ABSOLUTE CONVERGENCE OF FOURIER EXPANSIONS

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1. On the absolute convergence of the Fourier expansion of a function $f(x)$ with period 2π, the Wiener theorem reads as follows:

If, for every $x_0 \in [0. 2\pi]$, there corresponds a neighbourhood I_{x_0} of x_0 in which $f(x)$ coincides with a function having the absolute convergent Fourier expansion, then the Fourier expansion of $f(x)$ itself converges absolutely.

The main object of this paper is to show that the theorem does not always remain true if there is an exceptional point in the hypothesis.

2. THEOREM 1. *Let $f(x)$ be a function with period π and vanish at $x = 0$ and $x = \pi$. Suppose that the sine expansion of $f(x)$ converges absolutely:*

(1) $$f(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

(2) $$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Under these conditions the cosine expansion of $f(x)$ is not always absolutely convergent.

From this theorem we can show that the existence of an exceptional point is not permissible in the hypothesis of the Wiener theorem. In fact, from Theorem 1 we may find a function $f(x)$ of period π such that

(3) $$f(x) = \sum_{n=1}^{\infty} a_n \sin nx, \qquad \sum_{n=1}^{\infty} |a_n| < \infty \qquad (0 \leqq x \leqq \pi)$$

and that

(4) $$f(x) \sim \sum_{n=0}^{\infty} b_n \cos nx, \qquad \sum_{n=0}^{\infty} |b_n| = \infty \qquad (0 \leqq x \leqq \pi).$$

Let $g(x)$ be the even function of period 2π which coincides with $f(x)$ in the interval $[0, \pi]$, then its Fourier series coincides with the expression (4) for $0 \leqq x \leqq 2\pi$, and so does not converge absolutely; meanwhile $g(x)$ coincides with $f(x)$ or $-f(x)$ in every neighbourhood I_{x_0}, $x_0 \neq 0 \,(\text{mod. } \pi)$. The Fourier series of the function $f(x)$ and $-f(x)$, regarding as functions of period 2π, are both absolutely convergent. The function $g(x)$ forms a required negative example with exceptional point $x = 0$.

3. To prove Theorem 1 we shall make some preliminary consideration. Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be the coefficients of sine and cosine expansions of $f(x)$ respectively. Then

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi \sum_{k=1}^\infty a_k \sin kx \cos nx\,dx$$
$$= \frac{2}{\pi}\sum_{k=1}^\infty a_k \int_0^\pi \sin kx \cos nx\,dx.$$

Since

$$\int_0^\pi \sin kx \cos nx\,dx = \begin{cases} 0 & \text{if } k-n \text{ is even,} \\ 2\left(\frac{1}{k+n} + \frac{1}{k-n}\right) & \text{if } k-n \text{ is odd,} \end{cases}$$

we have

$$b_n = \frac{4}{\pi}\sum_{k=1}^{\infty}{}' a_k \left(\frac{1}{k+n} + \frac{1}{k-n}\right)$$

where $\sum'$ means the summation in which $k-n$ is odd. Therefore

$$\sum_{n=1}^\infty |b_n| = \frac{4}{\pi}\sum_{n=1}^\infty \left|\sum_{k=1}^{\infty}{}' a_k\left(\frac{1}{k+n} + \frac{1}{k-n}\right)\right| \tag{5}$$
$$= \frac{4}{\pi}\sum_{n=1}^\infty |S_n|$$

say. Let us divide the inner sum S_n into several partial sums and the rest:

$$S_n = \sum_{k=1}^{[n/2]}{}' \frac{a_k}{k-n} + \sum_{k=[n/2]+1}^{n-1}{}' \frac{a_k}{k-n} + \sum_{k=n+1}^{[3n/2]}{}' \frac{a_k}{k-n}$$
$$+ \sum_{k=[3n/2]+1}^{\infty}{}' \frac{a_k}{k-n} + \sum_{k=1}^{[n/2]}{}' \frac{a_k}{k+n} + \sum_{k=[n/2]+1}^{\infty}{}' \frac{a_k}{k+n}$$
$$= S_n(1) + S_n(2) + S_n(3) + S_n(4) + S_n(5) + S_n(6)$$

