ON THE THEORY OF UNIVALENT FUNCTIONS

TOSHIO UMEZAWA

(Received November 22, 1954)

1. Introduction. It is the purpose of the present paper to obtain sufficient conditions that $f(z)$ regular or meromorphic in a given region be univalent or multivalent in the region. For this purpose the convexity and concavity of the image curves will be used efficiently.

A necessary and sufficient condition for the convexity of the function $f(z) = z + a_2z^2 + \dots$ regular for $|z| < r$ is known to be [1]

$$(1) \qquad 1 + \Re \frac{zf''(z)}{f'(z)} > 0 \text{ for } |z| < r.$$

However, for the univalency of the above function $f(z)$, it is sufficient that $f(z)$ is convex in one direction [2] and we have the following result: [3, 4. 8].

THEOREM *A*. *Let $f(z) = z + a_2z^2 + \dots$ be regular for $|z| \leqq 1$ and $f'(z) \neq 0$ on $|z| = 1$. If there holds the relation*

$$(2) \qquad \int_0^{2\pi} \left| 1 + \Re \frac{zf''(z)}{f'(z)} \right| d\theta < 4\pi, \quad |z| = 1,$$

then $f(z)$ is convex in one direction and hence $f(z)$ is univalent in $|z| \leqq 1$.

Some of the conditions which will be given in this paper contain the above theorem as a special case and have the form analogous to the above one. But in our present case $f(z)$ is not necessarily convex in one direction. The univalency of $f(z)$ will be deduced from a geometrical fact, more general than the convexity in one direction. This geometrical fact (Lemma 1) will be stated in §2 which is fundamental in our investigation.

Making use of the same lemma, we shall also extend or make more precise the following well-known:

THEOREM *B*. *If $f(z)$ is regular in a convex region D and if $\Re f'(z) > 0$ in D, then $f(z)$ is univalent in D.*

This is due to K. Noshiro [5] and J. Wolff [6], Their methods of proof were very elegant. However it seems to me that the methods are hardly useful for the purpose of extending Theorem B to the case of multiply connected domain. Our method is powerful enough to enable us to succeed in the work.

Furthermore we shall give a new generalization of Theorem B to the case of p-valence.

2. The fundamental lemma.

LEMMA 1. *Let $w = f(z)$ be regular in a simply connected closed region D_z whose boundary Γ_z consists of a regular curve and $f'(z) \neq 0$ on Γ_z. If there holds one of the following conditions;*

(i) *For arbitrary arcs C_z on Γ_z*

$$\int_{C_z} d \arg df(z) > -\pi \tag{2.1}$$

and

$$\int_{\Gamma_z} d \arg df(z) = 2\pi, \tag{2.2}$$

(ii) *For arbitrary arcs C_z on Γ_z*

$$\int_{C_z} d \arg df(z) < 3\pi, \tag{2.3}$$

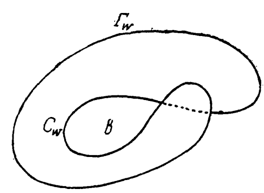
then $f(z)$ is univalent in D_z.

PROOF. Let D_w and Γ_w be the images of D_z and Γ_z respectively.

(i) There exists no branch-point in D_w since we have (2.2) by Morse-Heins' theorem [7].

Suppose that $f(z)$ is n-valent in D_z and that $n \geqq 2$, then D can be considered as a one-sheeted region on an at least n-sheeted Riemann surface S. It is evident that if D_w encircles no branch-point of S, namely if any curve connecting any two points in D_w encircles no branch-point of S, then D_w has no overlapping part. Hence D_w encircles at least one branch-point B of S without including it, since D has some overlapping parts and since D_w include no branch-point by our assumption. The inner boundary of this encircling part of D_w makes an arc (or a loop) C_w for which

$$\int_{C_w} d \arg dw \leqq -\pi$$



holds, since the positive direction on Γ_w coincides with the clockwise direction on C_w. Namely, it is necessary that there exists at least an arc C_z on Γ_z for which

$$\int_{C_z} d \arg df(z) \leqq -\pi$$

if $f(z)$ is at least two valent and if we have (2.2). Hence $f(z)$ is univalent if we have the condition (i).

(ii) Since we have (2.3) and since $\frac{1}{2\pi} \int_{\Gamma_z} d \arg df(z)$ is a positive integer, we obtain (2.2). Subtracting (2.3) from (2.2) we obtain (2.1) for arbitrary arcs C_z on Γ_z, which proves the case (ii) by using the condition (i).

3. The fundamental theorems.

THEOREM 1. *Let $w=f(z)$ be regular for a closed domain D_z whose boundary Γ_z be a simple closed regular curve and $f'(z)\neq 0$ on Γ_z. Let z_i and z_j, $i,j=1,2,\dots$ be the roots of the equation*

$$d \arg df(z)=0 \text{ on } \Gamma_z.$$

If there holds one of the following conditions:

(i) $$\operatorname*{Max}_{i,j}\int_{z_j}^{z_i} d\arg df(z)<3\pi,\ z\in\Gamma_z:$$

(ii) $$\int_{\Gamma_z} d\arg df(z)=2\pi$$

and

$$\operatorname*{Min}_{i,}\int_{z_i}^{z_j} d\arg df(z)>-\pi,$$

then $f(z)$ is univalent in D_z.

