ZERO-DIMENSIONAL SPACES*

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Introduction. This paper contains some topological results principally concerning zero-dimensional spaces. We observe first that the dimension of a uniform space can be so defined that invariance under completion is trivial. Katětov's result [10], that the covering dimension of normal spaces is preserved under Stone-Čech compactification, follows as a corollary. We obtain sharper results in the zero-dimensional case, including a representation of the completion of uX as the structure space of the Boolean algebra of uniformly continuous functions on uX to the two-element field. An example shows that inductive dimension zero need not be preserved under Stone-Čech compactification.

The concluding section of the paper answers three questions raised in [9], by counterexamples, and contributes some propositions in continuation. Two of the examples were communicated by Professor V. L. Klee. Others who have had a hand in the paper are M. Henriksen, T. Shirota, and H. Trotter, in the counterexamples, and especially S. Ginsburg. The paper grew out of our collaboration [5] with Professor Ginsburg, which involves closely related ideas.

1. Dimension and Completion. The term *dimension*, unqualified, will refer to the Menger-Urysohn inductive dimension [8]. The Lebesgue *covering dimension* is known to coincide for metric spaces [11]. We define below two "covering" dimensions for uniform spaces. In this connection we note the special uniformities f and e [18, 14] consisting of all normal coverings having finite resp. countable normal subcoverings. For any uniformity u, the finite coverings in u define a uniformity fu. (The same is true for countable coverings [5]). The completion of a uniform space fuX is the *Samuel compactification* of uX [13].

DEFENITION. The (finite) *large dimension* $dl(uX)$ of a uniform space uX is the least integer m, if such exists, such that every covering in u has a refinement in u, no $m+2$ elements of which have a common point. The *uniform dimension* $du(uX)$ is $dl(fuX)$.

We remark that $du(uX) \leqq dl(uX)$, if both are defined. The problem of the reverse inequality seems quite difficult and is not touched on below.

1.1. THEOREM. *Large dimension is invariant under completion.*

PROOF. Let $\overline{uX}$ be the completion of uX. That $dl(uX) \leqq dl(\overline{uX})$ is obvious.

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Suppose $dl(uX) = m$. Let $\{U_\alpha\}$ be a large covering of $\overline{uX}$ and $\{V_\beta\}$ a large star-refinement of $\{U_\alpha\}$. Then $\{\bar{V}^0_\beta\}$ is a large refinement of $\{U_\alpha\}$. Since $\{V_\beta \cap X\}$ is large on uX, there is a large refinement $\{W_\gamma\}$ such that no $m+2$ sets W_γ have a common point. In $\overline{uX}$ form $\bar{W}^0_\gamma$. Then $\{\bar{W}^0_\gamma\}$ is a large covering of $\overline{uX}$ [12], refining $\{U_\alpha\}$, and no finite subfamily has a common point unless the corresponding W_γ already have a common point [12]. Hence the theorem follows.

1. 2. COROLLARY. *Uniform dimension is invariant under Samuel compactification.*

1. 3. COROLLARY. *Uniform dimension is invariant under completion.*

1. 4. COROLLARY (Katětov). *The covering dimension of normal spaces is invariant under Stone-Čech compactification.*

For the covering dimension of a normal space X is the uniform demension of fX, and the Stone-Čech compactification is $\overline{fX}$.

Every zero-dimensional space X can be embedded in a zero-dimensional compact space, namely the structure space of the Boolean algebra $M(X)$ of open-closed subsets of the given space [16]. Stone's construction is equivalent to completion under the uniformity defined by all finite open partitions. (Note that every zero-dimensional space X has a uniformity u such that uX has uniform dimension zero, namely that just mentioned). In general, let $C(uX, T)$ be the algebra of all uniformly continuous functions on uX to the two-element field T. The structure space $H(C(uX, T))$, the space of maximal ideals in the hull-kernel topology, is always compact and zero-dimensional [16].