say. If $\sum |a_k| < \infty$, then, denoting by A a positive constant not necessarily the same in every occurrence, we get

$$\sum_{n=1}^\infty |S_n(4)| = \sum_{n=1}^\infty \left|\sum_{k=[3n/2]+1}^{\infty}{}' \frac{a_k}{k-n}\right|$$
$$\leqq \sum_{n=1}^\infty \sum_{k=[3n/2]+1}^\infty \frac{|a_k|}{k-n} = \sum_{k=1}^\infty |a_k| \sum_{n=1}^{[2(k-1)/3]} \frac{1}{k-n}$$
$$\leqq A\sum_{k=1}^\infty |a_k| < \infty,$$

similarly

$$\sum_{n=1}^\infty |S_n(6)| = \sum_{n=1}^\infty \left|\sum_{k=[n/2]+1}^{\infty}{}' \frac{a_k}{k+n}\right|$$

$$\leqq A\sum_{k=1}^{\infty}|a_k| < \infty.$$

On the other hand

$$\sum_{n=1}^{\infty}|S_n(1)+S_n(5)| = 2\sum_{n=1}^{\infty}\left|\sum_{k=1}^{[n/2]}{}' a_k \frac{k}{n^2-k^2}\right|$$

$$\leqq 2\sum_{k=1}^{\infty}k|a_k|\sum_{n=2k}^{\infty}\frac{1}{n^2-k^2}$$

$$\leqq A\sum_{k=1}^{\infty}|a_k| < \infty.$$

Hence the convergence of $\sum|b_n|$ will be, under the assumption $\sum|a_n| < \infty$, equivalent to the convergence of the series

$$\sum_{n=1}^{\infty}|S_n(2)+S_n(3)|$$

which will be reduced easily to the convergence of:

(6) $$\sum_{n=1}^{\infty}\left|\sum_{k=1}^{[n/2]}{}' \frac{a_{n-k}-a_{n+k}}{k}\right|.$$

4. We are now in a position to prove Theorem 1. From the above consideration it is sufficient to construct a sequence $\{a_n\}$ such that $\sum|a_n|$ converges and the series (6) diverges.

Put

$$a_{4^n} = \frac{1}{n^2} \qquad \text{for } n = 1, 2, 3, \ldots.$$

and $a_m = 0$ if m is not of the form 4^n.

Obviously

$$\sum_{k=1}^{\infty}|a_k| = \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

and

$$\sum_{n=1}^{\infty}\left|\sum_{k=1}^{[n/2]}{}' \frac{a_{n-k}-a_{n+k}}{k}\right| = \sum_{l=0}^{\infty}\sum_{n=4^l}^{4^{l+1}-1}\left|\sum_{k=1}^{[n/2]}{}' \frac{a_{n-k}-a_{n+k}}{k}\right|$$

$$\geqq \sum_{l=1}^{\infty}\sum_{n=4^l+1}^{2\cdot 4^l}\left|\sum_{k=1}^{[n/2]}{}' \frac{a_{n-k}-a_{n+k}}{k}\right|$$

$$= \sum_{l=1}^{\infty}\sum_{n=4^l+1}^{2\cdot 4^l}|a_{4^l}|\frac{1}{n-4^l}$$

$$= \sum_{l=1}^{\infty}a_{4^l}\sum_{n=4^l+1}^{2\cdot 4^l}\frac{1}{n-4^l}$$

$$= \sum_{l=1}^{\infty} \frac{1}{l^2} \sum_{n=1}^{4^l} \frac{1}{n} \geqq \sum_{l=1}^{\infty} \frac{1}{l^2} \cdot \log 4^l \geqq \sum_{l=1}^{\infty} \frac{1}{l} = \infty.$$

Thus the theorem was proved.

5. Now we shall consider the problem: Under what condition does the absolute convergence of sine expansion imply that of cosine expansion?