PROOF. By Lemma 1, $f(z)$ is univalent, if we have

(3.1) $$\int_{y}^{x} d\arg df(z)<3\pi \qquad z\in\Gamma_z$$

for arbitrary x and y belonging to Γ_z.

On the other hand, the maximum of the integral in (3.1) occurs only when $f(x)$ and $f(y)$ are points of inflexion on Γ_w, the image of Γ_z. Namely it occurs when x and y are the zeros of $d\arg df(z)=0$ on Γ_z. Hence $f(z)$ is univalent if we have the condition (i).

Analogous reasoning with condition (i) of Lemma 1 yields the proof of the case (ii), which may be omitted here.

THEOREM 2. *Let $f(z)$ be regular and $f'(z)\neq 0$ in a closed convex domain D whose boundary L be a regular curve. Further let z_i, $i=1,2,\dots\dots$ be the roots of the equation*

(3.2) $$\frac{d\arg df(z)}{d\arg dz}=0 \text{ on } L.$$

If there hold the relations

(3.3) $$\Re\, e^{i\alpha} f'(z_i)>0 \qquad (\alpha: \text{a real constant})$$

for all z_i, then $f(z)$ is univalent in D.

PROOF. Since D is a convex domain, $d\arg dz\geqq 0$ on L. Hence the equation (3.2) is equivalent to the equation $d\arg df(z)=0$ on L.

Now since $f'(z)\neq 0$ in D, $\arg f'(z)$ is one-valued in D. Accordingly $\arg df(z)$ is also one-valued on L if we take a suitable branch of $\arg dz$ since

$$\arg df(z)=\arg f'(z)+\arg dz.$$

By noticing this fact and by the assumption (3.3), we have

(3.4) $$-\frac{\pi}{2} < \alpha + \arg df(z_i) - \arg dz_i < \frac{\pi}{2}$$

and

(3.5) $$-\frac{\pi}{2} < -\alpha - \arg df(z_j) + \arg dz_j < \frac{\pi}{2}$$

for every z_i and z_j, $i > j$ satisfying (3.3). Hence we have

$$-\pi < \arg df(z_i) - \arg df(z_j) + \arg dz_j - \arg dz_i < \pi$$

where $2\pi \geqq \arg dz_i - \arg dz_j \geqq 0$ since D is a convex domain. Thus we have

$$-\pi < \arg df(z_i) - \arg df(z_j) < 3\pi$$

which is equivalent to

$$-\pi < \int_{z_j}^{z_i} d \arg df(z) < 3\pi.$$

This inequality shows that $f(z)$ is univalent in D by Theorem 1.

COROLLARY 1. *Let $f(z)$ be regular for $|z| \leqq r$. Let θ_i and $\theta_j, i, j = 1, 2, \ldots.$ be the roots of the equation*

(3.6) $$1 + \Re \frac{zf''(z)}{f'(z)} = 0, \quad |z| = r, \quad 0 \leqq \theta \leqq 2\pi.$$

If there holds one of the following conditions ;

(i) $\Re e^{i\alpha} f'(re^{i\theta_j}) > 0$ (α : *a real constant*) *for all θ_j satisfying* (3.6),

(ii) $$\underset{i,j}{Max} \int_{\theta_j}^{\theta_i} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta < 3\pi, \qquad |z| = r,$$

(iii) $f'(z) \neq 0$ *in* $|z| \leqq r$ *and*

$$\underset{i,j}{Min} \int_{\theta_j}^{\theta_i} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi, \qquad |z| = r$$

then $f(z)$ is univalent in $|z| \leqq r$.

PROOF. By a simple calculation we see that

$$1 + \Re \frac{zf''(z)}{f'(z)} = \frac{d \arg df(z)}{d \arg dz} \qquad \text{on } |z| = r,$$

which proves this corollary by Theorems 1 and 2.

REMARK. As an immediate result of Corollary 1 with condition (i) we obtain Theorem B stated in §1.

4. Special cases.

THEOREM 3. *Let $w = f(z)$ be regular for a simply connected closed domain D whose boundary Γ consists of a regular curve and $f'(z) \neq 0$ on Γ. Further let C_1 be the part of Γ on which d arg $df > 0$ and C_2 be the part of Γ on which d arg $df \leqq 0$. If $f(z)$ satisfies for Γ one of the following conditions*

(i)
$$\int_{C_1} d \arg df(z) < 3\pi$$

(ii) $f'(z) \neq 0$ *in D and*
$$\int_{C_2} d \arg df(z) > -\pi.$$

(iii) $f'(z) \neq 0$ *in D and*
$$\int_{\Gamma} |d \arg df(z)| < 4\pi$$

then $f(z)$ *is univalent in D.*

PROOF. Under our assumption and the conditions (i) and (ii) we obtain (2.3) and (2.2) with (2.1) in Lemma 1 for arbitrary arcs C_ν since
$$\int_{C_\nu} d \arg df < \int_{C_1} d \arg df < 3\pi$$
and since
$$\int_{C_\nu} d \arg df > \int_{C_2} d \arg df > -\pi$$
and
$$\int_{\Gamma} d \arg df = 2\pi$$
in view of $f'(z) \neq 0$ in D. Thus $f(z)$ is univalent by Lemma 1, if we have (i) and (ii).

