha \sum_{k=0}^{N-1} s_k^1 \sum_{n=N+1}^{\infty} A_{n-k-1}^{\alpha-2} \frac{\sin nt}{n} + t^\alpha s_N^1 \sum_{n=N+1}^{\infty} A_{n-N-1}^{\alpha-1} \frac{\sin nt}{n}$$
$$= o\left(\sum_{k=0}^{N-1} k \cdot N^{-1}(N-k+1)^{\alpha-1} \right) + o(N \cdot N^{-1})$$
$$= o(1),$$

and (6.4) is proved.

Here we remark that *the rearrangement* (3.2) *is permissble when* $s_n^1 = o(n)$.

Therefore, for the proof, it is sufficient to prove that the series (6.3) converges in some interval $0 < t < t_0$, and its sum tends to zero as $t \to 0$. Convergence of the series (6.3) follows from the estimation of $V(t)$ below. Let us write

$$\sum_{k=1}^{\infty} s_k H_k(t) = \left(\sum_{k=1}^{\rho} + \sum_{k=\rho+1}^{\infty} \right) = U(t) + V(t),$$

where $\rho = [(\varepsilon t)^{-1}]$, ε being an arbitrary fixed positive number. By (2. 17), (2. 18) and (6. 1), we have

$$\begin{aligned} U(t) &= \sum_{k=1}^{\rho-1} s_k^1 \, \Delta H_k(t) + s_\rho^1 H_\rho(t) \\ &= o\Big(\sum_{k=1}^{\rho-1} k\cdot k^{-1}t\Big) + o(\rho\cdot\rho^{-1}) \\ &= o(\rho t) + o(1) \\ &= o(1). \end{aligned}$$

Since $\eta_n(t) = \sum_{k=n}^{\infty} H_k(t) = O(n^{-1}t^{-1})$ by (2. 19), we have

$$\begin{aligned} V(t) &= \sum_{k=\rho+1}^{\infty} s_k H_k(t) \\ &= -\sum_{k=\rho+1}^{\infty} a_{k+1}\, \eta_{k+1}(t) + s_{\rho+1}\eta_{\rho+1}(t) \\ &= O\Big(\sum_{k=\rho+1}^{\infty} \frac{|a_{k+1}|}{k+1}\, t^{-1}\Big) + O\Big(\frac{1}{\rho t}\Big) \\ &= O(\rho^{-1}t^{-1}) \\ &= O(\varepsilon), \end{aligned}$$

by Lemma 5. Thus, we have

$$\limsup_{t\to 0} |U(t) + V(t)| = O(\varepsilon).$$

Since ε is arbitrary, we have

$$\lim_{t\to 0} (U(t) + V(t)) = 0,$$

and the proof is complete.

7. **Proof of Theorem** 5. We may suppose, without loss of generality, that $s = 0$. By (1. 8), the series $\sum_{k=1}^{\infty} a_k$ is evaluable $(C, 1)$ to zero; hence $s_n^1 = \sum_{k=1}^{n} s_k = o(n)$ and, by the remark in § 6, the rearrangement (6. 2) is permissible. Convergence of the series (6. 3) follows from the estimation of $W(t)$ below. Let us write

$$\sum_{k=1}^{\infty} s_k H_k(t) = \Big(\sum_{k=1}^{h} + \sum_{k=h+1}^{j} + \sum_{k=j+1}^{\infty}\Big) = U(t) + V(t) + W(t),$$

where $h = [t^{-1}]$, $j = [t^{-\beta}]$ and $r\beta > 1$. Then we have, by Lemmas 7 and 8,

$$\begin{aligned} U(t) &= \sum_{k=1}^{h} s_k H_k(t) \\ &= \sum_{k=1}^{h-1} s_k^1 \, \Delta H_k(t) + s_h^1 H_h(t) \end{aligned}$$

$$= o\Big(\sum_{k=1}^{h-1} \frac{k}{\log (k+1)} \cdot \frac{t}{k}\Big) + o\Big(\frac{h}{\log h} \cdot \frac{1}{h}\Big)$$
$$= o(ht/\log h) + o(1)$$
$$= o(1).$$

If we put $S_n = \sum_{k=1}^{n} |s_k|$ we have using (1.8) and (2.17)

$$|V(t)| \leqq \sum_{k=h+1}^{j} |s_k|\,|H_k(t)|$$
$$= O\Big(\sum_{k=h+1}^{j} |s_k| k^{-1}\Big)$$
$$= O\Big(\sum_{k=h+1}^{j-1} S_k \Big(\frac{1}{k} - \frac{1}{k+1}\Big) + S_j \frac{1}{j} - S_h \frac{1}{h+1}\Big)$$
$$= o\Big(\sum_{k=h+1}^{j-1} \frac{k}{\log k} \cdot \frac{1}{k^2}\Big) + o\Big(\frac{j}{\log j} \cdot \frac{1}{j}\Big) + o\Big(\frac{h}{\log h} \cdot \frac{1}{h+1}\Big)$$
$$= o(\log \beta) + o(1)$$
$$= o(1).$$

Lastly, by (2.16) and (2.19), we have

$$W(t) = \sum_{k=j+1}^{\infty} s_k H_k(t)$$
$$= -\sum_{k=j+1}^{\infty} a_{k+1}\, \eta_{k+1}(t) + s_{j+1}\eta_{j+1}(t)$$
$$= O\Big(t^{-1} \sum_{k=j}^{\infty} \frac{|a_k|}{k}\Big) + O\Big(j^{1-r} \cdot \frac{1}{jt}\Big)$$
$$= O(t^{-1} j^{-r})$$
$$= O(t^{\beta r - 1})$$
$$= o(1),$$

and the proof of Theorem is complete.

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