THEOREM 2. *Let $f(x)$ be of period π and vanish at $x = 0$ and $x = \pi$. Suppose that the sine expansion of $f(x)$ converges absolutely, that is, the expressions* (1) *and* (2) *hold. If one of the series*

(i) $$\sum_{n=1}^{\infty} |a_n| \log n,$$

(ii) $$\sum_{n=1}^{\infty} n |\Delta a_n| \qquad \text{where } \Delta a_n = a_n - a_{n+2},$$

is convergent, then the cosine expansion of $f(x)$ converges absolutely.

PROOF. It is also sufficient to show the convergence of the series (6). In case (i) we have

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{[n/2]}{}' \frac{a_{n-k} - a_{n+k}}{k} \right| \leqq \sum_{n=1}^{\infty} \left| \sum_{k=[n/2]}^{[3n/2]+1}{}' \frac{a_k}{k-n} \right|$$
$$\leqq \sum_{n=1}^{\infty} \sum_{k=[n/2]}^{[3n/2]+1}{}' \frac{|a_k|}{k-n}$$
$$\leqq \sum_{k=1}^{\infty} |a_k| \sum_{n=[2(k-1)/3]}^{2(k+1)}{}' \frac{1}{k-n}$$
$$\leqq A \sum_{k=1}^{\infty} |a_k| \log k.$$

In case (ii) we have

$$\left| \sum_{k=1}^{[n/2]}{}' \frac{a_{n-k} - a_{n+k}}{k} \right| = \left| \sum_{k=1}^{[n/2]}{}' \frac{1}{k} \sum_{j=-k}^{k-1} \Delta a_{n+j} \right|$$
$$\leqq \left| \sum_{j=1}^{[n/2]} \Delta a_{n+j} \sum_{k=j+1}^{[n/2]} \frac{1}{k} \right| + \left| \sum_{j=-[n/2]}^{-1} \Delta a_{n+j} \sum_{k=-j}^{[n/2]} \frac{1}{k} \right|$$
$$\leqq \sum_{j=n+1}^{[3n/2]} |\Delta a_j| \sum_{k=j-n+1}^{[n/2]} \frac{1}{k} + \sum_{j=[n/2]}^{n-1} |\Delta a_j| \sum_{k=n-j}^{[n/2]} \frac{1}{k}$$
$$\leqq \sum_{j=n+1}^{[3n/2]} |\Delta a_j| \frac{\frac{3n}{2} - j - 1}{j - n + 1} + \sum_{j=[n/2]}^{n-1} |\Delta a_j| \frac{j - \frac{n}{2} + 1}{n - j}.$$

Hence

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{[n/2]}{}' \frac{a_{n-k} - a_{n+k}}{k} \right|$$

$$\leqq A\sum_{n=1}^{\infty}\sum_{j=n+1}^{[3n/2]}|\Delta a_j|\frac{n}{2(j-n+1)}+A\sum_{n=1}^{\infty}\sum_{j=[n/2]}^{n-1}|\Delta a_j|\frac{n+2}{2(n-j)}$$
$$\leqq A\sum_{j=1}^{\infty}|\Delta a_j|\sum_{n=[2j/3]}^{j-1}\frac{n}{j-n}+A\sum_{j=1}^{\infty}|\Delta a_j|\sum_{n=j+1}^{2j}\frac{n}{n-j}$$
$$\leqq A\sum_{j=1}^{\infty}j|\Delta a_j|.$$

Thus the convergent majorants of the series (6) were obtained in both cases, and the theorem was proved.

We remark that if in Theorem 2 the coefficients $\{a_n\}$ forms a decreasing sequence, then the series (ii) converges and so we get the same conclusion. In fact

$$\sum_{n=1}^{\infty}n|\Delta a_n| = \sum_{n=1}^{\infty}n(a_n-a_{n+2})$$
$$=2\sum_{n=1}^{\infty}a_n-a_1-\lim_{N\to\infty}(N-1)a_{N+1}-\lim_{N\to\infty}N\,a_{N+2}$$
$$=2\sum_{n=1}^{\infty}a_n-a_1$$

since $a_n=o(n)$ as $n\to\infty$ by Abel's lemma.