The proof of the case (iii) is as follows:

We have
$$\int_{\Gamma} |d \arg df| = \int_{C_1} d \arg df - \int_{C_2} d \arg df < 4\pi.$$
On the other hand we have
$$\int_{\Gamma} d \arg df = \int_{C_1} d \arg df + \int_{C_2} d \arg df = 2\pi.$$
Hence we have
$$\int_{C_1} d \arg df < 3\pi$$
which completes our proof by the condition (i).

REMARK. As a generalization of Theorem A, Theorem 3 may not be so important. But it should be noted that the method of proof is quite different from that of Theorem A.

As an application of Theorem A we can obtain the following

THEOREM 4. *Let* $f(z) = z + a_2 z^2 + \dots$ *be regular for* $|z| \leqq 1$ *and let*

(4.1) $$|f''(z)| < \sqrt{6}\, |f'(z)| \qquad in\ |z| \leqq 1$$

then $f(z)$ *is univalent in* $|z| \leqq 1$.

PROOF. By our hypothesis (4.1) we have $f'(z) \neq 0$ in $|z| \leqq 1$ and

$$\int_0^{2\pi} \left| \frac{zf''(z)}{f'(z)} \right|^2 d\theta < 12\pi \qquad \text{for } |z| = 1.$$

Hence we have

(4.2) $$\frac{1}{4} \int_0^{2\pi} \left[\left(\frac{zf''}{f'} \right)^2 + 2 \left| \frac{zf''}{f'} \right|^2 + \left(\overline{\frac{zf''}{f'}} \right)^2 \right] d\theta < 6\pi$$

since

$$\int_0^{2\pi} \left(\frac{zf''}{f'} \right)^2 d\theta = 0.$$

(4.2) is equivalent to the following

(4.3) $$\int_0^{2\pi} \left(\Re \frac{zf''}{f'} \right)^2 d\theta < 6\pi$$

which is also equivalent to the following inequality

$$\int_0^{2\pi} \left(1 + \Re \frac{zf''}{f'} \right)^2 d\theta < 8\pi$$

since $\int_0^{2\pi} \Re \frac{zf''}{f'}\, d\theta = 0$ in view of the fact that $f'(z) \neq 0$ in $|z| \leqq 1$. By employing Schwarz' inequality we obtain

$$\int_0^{2\pi} \left| 1 + \Re \frac{zf''}{f'} \right| d\theta \leqq \sqrt{2\pi \int_0^{2\pi} \left(1 + \Re \frac{zf''}{f'} \right)^2 d\theta} < 4\pi,$$

which shows that $f(z)$ is univalent in $|z| \leqq 1$ by Theorem A.

5. On the case of meromorphic functions.

THEOREM 5. *Let* $f(z)$ *be regular for a simply connected closed region* D *except for a simple pole. Let the boundary* Γ *of* D *consist of a regular curve and* $f'(z) \neq 0$ *on* Γ. *If there hold the following relations*

(5.1) $$d \arg df(z) < 0 \qquad on\ \Gamma$$

and

(5.2) $$\int_\Gamma d \arg df(z) = -2\pi$$

then $f(z)$ is univalent in D.

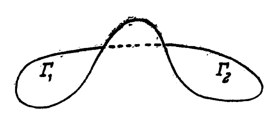
PROOF. Since $f(z)$ has a simple pole in D and since we have (5.2), there exists no branch-point in the image domain D_w of D by Morse-Heins' theorem [7].

If $f(z)$ is n-valent and if $n \geqq 2$, then the image region D_w can be considered as a one-sheeted region on an at least n-sheeted Riemann surface S. Since $f(z)$ has only one pole, only one sheet of S has the point at infinity which is an interior point of D_w. Let us take up this sheet in particular. The point at infinity is not a branch-point on this sheet and hence there exist at least two branch-points which are exterior points of D_w. So long as D_w does not encircle two branch-points in pairs, there exists no overlapping part in D_w. Hence D_w encircles these points respectively without including them since D_w has some overlapping parts. The inner boundary of these encircling parts make loops (or arcs) Γ_1 and Γ_2 for which

$$\int_{\Gamma_i} d \arg df \leqq -\pi, \quad i = 1, 2, \ldots$$

hold as we see in Lemma 1. Hence

$$\int_{\Gamma_1+\Gamma_2} d \arg df \leqq -2\pi.$$



Consequently for the complementary arcs C of $\Gamma_1 + \Gamma_2$ we have

$$\int_C d \arg df > 0 \qquad C \neq 0 \tag{5.3}$$

since we have (5.2). Namely, if $f(z)$ is at least two valent, then there exist at least two arcs whose sum satisfies (5.3).

On the other hand we have (5.1) and hence there exists no arcs satisfying (5.3). Hence $f(z)$ is univalent in D.

COROLLARY 2. *Let $f(z) = \dfrac{1}{z} + a_0 + a_1 z + \ldots$ be regular for $0 < |z| \leqq 1$. If there holds the relation*

$$-2 < 1 + \Re \frac{zf''(z)}{f'(z)} < 0 \qquad on\ |z| = 1, \tag{5.4}$$

then $f(z)$ is univalent in $|z| \leqq 1$.