1. 5. THEOREM. *The Samuel compactification of uX is homeomorphic with $H(C(uX, T))$ if and only if uX has uniform dimension zero; in this case there is a natural homeomorphism.*

PROOF. For any point x in $\sigma X = \overline{fuX}$, let I_x be the ideal in $C(uX, T)$ consisting of all functions f such that x is a limit point of $f^{-1}(0)$. Clearly I_x is a maximal ideal.

Let M be a maximal ideal in $C(uX, T)$. Let D_M be the filter or all open-closed subsets Q of X such that the function f_Q sending Q to 0 and $X - Q$ to 1 is in M. (Precisely, D_M is filter base; it is that because a maximal ideal is prime). If D_M is Cauchy then there is precisely one point x in σX which is a limit point of every element of D_M. Let g be any element of I_x. Then $\{g^{-1}(0), g^{-1}(1)\}$ is in u, and since x is not a limit point of $g^{-1}(1)$, the Cauchy filter D_M must contain $g^{-1}(0)$. Therefore $I_x \subseteq M$, and since M is a proper ideal, $I_x = M$.

Suppose D_M is not Cauchy and uX satisfies the condition. Then there is a finite partition $\{Q_1, \dots . Q_n\}$ in u such that no Q_i is in D_M. We can assume $n = 2$; for 0 is a finite intersection of the sets $X - Q_i$, and hence not all these are in D_M. But then neither f_{Q_1} nor $1 - f_{Q_1} = f_{Q_2}$ is in M. Hence M is not maximal.

Thus if $du(uX) = 0$, there is a natural one-to-one correspondence between the points of σX and the maximal ideals of $C(uX, T)$. Since $H = H(C(uX, T))$ is compact, it suffices to show that a neighborhood of a point in σX is a neighborhood in H. But every neighborhood of x contains an open and closed neighborhood Q such that if $Q \cap X = R$, f_R is in $C(uX, T)$; for every Cauchy filter has a basis consisting of such sets. The set of all maximal ideals not containing f_R is open and closed in H, and the proof of sufficiency is complete.

If uX has uniform dimension non-zero, so does σX, by 1.3. The uniform dimension of a compact space is the covering dimension, which coincides with the inductive dimension [16]. Hence the proof is complete.

1.6. COROLLARY. *The Stone-Čech compactification of X is zero-dimensional and naturally homeomorphic with $H(M(X))$ if and only if fX has uniform dimension zero.*

1.7. COROLLARY. *The Wallman compactification of X is homeomorphic with $H(M(X))$ if and only if X has covering dimension zero.*

PROOF. Samuel showed [13] that the Wallman and Stone-Čech compactifications are equivalent precisely for normal spaces. If X is normal, then the uniform dimension fX is the covering dimension of X. If X is not normal, then its Wallman compactification is not Hausdorff [19]. And a space of covering dimension zero is clearly normal.

2. **Disconnection**, Consider the following five possible properties of completely regular spaces.

(a) The open-closed sets form an open basis.

(b) Every finite normal covering is refined by an open partition.

(c) Every finite open covering is refined by an open partition.

(d) For every continuous real-valued function f, $f^{-1}(0)$ is open.

(e) The closure of every open set is open.

All the open sets concluded to exist in (b) — (e) are obviously closed. Properties (a), (b), (c) express that the inductive dimension of X, uniform dimensin of fX, covering dimension of X, respectively, vanish. Property (d) defines pseudo-discrete or P-spaces [2]; (e), extremally disconnected spaces [6].

Evidently (c) or (e) implies (b). Since the stars of points in finite normal coverings form an open basis [18], (b) implies (a). From the definition of complete regularity, (d) implies (a). In normal spaces (b) and (c) are equivalent (every finite open covering is normal [18]); and obviously (c) implies normality. It is well known that (a) and (c) are equivalent in compact spaces [16] and in separable spaces [8].

2.1. THEOREM. *Property* (d) *implies* (b).

PROOF. In view of 1.6, it suffices to show that $\beta X = \overline{fX}$ satisfies (a). Now the proof of the Gelfand-Kolmogoroff theorem in [3] shows that every neighborhood of a point p in βX contains an inverse image of 0, under a continuous real function of X, whose closure in βX contains p. With (d), such a set is open and closed; since its characteristic function extends continuously to βX, its closure in βX is open and closed. Since βX is regular, the theorem follows.