6. We discussed hitherto the absolute convergence of the cosine expansion of function which has an absolute convergent sine expansion. Let us now consider the case where the situations of "sine" and "cosine" are exchanged with each other. The analogues of Theorem 1 and Theorem 2 (i) are valid but Theorem 2 (ii) is not the case. We shall prove the following theorems.

THEOREM 3. *Let $f(x)$ be a function of period π and vanish at $x=0$ and $x=\pi$. Suppose that the cosine expansion of $f(x)$ converges absolutely, that is,*

$$f(x)=\sum_{n=0}^{\infty}a_n\cos nx, \tag{7}$$

$$\sum_{n=0}^{\infty}|a_n|<\infty. \tag{8}$$

Under these conditions the sine expansion of $f(x)$ is not always absolutely convergent.

THEOREM 4. *Let $f(x)$ be of period π and vanish at $x=0$ and $x=\pi$. Suppose that the cosine expansion of $f(x)$ converges absolutely, that is, the expressions* (7) *and* (8) *hold, and that*

$$\sum_{n=0}^{\infty}|a_n|\log(n+1)<\infty. \tag{9}$$

Then the sine expansion of $f(x)$ converges absolutely.

If $f(x)$ does not vanish at $x=0$ or at $x=\pi$, its odd extension of period

2π is not continuous and so its sine expansion is evidently not absolutely convergent. Hence the condition $f(0)=f(\pi)=0$ is indispensable, and then we get easily

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (-1)^n a_n = 0, \tag{10}$$

or equivalently

$$\sum_{n=0}^{\infty} a_{2n} = \sum_{n=0}^{\infty} a_{2n+1} = 0.$$

If we consider the analogue of Theorem 2 (ii), with stronger condition $a_n \downarrow 0$, the monotonity of the sequence of coefficients $\{a_n\}$ needs therefore some modification. For this it will be natural to suppose that the sequence $\{a_n\}$ is monotone except the first two terms a_0 and a_1, that is,

$$a_3 \geqq a_4 \geqq a_5 \geqq \dots,$$

and

$$a_0 = -\sum_{n=1}^{\infty} a_{2n}, \quad a_1 = -\sum_{n=1}^{\infty} a_{2n+1}. \tag{11}$$

But even under this condition of the sequence $\{a_n\}$ the analogue of Theorem 2 (ii) does not valid, this fact will be shown later in the proof of Theorem 3.

7. Let the expression (7) and (8) hold, and let

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx.$$

By easy calculation we get

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi}\sum_{k=0}^{\infty} a_k \int_0^{\pi} \cos kx \sin nx \, dx \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty}{}' a_k \left(\frac{1}{k+n} + \frac{1}{n-k}\right) \end{aligned} \tag{12}$$

where $\sum'$ has the same meaning as in §3. Hence if n is odd, putting $n = 2m+1$, we have

$$b_{2m+1} = \frac{4}{\pi}\sum_{k=0}^{\infty} a_{2k}\left(\frac{1}{2k+2m+1} + \frac{1}{2m-2k+1}\right).$$

Substituting the first of (11) into this formula we get

$$b_{2m+1} = \frac{4}{\pi}\left\{\sum_{k=1}^{\infty} a_{2k}\left(\frac{1}{2k+2m+1} + \frac{1}{2m-2k+1}\right) - \sum_{k=1}^{\infty} a_{2k}\frac{2}{2m+1}\right\} \tag{13}$$

$$= \frac{32}{\pi(2m+1)} \sum_{k=1}^{\infty} \frac{k^2 a_{2k}}{(2m+2k+1)(2m-2k+1)} .$$