6. On the case of regular functions defined in an annulus.

LEMMA 2. *Let $w = f(z)$ be regular and single-valued in a doubly connected closed domain D which does not contain the point at infinity and bounded by two simple closed regular curves C_1 and C_2 (C_2 is inside C_1)*

$$C_i : z = z_i(t) \ (0 \leqq t \leqq 1) \qquad i = 1, 2.$$

Further let $f(z_1) \neq f(z_2)$ for arbitrary two points z_1 and $z_2, z_1 \neq z_2$ on C_i, $i = 1, 2$. Suppose that $f'(z) \neq 0$ in D. Then the image region Δ of D mapped by $f(z)$

is bounded by two simple closed regular curves

$$\Gamma_i : w = w(z_i(t)), \qquad i = 1, 2$$

and this function maps D univalently onto Δ.

PROOF. By our hypothesis, Γ_i, $i = 1, 2$ are simple and closed and

$$w'(t) = f'(z_i(t))z_i'(t) \neq 0, \qquad i = 1, 2.$$

Hence Γ_i, $i = 1, 2$ are also regular.

By the assumption that $f'(z) \neq 0$ in D, we obtain

$$\int_{C_1} d \arg df(z) = \int_{C_2} d \arg df(z) = 2\pi \text{ or } -2\pi, \tag{6.1}$$

by Morse-Heins' theorem [7], where the integrals are taken so that z moves round on C_1 and C_2 in the counter-clockwise direction.

First we consider the case where the integrals in (6.1) are equal to 2π. Then w moves round on Γ_1 and Γ_2 in the counter-clockwise direction when z moves round on C_1 and C_2 in the counter-clockwise direction. Now let ω be an arbitrary point which does not lie on Γ_i, $i = 1, 2$ and let $n(D, \omega)$ denote the number of ω-points of $f(z)$ in D. Then we have

$$\begin{aligned} n(D, \omega) &= \frac{1}{2\pi}\int_{C_1} d \arg (f(z) - \omega) - \frac{1}{2\pi}\int_{C_2} d \arg(f(z) - \omega) \\ &= \frac{1}{2\pi i}\int_{C_1} \frac{f'(z)}{f(z) - \omega}\, dz - \frac{1}{2\pi i}\int_{C_2} \frac{f'(z)}{f(z) - \omega}\, dz \\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'(z_1(t))z_1'(t)}{f(z_1(t)) - \omega} - \frac{\quad}{2\pi i}\int_0^1 \frac{f'(z_2(t)) \quad t)}{f(z_2(t)) - \omega}\, dt \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{dw}{w - \omega} - \frac{1}{2\pi i}\int_{\Gamma_2} \frac{dw}{w - \omega} \end{aligned}$$

where the integrals on Γ_1 and Γ_2 are also taken in the counter-clockwise direction by the above statement.

Thus we see that (i) $n(D, \omega) = 0$ when ω is inside Γ_1 and Γ_2 (ii) $n(D, \omega) = 1$ when ω is inside Γ_1 and outside Γ_2 (iii) $n(D, \omega) = -1$ when ω is inside Γ_1 and outside Γ_2 (iv) $n(D, \omega) = 0$ when ω is outside Γ_1 and Γ_2.

Since $n(D, \omega) \geqq 0$, the case (iii) is a contradiction, unless there exists no point outside Γ_1 and inside Γ_2.

Hence D is mapped by $f(z)$ univalently onto Δ bounded by Γ_1 and Γ_2 where Γ_2 lies inside Γ_1.

In the case where the integrals in (6.1) are equal to -2π the proof is quite analogous to the above one and may be omitted. We merely note that in this case the image curves Γ_1 and Γ_2 change their position.

THEOREM 6. *Let $f(z)$ be regular, single-valued and $f'(z) \neq 0$ for $r \leqq |z| \leqq R$. Suppose that*

$$2 > 1 + \Re \frac{zf''(z)}{f'(z)} > 0 \qquad on\ |z| = r. \tag{6.2}$$

Let θ_i and θ_j $(i, j = 1, 2, \ldots.)$ be the roots of the equation

$$1 + \Re \frac{zf''(z)}{f'(z)} = 0, \qquad |z| = R,\ 0 \leqq \theta \leqq 2\pi. \tag{6.3}$$

If there holds one of the following conditions:

(A) *For all θ_j satisfying* (6.3)

$$\Re e^{i\alpha} f'(Re^{i\theta_j}) > 0 \qquad (\alpha\text{: a real constant})$$

(B)
$$\operatorname*{Max}_{i,j} \int_{\theta_j}^{\theta_i} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta < 3\pi, \qquad |z| = R,$$

(C)
$$\int_0^{2\pi} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta = 2\pi, \qquad |z| = R$$

and

$$\operatorname*{Min}_{i,j} \int_{\theta_j}^{\theta_i} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi, \qquad |z| = R$$

then $f(z)$ is univalent in $r \leqq |z| \leqq R$.

PROOF. Since $f'(z) \neq 0$ for $r \leqq |z| \leqq R$, there exists no branch-point in the image region of $r \leqq |z| \leqq R$ and hence we have

$$\int_{|z|=R} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta = \int_{|z|=r} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta = 2\pi \tag{6.4}$$

by making use of (6.2).

Let L_1 and L_2 be the image curves of $|z| = r$ and $|z| = R$, respectively. The positive direction on L_1 and L_2 are decided by (6.4). Let R be the image region of $r \leqq |z| \leqq R$ which is of course bounded by L_1 and L_2. Since we have (6.2) and (6.4), L_1 is a simple closed regular curve and the image region of $r \leqq |z| \leqq r + \varepsilon$ exists outside L_1.

Now let us show that L_2 is also a simple closed regular curve.

