ON THE GROUP ISOMORPHISM OF UNITARY GROUPS IN AW^*-ALGEBRAS

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1. Introduction. The connection between the algebraic isomorphism of W^*-algebras and the group isomorphism of their unitary groups was studied by H. A. Dye [1] under the restriction of the *weak bicontinuity* of the group isomorphism. However as his restriction is strong in some part, it seems that an analogous study under another restriction is necessary.

The purpose of this paper is to study the group isomorphism of unitary groups in factors of *all types* under the assumption of the *one-sided uniform continuity*. Since the uniform topology is free from a underlying Hilbert space, it is meaningful that we extend our objects to AW^*-algebras. Therfore we shall always consider AW^*-algebras. Then our main assertion in this paper is as follows: Any (one-sided) uniformly continuous group isomorphism between the unitary groups of two AW^*-factors is implemented by a (linear or conjugate linear) $*$-isomorphism of the factors themselves.

2. Group isomorphisms. Let M and N be two AW^*-algebras in the sense of Kaplansky [cf. 3], M_u and N_u their unitary groups. We shall prove the following theorem.

THEOREM. *Any (one-sided) uniformly continuous group isomorphism between the unitary groups in two AW^*-factors is implemented by a (linear or conjugate linear) $*$-isomorphism of the factors themselves.*

The proof will follow from the following considerations and lemmas.

Let M be an AW^*-algebra, M_u the unitary group of all unitary elements in M and M_s the real vector space of all self-adjoint elements of M.

Then $\exp(ith)$ $(-\infty < t < \infty,\ h \in M_s)$ is a uniformly continuous one-parameter subgroup in M_u; conversely, if $U(t)$ $(-\infty < t < \infty)$ is a uniformly continuous one-parameter subgroup in M_u, then there exists an element h $(\in M_s)$ such that $U(t) = \exp(ith)$ [cf. 2]. Moreover if $\exp(ith_1) = \exp(ith_2)$ $(-\infty < t < \infty)$, then $h_1 = h_2$. Therfore there exists a one-to-one correspondence between self-adjoint elements of M and uniformly continuous one-parameter subgroups in M_u, in which we have

$$\lim_{t \to 0} \frac{\| \exp(ith) - I \|}{|t|} = \| h \|.$$

Next, let M and N be two AW^*-algebras, M_u and N_u their unitary groups, and M_s and N_s the real vector spaces of all self-adjoint elements of M and N respectively, and suppose that there exists a uniformly continuous group isomorphism ρ of M_u onto N_u. Then by an analogous method as in the case of a compact linear group [cf. 4, p. 177, Satz 1], we can easily show that there exists a fixed positive number K such that

$$\| \rho(u) - \rho(v) \| \leqq K \| u - v \| \quad \text{for } u, v \in M_u.$$

If h belongs to M_s, $\rho(\exp(ith))$ is uniformly continuous and $\rho(\exp(ith)) = \exp(ith')$ $(h' \in N_s)$. If we define $f(h) = h'$, then f is a mapping of M_s into N_s, and it is easily shown that the mapping f satisfies the following relations: (1) $\|f(h)\| \leqq K\|h\|$, (2) $f(\alpha h_1 + \beta h_2) = \alpha f(h_1) + \beta f(h_2)$, where α and β are real numbers, (3) $f(u^*hu) = \rho(u)^* f(h)\rho(u)$; where u is any unitary element of M_u, and (4) $f(i[h_1, h_2]) = i[f(h_1), f(h_2)]$, where $[h_1, h_2] = h_1h_2 - h_2h_1$.

Now we shall extend the mapping f to a mapping of M into N by $f(h_1 + ih_2) = f(h_1) + if(h_2)$, where h_1 and h_2 are elements of M_s.

It is easily shown that the extended mapping f satisfies the following relations: (1)′ $\|f(a)\| \leqq 2K\|a\|$, (2)′ $f(\lambda a_1 + \mu a_2) = \lambda f(a_1) + \mu f(a_2)$, where λ and μ are complex numbers, (3)′ $f(u^*au) = \rho(u)^* f(u)\rho(u)$, where u is any unitary element of M, (4)′ $f([a_1, a_2]) = [f(a_1), f(a_2)]$, and (5)′ $f(u^*) = f(u)^*$.