Further we shall divide the sum into several parts:

$$b_{2m+1} = \frac{32}{\pi(2m+1)} \left(\sum_{k=1}^{2m} + \sum_{k=2m+1}^{\infty} \right) \frac{k^2 a_{2k}}{(2m+2k+1)(2m-2k+1)} \tag{14}$$

$$= \frac{8}{\pi(2m+1)} \sum_{k=1}^{2m} k\, a_{2k} \left(\frac{1}{2m-2k+1} - \frac{1}{2m+2k+1} \right)$$

$$+ \frac{32}{\pi(2m+1)} \sum_{k=2m+1}^{\infty} \frac{k^2 a_{2k}}{(2m+2k+1)(2m-2k+1)}$$

$$= \frac{8}{\pi(2m+1)} \sum_{k=1}^{m} \frac{k\, a_{2k}}{2m-2k+1} + \frac{8}{\pi(2m+1)} \sum_{k=m+1}^{2m} \frac{k\, a_{2k}}{2m-2k+1}$$

$$- \frac{8}{\pi(2m+1)} \sum_{k=1}^{2m} \frac{k\, a_{2k}}{2m+2k+1} + \frac{32}{\pi(2m+1)} \sum_{k=2m+1}^{\infty} \frac{k^2 a_{2k}}{(2m+2k+1)(2m-2k+1)}$$

$$= \frac{8}{\pi(2m+1)} \sum_{k=1}^{m} \frac{1}{2m-2k+1} \{k a_{2k} - (2m-k+1)\, a_{2(2m-k+1)}\}$$

$$- \frac{8}{\pi(2m+1)} \sum_{k=1}^{2m} \frac{k\, a_{2k}}{2m+2k+1} - \frac{32}{\pi(2m+1)} \sum_{k=2m+1}^{\infty} \frac{k^2 a_{2k}}{(2k+2m+1)(2k-2m-1)}$$

$$= \frac{8}{\pi(2m+1)} P_m - \frac{8}{\pi(2m+1)} Q_m - \frac{32}{\pi(2m+1)} R_m$$

say. For the proof of Theorem 3 it is sufficent to show the divergence of the series

$$\sum_{m=0}^{\infty} |b_{2m+1}| = \sum_{m=0}^{\infty} \frac{1}{2m+1} \left| \frac{8}{\pi} P_m - \frac{8}{\pi} Q_m - \frac{32}{\pi} R_m \right| . \tag{15}$$

We shall now construct a counter example of the coefficients $\{a_n\}$.

Put

$$a_n = \frac{1}{n(\log n)^2} \qquad \text{for } n = 2, 3, 4, \ldots\ , \tag{16}$$

and

$$a_0 = -\sum_{n=1}^{\infty} a_{2n}, \qquad a_1 = -\sum_{n=1}^{\infty} a_{2n+1}. \tag{17}$$

Obviously $\sum |a_n| < \infty$ and the condition of Theorem 3 is satisfied.

Now we shall estimate the series (15), substituting (16) into P_m we have

$$P_m = \sum_{k=1}^{m} \frac{1}{2m-2k+1} \left\{ \frac{1}{2(\log 2k)^2} - \frac{1}{2(\log(2m-k+1))^2} \right\}.$$

By elementary estimation, if $1 \leqq k \leqq m$, we get

$$\left| \frac{1}{(\log 2k)^2} - \frac{1}{(\log(2m-k+1))^2} \right|$$

$$= \left| \frac{\log (2k(2m-k+1)) \log \dfrac{2m-k+1}{2k}}{(\log 2k)^2 (\log (2m-k+1))^2} \right|$$

$$\leqq \frac{A \log \dfrac{2m-k+1}{2k}}{(\log k)^2 \log m}$$

$$\leqq \begin{cases} \dfrac{A}{(\log k)^2}, & \text{if } 1 \leqq k \leqq \dfrac{2m+1}{3}; \\ \dfrac{A}{(\log k)^2 \log m}, & \text{if } \dfrac{2m+1}{3} \leqq k \leqq n, \end{cases}$$

since

$$\left| \log \frac{2m-k+1}{2k} \right| \leqq \begin{cases} \log m, & \text{if } 1 \leqq k \leqq \dfrac{2m+1}{3}; \\ \log \dfrac{m+1}{2m} \leqq A, & \text{if } \dfrac{2m+1}{3} \leqq k \leqq n. \end{cases}$$