The completion of eX is called υX. If $\upsilon X = X$, then X is a *Q-space*. [7, 14]. The structure space of the algebra of all continuous real-valued functions on X is βX [3] ; υX is the space of homomorphisms of that algebra onto the reals [7].

P-spaces are characterized [2] by the property

(d′) *Every G_δ-set is open.*

Thus if X is a P-space, then $M(X)$ is an $\aleph_0$-additive field of sets. For any $\aleph_0$-complete Boolean algebra B, let $K(B)$ be the subspace of $H(B)$ consisting of all maximal ideals closed under countable union.

2. 2. THEOREM. *If B is an $\aleph_0$-complete Boolean algebra, then $K(B)$ is a P-space and a Q-space. If X is a P-space then $K(M(X)) = \upsilon X$.*

PROOF. That every G_δ in $K(B)$ is open is clear. Using results of Hewitt and Shirota, (1) $K(B)$ is a Q-space because every CZ-maximal family of zero sets has a common point [14] ; (2) to show that $K(M(X))$ is υX it suffices to show that X is dense in $K(M(X))$ and that every continuous real-valued function on X has a continuous extension on X has a continuous extension on $K(M(X))$ [7], both of which are easily seen.

If X is not a P-space then neither is υX, for the class is hereditary. Thus $K(M(X)) = \upsilon X$ precisely for P-spaces.

2. 3. THEOREM. *A zero-dimensional space X satisfies* (e) *if and only if $M(X)$ is complete.*

PROOF. The necessity is trivial. But if X satisfies (a) but not (e), consider any open set U for which $\overline{U}$ is not open. The family of all open-closed sets contained in $\overline{U}$ can have no supremum in $M(X)$.

Stone [15] demonstrated 2. 3 for compact spaces, together with numerous further results on these spaces. See also Hewitt's paper [6]. The basic reference on P-spaces is [2].

2. 4. THEOREM. *An extremally disconnected P-space of non-measurable power is discrete.*

PROOF. $M(X)$ is a complete Boolean algebra or non-measurable power. Its maximal ideals I closed under countable union are precisely those such that $M(X) - I$ contains an atom [4]. Hence $K(M(X))$ is discrete (the complement of a point is the hull of its kernel and hence is closed). Hence, by 2. 2, X is discrete.

We have shown (c) or (d) or (e) implies (b), which implies (a). If there is a non-discrete space simultaneously satisfying (d) and (e), then there may be a non-normal one; but it is consistent with the usual axioms for set theory to assume (d) and (e) imply (c). (For it is consistent to assume all cardinal number are non-measurable [17]). No other implications hold among these properties in completely regular spaces.

We shall not give all the examples. The well known Tychonoff plane, described e. g. in [8] and incidentally below, can be shown to satisfy (b) but no more. Modifications of this example give several others. A non-normal P-space is given in [2] (hence (d) does not imply (c)). Theorems in [2] facilitate the construction of a linearly ordered space satisfying (d) (and hence (c)) but not (e).

2.5. **Example.** (Henriksen-Shirota). The space X will be zero-dimensional, but βX is not zero-dimensional, and hence X does not satisfy (b). For convenience we use redundant coordinates. X consists of all ordered quadruples (x, y, α, β), where x and y are real numbers in $[0,1]$, if x is rational then α is any ordinal $< \omega_1$ if x is irrational then α is any ordinal $< \omega_2$, and β depends on y by the same rules, provided that if $x = y$ then, $\alpha = \beta$.