We shall find the properties of the mapping f and characterise the group isomorphism ρ. Henceforward we shall assume that the AW^*-algebras M and N are *factors*, that is, their centers are the scalar multiples of the identity I; however, certains of the following discussions are extended to general AW^*-algebras under suitable restrictions.

We shall, at first, show that if e is a projection of M, then either $f(e)$ is a projection of N, or $-f(e)$ is a projection of N (Lemma 5). It is shown that f or $-f$ preserves the power structure of normal elements (Lemma 7), and finally we shall give the proof of the theorem.

Let e be a projection of M, then $\exp(ite)$ and $\rho(\exp(ite))$ are uniformly continuous representations of the one-dimensional torus group, since $\exp(ite) = (I - e) + \exp(it)e$. Therefore by the complete reducibility of representation we have

$$\rho(\exp(ite)) = \sum_{n=-\infty}^{\infty} \exp(itn)p'_n,$$

where p'_n are mutually orthogonal projections of N, $\sum_{n=-\infty}^{\infty} p'_n = I$ and $f(e) = \sum_{n=-\infty}^{\infty} np'_n$. It is clear that $p'_n = 0$ if $|n|$ is large. We shall denote p'_n as $f(e)_n$ in notation.

LEMMA 1. *If* $f(e)_p \neq 0$, $f(e)_n = 0$ *for all* n $(|n - p| \geqq 2)$.

PROOF. Now suppose that $f(e)_n \neq 0$ for some n $(|n - p| \geqq 2)$. Then, by the comparability of projections in AW^*-factors, one of three relations $f(e)_n \gtreqless f(e)_p$ holds. Suppose that $f(e)_n \gtrsim f(e)_p$, then there exists a projection $e' \sim f(e)_p$. Let v be a partially isometric operator of N which gives the equivalence $e' \sim f(e)_p$, that is, $v^*v = e'$ and $vv^* = f(e)_p$, and define an operator u by $u = v + v^* + (I - e' - f(e)_p)$. At first, we say that u is a unitary operator of

N. In fact, since e' and $f(e)_p$ are mutually orthogonal, $v^{*2} = v^2 = v^*(I - e' - f(e)_p) = v(I - e' - f(e)_p) = (I - e' - f(e)_p)v = (I - e' - f(e)_p)v^* = 0$. Hence we shall have $u^*u = uu^* = v^*v + vv^* + (I - e' - f(e)_p) = e' + f(e)_p + (I - e' - f(e)_p) = I$, so that u is a unitary operator of N. Next, we shall show that $f(e)$ and $u^*f(e)u$ are mutually commutative. In fact, we have

$$u^*(f(e))u = u^*\Big(\sum_{m=-\infty}^{\infty} mf(e)_m\Big)u = u^*(pf(e)_p + ne' + (nf(e)_n - ne')$$

$$+ \sum_{m\neq p,n} mf(e)_m)u = pe' + nf(e)_p + n(f(e)_n - e') + \sum_{m\neq p,n} mf(e)_m.$$

Therefore we have

$$(5) \qquad f(e) - u^*f(e)u = f(e) - f(\rho^{-1}(u)^*e\rho^{-1}(u)) = f(e - \rho^{-1}(u)^*e\rho^{-1}(u)) \\ = (p-n)f(e)_p + (n-p)e'.$$

Since $f(e)_p, e'$, $(f(e)_n - e')$, and $f(e)_m$ ($m \neq p$ and n) are mutually orthogonal, $f(e)$ and $u^*f(e)u$ are mutually commutative. Hence $f([e, \rho^{-1}(u)^*e\rho^{-1}(u)]) = [f(e), f(\rho^{-1}(u)^*e\rho^{-1}(u))] = 0$ and so, by the isomorphism of f, $[e, \rho^{-1}(u)^*e\,\rho^{-1}(u)] = 0$. Therefore e and $\rho^{-1}(u)^*e\rho^{-1}(u)$ are mutually commutative.

