AFFINE TRANSFORMATIONS IN A RIEMANNIAN MANIFOLD

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0. Introduction[1]. It has been proved by K. Yano [7] that in a compact orientable Riemannian manifold an infinitesimal affine transformation is an infinitesimal motion. But it seems that the assumptions of "compactness" and "orientability" are too strong or unnecessary. In this respect, K. Nomizu [5] has recently given very suggestive lemmas on the relations between the group $A(M)$ of all affine transformations and the group $I(M)$ of all isometries in a Riemannian manifold M. According to one of his lemmas, in an irreducible Riemannian manifold the affine transformation may be considered as the homothetic one. Furthermore so far as the identity component $A_0(M)$ of $A(M)$ is concerned, we may restrict our consideration to the irreducible or locally flat parts of the Riemannian manifold M. Thereby we shall first treat of the properties of the homothetic transformation and show that in some complete Riemannian manifold a homothetic transformation is necessarily an isometry. We shall next apply this to affine transformations in an irreducible Riemannian manifold. Afterwards we shall consider the locally flat case. In the last section we shall give some examples which show that the assumptions in our theorems can not be made weak.

1. Preliminaries. If M is a differentiable Riemannian manifold with a fundamental metric tensor field G which is positive definite, for any vector field X we denote by $\nabla(X)$ the covariant differentiation in the direction of X with respect to the Riemannian connection.

Now let M_1 and M_2 be two Riemannian manifolds with G_1 and G_2 as their fundamental metric tensor fields and denote by $\nabla_1(X_1)$ and $\nabla_2(X_2)$ the corresponding covariant differentiations respectively. Let φ be a differentiable homeomorphism of M_1 onto M_2. If φ commutes with the covariant differentiations, i. e. for any vector field X on M_1

$$\varphi(\nabla_1 X) = \nabla_2(\varphi X)\varphi,$$ [2]

φ is called an *affine transformation*. If we have $\varphi G_1 = G_2$, then φ is said to be an *isometric transformation* or an *isometry*. If for some real constant $\rho > 0$ we have $\varphi G_1 = \rho G_2$, φ is called a *homothetic transformation*.

For a connected Riemannian manifold M, we denote by $A(M)$, $I(M)$ and

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1) Similar results to ours were proved independently by T. Nagano, J. Hano and S. Kobayashi though not simultaneously. Cf. Nagoya Math J., Vol. 9(1955), pp. 39-41 and 99-106

2) The definition of φ is as follows: If f is a function on M_1 then $\varphi f = f\circ\varphi^{-1}$;
If X is a contravariant vector field on M_1 and f is a function on M_2, then $(\varphi X)f = \varphi(X(\varphi^{-1}f))$;
If ω is a covariant vector field on M_1 and X is a contravariant vector field on M_2, then $(\varphi\omega)X = \varphi(\omega(\varphi^{-1}))$; and so on.

$H(M)$ the group of all affine transformations $M \to M$, that of all isometries $M \to M$ and that of all homothetic transformations $M \to M$ respectively. It is then evident that we have $A(M) \supset H(M) \supset I(M)$. Furthermore it has been proved [2, 3, 4] that they are Lie groups.

M is said *reducible* or *irreducible* if the *restricted* homogeneous holonomy group of M is reducible or not. It suits our convenience to call one-dimensional Riemannian manifolds reducible.

2. On homothetic transformations.

LEMMA 1. *Let M be a connected Riemannian manifold which is not locally flat and φ a homothetic transformation in M which is not an isometry. Then φ has no fixed point.*

PROOF. Without loss of generality we may assume that $\varphi G = \rho^{-2}G$ with a real constant ρ such that $0 < \rho < 1$. Suppose that φ fixes a point $p_0 \in M$. If p is any point of M at which the curvature tensor field R does not vanish, i. e. $R_p \neq 0$, then the sequence $\{\varphi^k p\}_{k=1,2,\dots}$ must converge to the point p_0 because the distance $d(p_0, \varphi^k p)$ tends to zero. For any unit vectors $X, Y, Z \in T_p$ and $\omega \in T_p^*$, $\varphi^k X$, $\varphi^k Y$, $\varphi^k Z$ are vectors of length ρ^k in $T_{\varphi^k p}$ and $\varphi^k \omega$ is a vector of length ρ^{-k} in $T^*_{\varphi^k p}$, where T_p^* is the dual space of T_p. If we put

$$X_k = \rho^{-k}\varphi^k X, \quad Y_k = \rho^{-k}\varphi^k Y, \quad Z_k = \rho^{-k}\varphi^k Z, \quad \omega_k = \rho^k \varphi^k \omega,$$

then they are unit vectors and we may regard the pairs $(\varphi^k p, X_k)$, $(\varphi^k p, Y_k)$, $(\varphi^k p, Z_k)$ and $(\varphi^k p, \omega_k)$ as points of the tangent sphere bundle of M[3]. It is easily seen that there exist unit vectors $X_0, Y_0, Z_0 \in T_{p_0}$ and $\omega_0 \in T^*_{p_0}$ such that

$$\lim_{\nu\to\infty}(\varphi^{k_\nu}p, X_{k_\nu}) = (p_0, X_0), \qquad \lim_{\nu\to\infty}(\varphi^{k_\nu}p, Y_{k_\nu}) = (p_0, Y_0), \text{ [4]}$$
$$\lim_{\nu\to\infty}(\varphi^{k_\nu}p, Z_{k_\nu}) = (p_0, Z_0), \qquad \lim_{\nu\to\infty}(\varphi^{k_\nu}p, \omega_{k_\nu}) = (p_0, \omega_0).$$