A neighborhood of $(x, y, \alpha, \beta,)$ $x \neq y$, is required to contain an order neighborhood in the set of all (x, y, γ, δ), $\gamma \leqq \alpha, \delta \leqq \beta$. A set U is a neighborhood of (x, x, α, α) if (i) there is $\beta < \alpha$ such that $\alpha \geqq \gamma > \beta$ implies (x, x, γ, γ) in U, (ii) for every rational y there is n ordinal $\delta(y) < \omega_1$, and for every irrational y there is an ordinal $\delta(y) < \omega_2$, such thatall $(x, y, \varepsilon, \zeta)$ with $\alpha \geqq \varepsilon > \beta$ and $\zeta > \delta(y)$ are in U, and (iii) there exists a neighborhood V of x in the real line such that all $(y, x, \zeta, \varepsilon)$ with y in V, $\zeta > \delta(y)$, and $\alpha \geqq \varepsilon > \beta$ are in U.

Evidently X is a T_1 space satisfying (a) and hence is completely regular. Furthermore, the real valued function φ on X defined by $\varphi(x, y, \alpha, \beta) = x$ is easily seen to be continuous at each point. Hence the sets A consisting of all $(1, y, \alpha, \beta)$ and B consisting of all $(0, y, \alpha, \beta)$ are closed sets separated by a continuous functions; therefore the closures in βX of A and B are disjoint [1]. However, A and B cannot be separated by an open partition.

Suppose S is an open and closed subset of X containing A. In particular, S contains $\{(1, 1, \alpha, \alpha)\}$. Hence for each countable ordinal, for some neighborhood V of 1 in the reals, for every y in V, S contains all $(y, 1, \zeta, \varepsilon)$ with $\zeta > \delta(y, \varepsilon)$. Uncountably many V's contain some fixed neighborhood U. If y is irrational then $\liminf_\varepsilon \delta(y, \varepsilon) < \omega_1$ and the closed set S must contain all (y, y, β, β) with $\beta > \lambda$ for some λ. Then the argument reverses; if S contains a cofinal subset of $T_y = \{(y, y, \beta, \beta)\}$, with y rational resp. irrational, then S contains a non-void residual subset of T_z for all irrational resp. rational z in some neighborhood of y.

Concluding, $X - S$ is supposed to be an open and closed set containing T_0. Let the real number z be the supremum of those γ for which $X - S$ contains a cofinal subset of T_γ. Either S or $X - S$, say S, contains a cofinal subset of T_z. Say z is rational. Then for some $\theta > 0$, for every irrational $t > z - \theta$, almost all of T_t is in S. Then there exists rational $u > z - \theta$ such that a cofinal subset of T_u is in $X - S$, by definition of z; but this is impossible. Similarly for the other cases; thus A and B cannot be separated. Therefore X does not have the property (b).

3. **Homogeneity**. In this section terms and abbreviations of [9] will be used. Generically S will be a topological space; a *figure* F is an ordered n-tuple $(F_0, \dots . F_{n-1})$ with $F_0 \subseteqq S$ and $F_i \subseteqq F_0$. The definitions of topological equivalence (*t-type*) and Frechet equivalence (*f type*) of figures are the obvious extensions of homeomorphism resp. equivalence of Frechet dimension. The points x, y, in S are m-(micro) resp. s-(semi) equivalent if they have bases of neighborhoods U_α, V_α, such that the figures (U_α, x), (V_α, y), resp. the open sets U_α, V_α, are pairwise of the same t type. Notice the weaker relations *mf*, *sf*, obtained by substituting Frechet equivalence in the preceding sentence. The space S is *packable* resp. *shrinkable about* x if there is a basis of neighborhoods U_α of x such that each figure (U_α, x) is of the same t type resp. f type as (S, x) ; the space is *locally packable* resp. *locally shrinkable at* x if x has a packable resp. shrinkable neighborhood. The phrase "at (about) x" will be omitted if this is true for all points of S.

The existence of a packable (shrinkable) basis follows from the definition of the local property. It does not suffice to assume a neighborhood of x packable (shrinkable) about x — witness the letter "T".

One of the most interesting questions on homogeneous spaces is when a microhomogeneous space is homogeneous. The main theorem of [9] rests heavily on shrinkability in showing that this implication holds in connected linearly ordered spaces. We notice

3.1. *A microhomogeneous zero-dimensional space is homogeneous.*

For if two points have equivalent open neighborhoods there are corresponding open-closed subneighborhoods U, V ; there is a mapping of the whole space sending U to V, V to U, and leaving $S - U - V$ fixed.