Put $e - \rho^{-1}(u)^*e\rho^{-1}(u) = e_1 - e_2$, where e_1 and e_2 are mutually orthogonal projections, then $\exp(it(e_1 - e_2)) = I - e_1 - e_2 + \exp(it)e_1 + \exp(-it)e_2$, so that

$$\exp\Big(i\frac{2\pi}{|p-n|}(e_1 - e_2)\Big) \neq I \text{ and } \rho\Big(\exp\Big(i\frac{2\pi}{|p-n|}(e_1 - e_2)\Big)\Big) \\ = \exp i\Big(\frac{2\pi}{|p-n|}f(e_1 - e_2)\Big) \neq I.$$

On the other hand, by the above relation (5),

$$\frac{2\pi}{|p-n|}f(e_1 - e_2) = \frac{2\pi}{|p-n|}f(e_1 - \rho^{-1}(u)^*e\rho^{-1}(u)) = \frac{2\pi}{|p-n|}((p-n)f(e)_p \\ + (n-p)e') = 2\pi\frac{p-n}{|p-n|}f(e)_p + 2\pi\frac{n-p}{|p-n|}e'.$$

So $$\exp\Big(i\frac{2\pi}{|p-n|}f(e_1 - e_2)\Big) = \exp i\Big(2\pi\frac{p-n}{|p-n|}f(e)_p + 2\pi\frac{n-p}{|p-n|}e'\Big) = I.$$

This contradicts to the preceding relation. The case that $f(e)_n \prec f(e)_p$ is quite analogous, and the lemma is proved.

Since $f(I)_n$ are central projections, $f(I) = f(I)_1 = I$ or $f(I) = -f(I)_{-1}$.

LEMMA 2. *$f(e)$ is positive or negative for any projection e ($\neq 0$) of M.*

PROOF. Suppose that $f(e)_m \neq 0$ for some positive integer m, then $|m-n| \geqq 2$ for all negative integers n, so that, by the lemma 1, $f(e)_n = 0$ for all negative integers n and we have $f(e) > 0$.

Next suppose that $f(e)_{m'} \neq 0$ for some negative integer m', then $|m' - n'| \geqq 2$ for all positive integers n', so that, by the same reason, we have $f(e)_{n'} = 0$. Since $f(e) \neq 0$, there exists an integer $m''(|m''| \geqq 1)$ such that $f(e)_{m''} \neq 0$. Hence we complete the proof of the lemma.

LEMMA 3. *If e ($\neq 0$) is a projection of M and there exist mutually ortho-*

gonal projections $(e_i | i = 1, 2, 3, \ldots, [K]+1)$ *each of which is equivalent to* e *and* $e_1 = e$, *then* $f(e) = f(e)_1$ *or* $-f(e)_{-1}$, *where* K *is the positive number in* (1) *and* $[K]$ *denotes the integral part of* K.

PROOF. Let v_i be a partially isometric operator which gives the equivalence $e \sim e_i$, and put $u_i = v_i + v_i^* + (I - e - e_i)$. Then, analogously as in the lemma 1, we can easily show that u_i is a unitary operator, and $u_i^* e u_i = e_i$ and $u_i^* e_i u_i = e$. Therefore we have

$$f(e_i)_m = f(u_i^* e u_i)_m = \rho(u_i)^* f(e)_m \rho(u_i).$$

Suppose that $f(e)_0 = 0$. Then $\sum_{m \neq 0} f(e)_m = I$ and $f(e) > 0$ or $f(e) < 0$ (Lemma 2). If $f(e) > 0$, then $f(e_i) > 0$ and $f(e_i) = \sum_{m \geq 1} m f(e_i)_m \geqq I$. Hence

$$f\left(\sum_{i=1}^{[K]+1} e_i\right) = \sum_{i=1}^{[K]+1} f(e_i) \geqq ([K]+1)I,$$

so that we have

$$\left\| f\left(\sum_{i=1}^{[K]+1} e_i\right) \right\| \geqq ([K]+1) \| I \| = [K]+1 > K.$$