The curvature tensor field R being continuous, we have then

$$\lim_{\nu\to\infty} R_{\varphi^{k_\nu}p}(X_{k_\nu}, Y_{k_\nu}, Z_{k_\nu}, \omega_{k_\nu}) = R_{p_0}(X_0, Y_0, Z_0, \omega_0),$$

On the other hand, R being invariant by all affine transformations

$$R_{\varphi^{k_\nu}p}(X_{k_\nu}, Y_{k_\nu}, Z_{k_\nu}, \omega_{k_\nu}) = \rho^{-2k_\nu} R_{\varphi^{k_\nu}p}(\varphi^{k_\nu}X, \varphi^{k_\nu}Y, \varphi^{k_\nu}Z, \varphi^{k_\nu}\omega)$$
$$= \rho^{-2k_\nu}R_p(X, Y, Z, \omega),$$

which cannot tend to the finite value $R_{p_0}(X_0, Y_0, Z_0, \omega_0)$ unless $R_p(X, Y, Z, \omega) = 0$ or $\rho > 1$. This leads to a contradiction and we conclude that φ has no fixed point.

LEMMA 2. *Let M be a complete and connected Riemannian manifold which is not locally flat, then $H(M) = I(M)$.*

3) We may restrict ourselves within a small neighborhood of the point p_0 and use a product representation of the bundle in this neighborhood.

4) $\{k_1, k_2, \dots\dots, k_\nu, \dots\dots\}$ is a suitable subsequence of $\{1, 2, \dots\dots, n, \dots\dots\}$.

PROOF. Suppose that there exists φ which is in $H(M)$ but not in $I(M)$. Then by Lemma 1 φ has no fixed point. We may assume that $\varphi G = \rho^{-2}G$, $0 < \rho < 1$. If p is an arbitrary point of M, then the sequence $\{\varphi^k p\}_{k=1,2,\ldots}$ is clearly a Cauchy sequence on M. M being complete, there exists a limit point $p_0 = \lim_{k\to\infty} \varphi^k p$ in M. It is then evident that φ must fix the point p_0, contrary to Lemma 1. Thus we have $H(M) = I(M)$.

The assumption of "completeness" seems to be really necessary. In the last section we shall show by an example that some irreducible but non-complete Riemannian manifold really admits homothetic transformations which are not isometries and have no fixed point.

3. Affine transformations in an irreducible Riemannian manifold.

Let M_1 and M_2 be two Riemannian manifolds with G_1 and G_2 as their fundamental metric tensor fields respectively. If φ is an affine transformation of M_1 onto M_2, by the definition for any vector field X on M_2 the relation $\varphi\nabla_1(\varphi^{-1}X) = \nabla_2(X)\varphi$ holds true. In particular, we have

$$\nabla_2(X)(\varphi G_1) = \varphi\nabla_1(\varphi^{-1}X)G_1 = 0$$

by virtue of the invariance of G_1 under the covariant differentiation. Therefore if M_2 is irreducible, it follows that we have $\varphi G_1 = \rho G_2$ for some constant $\rho > 0$.

The result established above becomes

LEMMA 3. *The notation being as above, if M is irreducible, then φ is a homothetic transformation* [5].

Combining Lemma 1 and 3, we have

THEOREM 1. *Let M be a connected irreducible Riemannian manifold and φ an affine transformation in M which is not an isometry. Then φ has no fixed point.*

Combining Lemma 2 and 3, we have

THEOREM 2. *Let M be a connected irreducible Riemannian manifold. If M is complete, then we have $A(M) = I(M)$.*

Let M be a connected and simply connected Riemannian manifold which is not irreducible and $T_p = T_p^{(0)} + T_p^{(1)} + T_p^{(2)} + \ldots + T_p^{(r)}$ the canonical decomposition of the tangent space T_p at any point p of M by the homogeneous holonomy group σ, where σ is trivial on $T_p^{(0)}$ and irreducible on $T_p^{(i)}$ $(1 \leqq i \leqq r)$. Then each $T_p^{(i)}$ $(0 \leqq i \leqq r)$ generates a completely integrable field $F^{(i)}$ of plane elements by the parallel displacement. We denote by $M_p^{(i)}$ the maximal integral manifold of $F^{(i)}$ through the point p. If moreover M is complete, by a theorem of G. de Rham [1] M is a (unique) direct product of $M_p^{(0)}, M_p^{(1)}, \ldots, M_p^{(r)}$. Then we have the following lemma due to K. Nomizu [5]:

LEMMA 4. *The notation being as above, let $A_0(M)$ be the identity component* *of the group* $A(M)$. *If* M *is simply connected, every element of* $A_0(M)$ *leaves invariant each field* $F^{(i)}$.