By (6.4), L_2 encloses a simply connected region D on the Riemann surface S generated by $f(z)$ containing the image of $R - \varepsilon \leqq |z| \leqq R$ if we neglect the existence of L_1.

Suppose that D contains the point at infinity, then R also contains the point at infinity, since R is the common part of D and the outside of L_1. But this is a contradiction since $f(z)$ is regular for $r \leqq |z| \leqq R$. Hence D does not contain the point at infinity. Consequently we can apply the discussion in §2 and §3 to D. Then the conditions (A), (B) and (C) are the sufficient conditions for L_2 not to have multiple points.

Thus all the hypotheses of Lemma 2 are satisfied and hence $f(z)$ is univalent for $r \leqq |z| \leqq R$.

As an immediate result, making use of the condition (A), we obtain the following theorem which is again an extension of Theorem B.

THEOREM 7. *Let $f(z)$ be regular and single-valued for $r \leqq |z| \leqq R$. Suppose that*

$$2 > 1 + \Re \frac{zf''(z)}{f'(z)} > 0 \qquad on \ |z| = r$$

and that in $r \leqq |z| \leqq R$

$$Re^{i\alpha} f'(z) > 0 \qquad (\alpha : a \ real \ constant)$$

then $f(z)$ is univalent for $r \leqq |z| \leqq R$.

Corresponding to the case where the integral in (6.1) is equal to -2π we obtain theorems analogous to Theorem 6 and 7.

THEOREM 8. *Let $f(z)$ be regular, single-valued and $f'(z) \neq 0$ in $r \leqq |z| \leqq R$. Suppose that*

$$-2 < 1 + \Re \frac{zf''(z)}{f'(z)} < 0 \text{ on } |z| = R. \tag{6.5}$$

Let θ_i and θ_j $(i, j = 1, 2, \ldots)$ be the roots of the equation

$$1 + \Re \frac{zf''(z)}{f'(z)} = 0, \qquad |z| = r, \quad 0 \leqq \theta \leqq 2\pi. \tag{6.6}$$

If there holds one of the following conditions:

(A') *For all θ_j satisfying* (6.6)

$$\Re e^{i\alpha} f'(re^{i\theta_j}) > 0 \qquad (\alpha: \text{a real constant})$$

(B')
$$\operatorname*{Min}_{i,j} \int_{\theta_j}^{\theta_i} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta > -3\pi, \qquad |z| = r$$

(C')
$$\int_0^{2\pi} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta = -2\pi, \qquad |z| = r$$

and

$$\operatorname*{Max}_{i,j} \int_{\theta j}^{\theta i} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta < \pi, \qquad |z| = r$$

then $f(z)$ is univalent in $r \leqq |z| \leqq R$.

The proof of this theorem is analogous to Theorem 6 and may be omitted.

As a direct consequence we have

THEOREM 9. *Let $f(z)$ be regular and single-valued for $r \leqq |z| \leqq R$. Suppose that*

$$-2 < 1 + \Re \frac{zf''(z)}{f'(z)} < 0 \qquad \text{on } |z| = R$$

and that for $r \leqq |z| \leqq R$

$$\Re e^{i\alpha} f'(z) > 0 \qquad (\alpha: \text{a real constant})$$

then $f(z)$ *is univalent for* $r \leqq |z| \leqq R$.

By making use of the results obtained in §4 we have

THEOREM 10. *Let* $f(z)$ *be regular, single-valued and* $f'(z) \neq 0$ *for* $r \leqq |z| \leqq R$. *Suppose that*

$$(6.7) \qquad 2 > 1 + \Re \frac{zf''(z)}{f'(z)} > 0 \qquad \text{on } |z| = r.$$

If there holds one of the following relations;

$$(a) \qquad \int_{|z|=R} \left| 1 + \Re \frac{zf''(z)}{f'(z)} \right| d\theta < 4\pi$$

$$(b) \qquad \alpha > 1 + \Re \frac{zf''(z)}{f'(z)} > -\frac{\alpha}{2\alpha - 3} \qquad \text{on } |z| = R$$

where α *is a certain number not less than* $3/2$, *then* $f(z)$ *is univalent for* $r \leqq |z| \leqq R$.

PROOF. As in the proof of Theorem 6 we have (6.4). In particular we have

$$(6.8) \qquad \int_{|z|=R} \left(1 + \Re \frac{zf''(z)}{f'(z)} \right) d\theta = 2\pi.$$

It was proved in [4] that if we have (6.8) the condition (b) is a sufficient condition for the condition (a). As for (a) the proof is obvious by Theorems 3 and 6.

7. Generalizations to the functions defined in an n-ply connected domain. If we apply the method of proof in Theorem 6 together with Morse-Heins' theorem to regular and single-valued functions defined in an n-ply connected domain which does not contain the point at infinity and bounded by n simple closed regular curves, then all the results in the preceding section can easily be extended. For example we obtain the following

THEOREM 11. *Let* $f(z)$ *be regular and single-valued in a closed convex domain* D *which has* $n-1$ *circular holes* $|z - a_i| < r_i$, $i = 1, 2, \ldots, n-1$, *in it. Suppose that*

$$2 > 1 + \Re \frac{(z - a_i)f''(z)}{f'(z)} > 0 \qquad \text{on } |z - a_i| = r_i$$
$$i = 1, 2, \ldots, n-1$$

and that in D

$$\Re e^{i\alpha} f'(z) > 0 \qquad (\alpha: \text{a real constant}),$$

then $f(z)$ *is univalent in* D.