Theorefore we have

$$|P_m| \leqq A \sum_{k=1}^{[(2m+1)/3]} \frac{1}{2m-2k+1} \frac{1}{(\log k)^2} \tag{18}$$

$$+ A \sum_{k=[(2m+1)/3]+1}^{m} \frac{1}{2m-2k+1} \frac{1}{(\log k)^2 \log m}$$

$$\leqq \frac{A}{(\log m)^2} + \frac{A}{(\log m)^2} \leqq \frac{A}{(\log m)^2}.$$

And easily

$$|Q_m| \leqq \sum_{k=1}^{2m} \frac{1}{2(\log 2k)^2 (2m+2k+1)} \leqq \frac{A}{(\log m)^2}, \tag{19}$$

$$|R_m| = \sum_{k=2m+1}^{\infty} \frac{k}{2(\log 2k)^2 (2k+2m+1)(2k-2m-1)} \tag{20}$$

$$\geqq \sum_{k=2m+1}^{\infty} \frac{k}{2(\log 2k)^2 3k(2k+1)}$$

$$\geqq \frac{C}{\log m}$$

where C is a positive constant independent of m.

From (15), (18), (19) and (20) we obtain

$$\sum_{m=0}^{\infty} |b_{2m+1}| \geqq \frac{32}{\pi} \sum_{m=0}^{\infty} \frac{|R_m|}{2m+1} - \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{|P_m|}{2m+1} - \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{|Q_m|}{2m+1}$$

$$\geqq \frac{32C}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)\log m} - \frac{8A}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)(\log m)^2}$$

$$= \infty,$$

and this proves Theorem 3.

We shall now prove Theorem 4. By the formula (12)

$$\sum_{n=1}^{\infty}|b_n| = \frac{4}{\pi}\sum_{n=1}^{\infty}\left|\sum_{k=0}^{\infty}{}' a_k\left(\frac{1}{k+n}+\frac{1}{n-k}\right)\right|$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\left|\sum_{k=0}^{\infty}{}' a_k\left(\frac{1}{k+n}+\frac{1}{n-k}-\frac{2}{n}\right)\right| \qquad \text{(by (10))}$$

$$= \frac{8}{\pi}\sum_{n=1}^{\infty}\left|\sum_{k=0}^{\infty}{}' \frac{k^2 a_k}{n(n+k)(n-k)}\right|$$

$$\leqq \frac{8}{\pi}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{k^2|a_k|}{n(n+k)(n-k)} + \frac{8}{\pi}\sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty}\frac{k^2|a_k|}{n(n+k)(k-n)}$$

$$= \frac{8}{\pi}M + \frac{8}{\pi}N$$

say. We have then

$$M = \sum_{k=0}^{\infty}k^2|a_k|\sum_{n=k+1}^{\infty}\frac{1}{n(n+k)(n-k)}$$

$$\leqq \sum_{k=0}^{\infty}k^2|a_k|\left(\sum_{n=k+1}^{2k}\frac{1}{n^2(n-k)}+\sum_{n=2k+1}^{\infty}\frac{2}{n^3}\right)$$

$$\leqq A\sum_{k=0}^{\infty}k^2|a_k|\left(\frac{\log k}{k^2}+\frac{1}{k^2}\right)$$

$$\leqq A\sum_{k=0}^{\infty}|a_k|\log k,$$

$$N = \sum_{k=2}^{\infty}k^2|a_k|\sum_{n=1}^{k-1}\frac{1}{n(n+k)(k-n)}$$

$$\leqq \sum_{k=2}^{\infty}k^2|a_k|\,\frac{1}{k}\left(\sum_{n=1}^{[k/2]}\frac{1}{n(k-n)}+\sum_{n=[k/2]+1}^{k-1}\frac{1}{n(k-n)}\right)$$

$$\leqq A\sum_{k=2}^{\infty}k|a_k|\left(\frac{\log k}{k}+\frac{\log k}{k}\right)$$

$$\leqq A\sum_{k=2}^{\infty}|a_k|\log k.$$

Hence we get immediately the conclusion: $\sum|b_n| < \infty$.

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