It is easily seen that if M is complete so also is each $M_p^{(i)}$. Thus, in case M is complete and simply connected, by Lemma 4, we have easily

$$A_0(M) = A_0(M_p^{(0)}) \times \dots \times A_0(M_p^{(r)}).$$

The following lemma follows from the above results.

LEMMA 5. *Let* M *be a simply connected Riemannian manifold. If* M *is complete and has no locally flat part, then* $A_0(M) = I_0(M)$, *where* $I_0(M)$ *is the identity component of* $I(M)$.

Now we do not assume that M is simply connected. Let $\widetilde{M}$ be the simply connected covering manifold of M and π the canonical projection of $\widetilde{M}$ onto M, then $\widetilde{G} = \pi^{-1}G$ is the fundamental metric tensor field of $\widetilde{M}$ which defines the same Riemannian geometry of $\widetilde{M}$ induced by that of M. If $\varphi(t)$is a one-parameter subgroup of $A_0(M)$, then there exists a one-parameter group $\widetilde{\varphi}(t)$ of affine transformations of $\widetilde{M}$ such that the relation $\pi\widetilde{\varphi}(t) = \varphi(t)\pi$ holds true. Furthermore it is easily seen that if $\widetilde{\varphi}(t)$ is isometric so also is $\varphi(t)$. Putting this fact and Lemma 5 together we have

THEOREM 3. *Let* M *be a connected Riemannian manifold. If* M *is complete and has no locally flat part, then* $A_0(M) = I_0(M)$.

4. **Locally flat case.** Let M be a simply connected Riemannian manifold which is locally flat and ξ an infinitesimal affine transformation in M. Then in a canonical coordinate system at any point of M we have

$$\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} = 0 \qquad (1 \leqq i, j, k \leqq n),$$

since the curvature tensor field vanishes [6]. Therefore we have

$$\xi^i = \sum_{j=1}^{n} a_j^i x^j + c^i \qquad (1 \leqq i \leqq n)$$

in this coordinate sytsem, where a_j^i and c^i are constants of which not all are zero.

Now, assume that M is complete. Then it is easily seen that if the length of ξ is bounded, we have $a_j^i = 0$ $(1 \leqq i, j \leqq n)$ in every canonical coordinate neighborhood and ξ defines an infinitesimal translation. Conversely, if ξ is an infinitesimal translation, then the length of ξ is obviously constant. By the remark which follows Lemma 5, we have the following theorem.

THEOREM 4. *Let* M *be a locally flat Riemannian manifold which is connected and complete. Then in order that an infinitesimal affine transformation* ξ *be a transaltion, it is necessary and sufficient that the length of* ξ *be bounded.*

By virtue of Theorem 3 and 4 we have easily the theorem of K. Yano [7].

5. **Examples.** (I) Let M be a product manifold $L\times L$ of the straight line L by itself. We give M a fundamental metric

$$ds^2=e^{2x}\{(y^2+1)^2dx^2+dy^2\},$$

where x and y are the usual coordinates on L. It is easy to see that the Riemannian manifold M is irreducible but not complete with respect to this metric.

Now, for any $x,y\in L$ we denote by φ_a the mapping $(x,y)\rightarrow(x-a,y)$ of M onto M, where a is an arbitrary real number. Then, if $a\neq 0$, φ_a is a homothetic transformation which is not isometry and has no fixed point. In fact we have

$$\varphi_a\cdot ds^2=e^{2(x+a)}\{(y^2+1)^2dx^2+dy^2\}=e^{2a}ds^2.$$

This example shows that the assumption of "completess" in Lemma 2 is really necessary.

(II) Let M_1 be a connected complete irreducible Riemannian manifold with a fundamental metric ds_1^2, M_2 a Riemannian manifold with the same underlying manifold M as M_1 but with another fundamental metric $ds_2^2=\rho ds_1^2$, where ρ is a positive constant $\neq 1$, and denote by $\widetilde{M}$ the product Riemannian manifold $M_1\times M_2$ with $ds^2=ds_1^2+ds_2^2$. Now for any points $x,y\in M$ the mapping $\varphi:(x,y)\rightarrow(y,x)$ of $\widetilde{M}$ onto $\widetilde{M}$ is obviously an affine transformation but not an isometry. Since the Jacobian of φ is negative, φ does not belong to the identity component $A_0(M)$ of the group $A(M)$.

As is easily seen by the definition of φ, this transformation leaves invariant neither of the fields $F^{(1)}$, $F^{(2)}$ of plane elements of $\widetilde{M}$[5].

This shows that Theorem 3 does not always hold true for $A(M)$ and $I(M)$ instead of $A_0(M)$ and $I_0(M)$.

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5) See the remark which follows Theorem 2.