THEOREM 12. *Let* $f(z)$ *be regular and single-valued in a closed domain* D

consisting of a circle $|z| \leqq R$ *with* $n-1$ *circular holes* $|z-a_i| < r_i$, $i = 1, 2, \ldots, n-1$, *in it. Suppose that*

$$2 > 1 + \Re \frac{(z-a_i)f''(z)}{f'(z)} > 0 \quad \text{on } |z-a_i| = r_i,\ i = 1, 2, \ldots, n-1.$$

If $f(z)$ *satisfies on* $|z| = R$ *one of the following conditions;*

$$\text{(i)}\ \ 1 + \Re \frac{zf''(z)}{f'(z)} > -\frac{1}{2} \qquad \text{(ii)}\ \ 1 + \Re \frac{zf''(z)}{f'(z)} < \frac{3}{2}$$

$$\text{(iii)}\ \ \left|1 + \Re \frac{zf''(z)}{f'(z)}\right| < 2 \qquad \text{(iv)}\ \ \left|\Re \frac{zf''(z)}{f'(z)}\right| < 2$$

then $f(z)$ *is univalent in* D.

In order to prove these theorems it will be sufficient to extend Lemma 2 to the following form.

LEMMA 3. *Let* $w = f(z)$ *be regular and single-valued for an n-ply connected closed domain* D *which does not contain the point at infinity and bounded by* n *simple closed regular curves* C_i, $i = 1, 2, \ldots, n$,

$$C_i : z = z_i(t) \quad (0 \leqq t \leqq 1), \qquad i = 1, 2, \ldots, n.$$

Further let $f(z_1) \neq f(z_2)$ *for arbitrary two points* z_1 *and* z_2, $z_1 \neq z_2$ *on* C_i $i = 1, 2, \ldots, n$. *Suppose that* $f'(z) \neq 0$ *for* D. *Then the image region* Δ *of* D *under* $f(z)$ *is bounded by* n *simple closed regular curves*

$$\Gamma_i : w = w(z_i(t)), \qquad i = 1, 2, \ldots, n$$

and this function maps D *univalently onto* Δ.

PROOF. We can easily see that Γ_i, $i = 1, 2, \ldots, n$ are simple closed regular curves. Let us suppose that C_j, $j = 1, 2, \ldots, n-1$ are inside C_n without loss of generality. By the assumption that $f'(z) \neq 0$ for D, we obtain

$$2(2-n)\pi = \int_{C_n} d \arg df(z) - \sum_{j=1}^{n-1} \int_{C_j} d \arg df(z) \tag{7.1}$$

by Morse-Heins' theorem [7], where the integral is taken so that z moves round on C_i, $i = 1, 2, \ldots, n$ in the counter-clockwise direction. Since Γ_i, $i = 1, 2, \ldots, n$ are simple the integrals $\int_{C_j} d \arg df(z)$, $j = 1, 2, \ldots, n$ are equal to 2π or -2π. Thus we have n cases (i) $\int_{C_j} d \arg df(z) = 2\pi$, $i = 1, 2, \ldots, n$, (ii) $\int_{C_n} d \arg df(z) = -2\pi$ and any one of the integrals $\int_{C_j} d \arg df$, $j = 1, 2, \ldots, n-1$ is equal to -2π and the others are equal to 2π, which gives $n-1$ cases.

Analogously to Lemma 2 we consider the value $n(D, \omega)$ for every ω and in each case stated above. Then we have our conclusion by similar consideration to that of Lemma 2. The detail may be omitted here.

We omit also the proofs of Theorems 11 and 12 noting only that the conditions (i)-(iv) can be obtained from (*b*) in Theorem 10 by choosing α suitably.

8. **Sufficient conditions for the p-valency of regular functions.** We shall generalize some of the results in §3 and §4 to the case of p-valency. For this purpose we need to extend Lemma 1 at first.

LEMMA 4. *Let $f(z)$ be regular for a simply connected closed domain D whose boundary Γ_z consists of a regular curve and $f'(z) \neq 0$ on Γ_z. Suppose that*

$$\int_{\Gamma_z} d \arg df(z) = 2k\pi. \tag{8.1}$$

If we have for arbitrary $p-k+1$ arcs $C_1, C_2, \ldots, C_{p-k+1}$ on the boundary Γ_z of D which do not overlap one another

$$\int_{C_1+C_2++\ldots+C_{p-k+1}} d \arg df(z) > -(p-k+1)\pi \tag{8.2}$$

or

$$\int_{C_1+C_2+\ldots+C_{p-k+1}} d \arg df(z) < (p+k+1)\pi \tag{8.3}$$

then $f(z)$ is at most p-valent in D.

PROOF. If $f(z)$ is n-valent where $n \geqq p+1$, then we can consider the image region D_w of D under $f(z)$ as a one-sheeted region on an at least n-sheeted Riemann surface S.

Since D_w has at least one n-valently overlapping part and since we must have a part of D_w encircling a branch-point of S in order to move from one sheet of the n-valently overlapping part to another, there exist at least p branch-points of S encircled by parts of D_w. Here and what follows the number of branch-points are counted in accordance with their multiplicities. Furthermore we say that a region R encircles a point P if there exists at least a curve in R connecting two points in R whose projection on the w-plane encircle the point P whether it is included in R or not.