3.2. *A packable space is s; a shrinkable space is sf.*

3.3. *The s-rooms (sf-rooms) of a locally packable (shrinkable) space are open and closed.*

PROOF. If S is packable, any two points have neighborhood bases all of the same t type, namely the type of the space. Hence the s-rooms of a locally packable space are open. Since they partition the space, they are closed. The arguments on shrinkable and locally shrinkable spaces are similar.

3.4. *If S is locally packable (shrinkable) at x, and x is s-equivalent to y, then S is locally packable (shrinkable) at y.*

3.5. *If the packable space S is compact Hausdorff, it is totally disconnected; if it is connected, it is locally connected.*

The proofs are evident.

Now a packable space may be Boolean; it may be connected, locally connected, and non-compact. If it contains a compact resp. connected open subset, it is locally compact resp. locally connected and not compact. The first part of this assertion follows formally from the fact that local compactness is s-invariant [9]. The second part is easily proved; it is worth noting that one has to do here with the not commonly considered property of local connetedness in a neighborhood of a point. Precisely,

3.6. *If S is locally connected in a neighborhood of x, and x is s-equivalent to y, then S is locally connected in a neighborhood of y.*

The parallel propositions 3.4 and 3.6 contrast with the counterexample (for weaker properties) 3.1 of [9].

3.7. **Example**. (Trotter). Let S be the quotient space of the product of the Cantor set with the interval $[0, 1]$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$, where f is an automorphism of fhe Cantor set under which the orbit of each point is everywhère dense. Such an automorphism is easily described in the representation of each point x in the Cantor set as a sequence (x_i) of zeros and ones. Regard the sequence (x_i) as the reversed formal dyadic "number" (x_{-i}); add one in the last place and carry, possibly to infinity.

It is easily verified that S is a locally packable homogeneous indecomposable continuum which is neither locally connected not totally disconnected.

It is also false that packbility or shrinkability about a point is invariant even under m-equivalance. This settles a question raised in [9].

3.8. **Example**. Let T be the triple $(X, Z, 0)$ in the complex plane, where $Z = \{1/n\}$, and $x + iy$ is in X if and only if (i) x and y are $\geqq 0$ and < 2 and either $y = 0$ or $y = nx$ for some integer n, or (ii) $x + iy$ is in $L = [1 + 2i, 2 + 2i]$. The set S is the set of all finite sequences $(s_1, \dots . s_n)$, with s_i in Z for $i < n$ and s_n in $X - Z$, less the set of all (s_1), s_1 in L. Let $U_j(q)$ designate the set of all points in S with j-th coordinate in the open set U, which contains q. A neighborhood basis at the point s, with last coordinate s_n, is given by (a) if $s_n \neq 0$, the set of all $U_n(s_n)$; (b) if $s_n = 0$, the set of all unions $U_n(s_n) \cup U_{n-1}(t)$, where t is the next to last coordinate of s. For the point 0 the last qualification is vacuous.

It can be shown that S is packable about 0 but not shrinkable about any of the m-equivalent points $(s_1, \dots . 0)$.

It was conjectured in [9] that every shrinkable connected space is locally connected. Professor V. L. Klee observes that the pseudo-arc refutes the conjecture. The product of any connected non-locally connected finite-dimensional space by the Hilbert cube is another counterexample. *) The point is perhaps now obvious; packability is the central property, and shrinkability serves chiefly when there is invariance of domain (and thus the properties

*) In his review of [9] in *Zentralblatt* 61, T. Genea gives still another counter-example to conjecture.

coincide). There is invariance of domain in the spaces most used in [9], the connected linearly ordered spaces and the locally Euclidean spaces. (Theorem 2.13 of [9], on locally Euclidean spaces, omits a hypothesis; the space must of course be Hausdorff). Finally, Klee points out that a product of two-point homogeneous spaces need not be two-point homogeneous; it suffices if one factor is connected and the other is not.

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