On the other hand there exist $k-1$ branch-points in D, of course encircled by parts of D_w by (8.1). Hence there exist at least $p-k+1$ branch-points of S exterior to D_w respectively. Let $B_i, i=1,2,\ldots,p-k+1$ be the projections on the w-plane of the branch-points stated above and let Q be a point on the w-plane overlapped by D_w n-valently. Since the projections of the inner boundaries of the encircling parts also encircles B_i, $i=1,2, \ldots, p-k+1$ respectively, we have arcs $C_i, i=1,2,\ldots, p-k+1$ on Γ_w, the boundary of D_w, which begin from a point on B_iQ, $i=1,2,\ldots,p-k+1$ and end also at a point on B_iQ, $i=1,2,\ldots,p-k+1$ encircling B_i, $i=1,2,\ldots,p-k+1$ once negatively, respectively.

Thus we have $p-k+1$ arcs on Γ_w which have no common parts except perhaps the ends of them and for which

$$\int_{\Gamma_i} d \arg dw \leqq -\pi, \quad i=1,2,\dots,p-k+1$$

hold. Hence we have

$$\int_{C_1+C_2+\dots+C_{p-k+1}} d \arg df(z) \leqq -(p-k+1)\pi$$

if $f(z)$ is at least $p+1$ valent in D.

Accordingly $f(z)$ is at most p-valent in D if we have (8.7) for arbitrary $p-k+1$ arcs $C_1, C_2, \dots, C_{p-k+1}$ on the boundary Γ_z of D which do not overlap one another. We note that (8.2) is equivalent to (8.3) since we have (8.1). Thus the proof is complete.

THEOREM 13. *Let $f(z)$ be regular for a closed convex domain D whose boundary L is a regular curve. Suppose that $f(z)$ has exactly $p-1$ critical points α_i, $i=1,2,\dots,p-1$ in D and no critical point on L. If there holds the inequality*

$$\Re\left[e^{i\alpha} f'(z)/\prod_{i-1}^{p-1}(z-\alpha_i)\right] > 0 \qquad (\alpha: \text{a real constant}) \tag{8.4}$$

on L, then $f(z)$ is at most p-valent in D.

PROOF. Now since $f'(z)/\prod_{i=1}^{p-1}(z-\alpha_i) \neq 0$ in D by (8.4), $\arg\left[f'(z)/\sum_{i=1}^{p-1}(z-\alpha_i)\right]$ is one-valued in D. Accordingly $\arg df(z)$ is also one-valued if we take suitable branches of $\arg(z-\alpha_i)$, $i=1,2,\dots,p-1$ and $\arg dz$ since

$$\arg df(z) = \arg\left[f'(z)/\sum_{i=1}^{p-1}(z-\alpha_i)\right] + \sum_{i=1}^{p-1}\arg(z-\alpha_i) + \arg dz.$$

By noticing this fact and by the assumption (8.4), we have

$$-\frac{\pi}{2} < \alpha + \arg df(z_i) - \arg dz_i - \sum_{k=1}^{p-1}\arg(z_i-\alpha_k) < \frac{\pi}{2}$$

and

$$-\frac{\pi}{2} < -\alpha - \arg df(z_j) + \arg dz_j + \sum_{k=1}^{p-1}\arg(z_j-\alpha_k) < \frac{\pi}{2}$$

for any z_i and z_j, $i>j$. Hence we have

$$-\pi < \arg df(z_i) - \arg df(z_j) + \arg dz_j - \arg dz_i$$

$$+\sum_{k=1}^{p-1}\arg(z_j-\alpha_k)-\sum_{k=1}^{p-1}\arg(z_i-\alpha_k)<\pi$$

where $2\pi \geqq \arg dz_i - \arg dz_j \geqq 0$ and $2\pi \geqq \arg(z_i-\alpha_k)-\arg(z_j-\alpha_k)\geqq 0$, $k=1,2,\dots,p-1$, since D is a convex region. Thus we have

$$-\pi < \arg f(z_i)-\arg f(z_j) < (2p+1)\pi$$

which is equivalent to

$$-\pi < \int_{z_j}^{z_i} d\arg df(z) < (2p+1)\pi.$$

This inequality shows that $f(z)$ is at most p-valent in D, by Lemma 4.

As an immediate result we obtain the following

THEOREM 14. *If $f(z)=z^p+a_{p+1}z^{p+1}+\dots$ is regular for $|z|\leqq r$ and if we have*

(8.5) $$R[e^{i\alpha}f'(z)/z^{p-1}]>0 \qquad (\alpha\text{: a real constant})$$

for $|z|\leqq r$, then $f(z)$ is p valent for $|z|\leqq r$,

REMARK. The condition (8.5) in Theorem 14 can be replaced by

(8.6) $$p>\sum_{n=p+1}^{\infty} n|a_n|r^{n-p}$$

which is seen as in the case $p=1$.

Making use of Theorem 14 we can extend many theorems on univalent functions to the case of p-valency. But we shall here enunciate only a generalization of Noshiro's theorem concerning the radius of univalence and the radius of convexity [9].

THEOREM 15. *Let $f(z)=z^p+a_{p+1}z^{p+1}+\dots$ be regular for $|z|\leqq 1$ and let $\left|\frac{f'(z)}{pz^{p-1}}\right|<M$ for $|z|\leqq 1$. Then $f(z)$ is p-valent in $|z|<\frac{1}{M}$ and $f(z)$ maps $|z|<M_2-\sqrt{M_2^2-1}$ where $M_2=\frac{1}{2}\left\{M\left(1+\frac{1}{p}\right)+\frac{1}{M}\left(1-\frac{1}{p}\right)\right\}$ onto a p-valently convex domain.*

PROOF. Let us put $g(z)=\frac{f'(z)}{pz^{p-1}}$. Then $g(0)=1$ and $|g(z)|<M$ for $|z|\leqq 1$. Hence we obtain

$$\left|\frac{g(z)-1}{M^2-g(z)}\right|\leqq\frac{r}{M}, \qquad (|z|\leqq r)$$

$$M\frac{1-Mr}{M-r}\leqq\Re\frac{f'(z)}{pz^{p-1}}\leqq M\frac{1+Mr}{M+r}, \qquad (|z|\leqq r)$$

by Schwarz' lemma. Hence we have (8.5) if $|z|<\frac{1}{M}$.

As for the convexity the proof is quite analogous to the case of univalence and may be omitted here.

THEOREM 16. *Let $f(z)$ be regular for a simply connected closed domain D whose boundary Γ consists of a regular curve and $f'(z) \neq 0$ on Γ. Suppose that*

$$\int_{\Gamma} d \arg df(z) = 2k\pi.$$

Further let C_1 be the part of Γ on which $d \arg df > 0$ and C_2 be the part of Γ on which $d \arg df \leqq 0$. If $f(z)$ satisfies one of the following conditions:

$$(A) \qquad \int_{C_1} d \arg df(z) < (p+k+1)\pi,$$

$$(B) \qquad \int_{C_2} d \arg df(z) > -(p-k+1)\pi,$$

$$(C) \qquad \int_{\Gamma} |d \arg df(z)| < 2(p+1)\pi$$

then $f(z)$ is at most p-valent in D.

We can prove this Theorem 16 analogously to Theorem 3 by making use of Lemma 4 and the proof may be omitted here.

COROLLARY 3. *Let $f(z)$ be regular for $|z| \leqq 1$. Suppose that $f(z)$ has $k-1$ critical points in $|z| < 1$ and no critical point on $|z| = 1$. Further let C_1 be the part of $|z| = 1$ on which*

$$1 + \Re \frac{zf''(z)}{f'(z)} > 0 \text{ and put } x = \int_{C_1} d \arg z$$

and C_2 be the part of $|z| = 1$ on which

$$1 + \Re \frac{zf''(z)}{f'(z)} \leqq 0 \qquad \text{and hence} \qquad 2\pi - x = \int_{C_2} d \arg z.$$

If $f(z)$ satisfies for $|z| = 1$ one of the following conditions:

$$(a) \qquad \int_{C_1} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta < (p+k+1)\pi,$$

$$(a') \qquad 1 + \Re \frac{zf''(z)}{f'(z)} < \frac{p+k+1}{x}\pi,$$

$$(b) \qquad \int_{C_2} \left(1 + \Re \frac{zf''(z)}{f'(z)}\right) d\theta > -(p-k+1)\pi,$$

$$(b') \qquad 1 + \Re \frac{zf''(z)}{f'(z)} > \frac{p-k+1}{2\pi - x}\pi,$$

$$(c) \qquad \int_{C_1+C_2} \left|1 + \Re \frac{zf''(z)}{f'(z)}\right| d\theta < 2(p+1)\pi,$$

$$(c') \quad \left|1 + \Re \frac{zf''(z)}{f'(z)}\right| < p + 1,$$

then $f(z)$ *is at most* p*-valent in* $|z| \leqq 1$.

This is an extension of a result given in [4, 3].

REFERENCES

[1] A. KOBORI, Über die notwendige und hinreichende Bedingung dafür, dass eine Potenzreihe den Kreisbereich auf den schlichten sternigen bzw. konvexen Bereich abbildet, Mem. Coll. Kyoto Imp. Univ., (A) 15(1932), 279-291.

[2] M. S. ROBERTSON, Analytic functions star-like in one direction, Amer. Journ. of Math., 58(1936), 465-472.
A representation of all analytic functions in term of functions with positive real parts, Ann. of Math., 38(1937), 770-783.

[3] S. OZAKI, On the theory of multivalent functions, II, Sci. Rep. Tokyo Bunrika Daigaku 4(1941), 45-86.

[4] T. UMEZAWA, Analytic functions convex in one direction, Journ. Math. Soc. Japan 4(1952), pp. 194-202.

[5] K. NOSHIRO, On the theory of schlicht functions, Journ. of the Faculty of Science, Hokkaido Imp. Univ., 2(1934), 129-155.

[6] J. WOLFF, L'intégrale d'une fonction holomorphe et a partie réelle positive dans un demi-plan est univalente, Comptes Rendus de l'Académie des Sciences, 198(1934), 1209-1210.

[7] M. MORSE, Topological methods in the theory of functions of a complex variable, Princeton Univ. Press(1947).

[8] T. UMEZAWA, Star-like theorems and convex-like theorems in an annulus, Journ. Math. Soc. Japan 6(1954), 68-75.

[9] K. NOSHIRO, On the starshaped mapping by an analytic function, Proc. Acad. Tokyo 8(1932), 275-277.

GUMMA UNIVERSITY.