Since (e_i) are mutually orthogonal, we get $\left\| \sum_{i=1}^{[K]+1} e_i \right\| = 1$. Therefore we have

$$\left\| f\left(\sum_{i=1}^{[K]+1} e_i\right) \right\| \geqq ([K]+1) \left\| \sum_{i=1}^{[K]+1} e_i \right\|.$$

This contradicts the relation (1), and $f(e) > 0$ is impossible. Analogously, we can find that $f(e) < 0$ is impossible. Hence it must be $f(e)_0 \neq 0$. Therefore by the lemma 1, we have $f(e) = f(e)_1$ or $-f(e)_{-1}$. This completes the proof of the lemma.

LEMMA 4. *Let* e *be a projection of* M *and* $e \sim I - e$, *then* $f(e) = f(e)_1$, *if* $f(I) = I$, *and* $f(e) = -f(e)_{-1}$ *if* $f(I) = -I$.

PROOF. Let v be a partially isometric operator which gives the equivalence $e \sim I - e$, that is, $v^*v = e$ and $vv^* = I - e$. Putting $u = v + v^*$ and by the analogous discussion as in the lemma 1, we can easily show that u is a unitary operator, and $u^*eu = I - e$ and $u^*(I-e)u = e$. On the other hand, if $f(I) = I$,

$$\begin{aligned} f(I) = I &= f(e) + f(I-e) = f(e) + f(u^*eu) \\ &= f(e) + \rho(u)^* f(e) \rho(u). \end{aligned}$$

By the lemma 2, $f(e)$ is positive or negative. If $f(e)$ is negative, then $f(I-e)$ is also negative, and this is impossible by the above equality. Hence $f(e) > 0$.

Moreover if $f(e)_{m_0} \neq 0$ $(m_0 > 1)$, then $\|f(I)\| \geqq \|f(e)\| \geqq m_0$. This is also impossible. Therefore $f(e) = f(e)_1$. The case $f(I) = -I$ is analogous. This completes the proof.

LEMMA 5. *$f(e) = f(e)_1$ for any projection e of M, if $f(I) = I$, and $f(e) = -f(e)_{-1}$ if $f(I) = -I$.*

PROOF. We shall prove the lemma only for the case $f(I) = I$, since our discussion is analogous in the case $f(I) = -I$.

CASE (1). Suppose that M is of type I_n $(n < \infty)$ and e_0 is a minimal projection of M. Then there exist mutually orthogonal projections $(e_i \mid i = 1, 2, 3, \ldots, n)$, each of which is equivalent to e_0 and $\sum_{i=1}^{n} e_i = I$. Since the equivalence relation is equivalent to the unitary equivalence in the AW^*-algebras of a finite class, we see that if $f(e_0) < 0$, then $f(e_i) < 0$ $(i = 1, 2, \ldots, n)$, and $f\left(\sum_{i=1}^{n} e_i\right) = \sum_{i=1}^{n} f(e_i) < 0$. This contradicts to $f\left(\sum_{i=1}^{n} e_i\right) = I$. Hence $f(e_0) > 0$.

Next let e be a projection of M, then there exist mutually orthogonal projections $(p_i \mid i = 1, 2, 3, \ldots, s)$ such that $p_i \sim e_0$ $(i = 1, 2, \ldots, s)$ and $\sum_{i=1}^{s} p_i = e$.

Therefore $f(p_i) > 0$, so that $f(e) = \sum_{i=1}^{s} f(p_i) > 0$.

Moreover,

$$f(I) = I = f(e + (I - e)) = f(e) + f(I - e)$$

$$= \sum_{m \geqq 0} m f(e)_m + \sum_{n \geqq 0} n f(I - e)_n.$$

Therefore from the above equality, $f(e)_m = 0$ $(m \geqq 2)$. Hence $f(e) = f(e)_1$.

CASE (2). Suppose that M is of type II_1.

Let e be a projection satisfying the assumption of the lemma 3, then $f(e) = f(e)_1$ or $-f(e)_{-1}$. Let $(e_i \mid i = 1, 2, \ldots, r)$ be a maximal family of mutually orthogonal projections which are equivalent to e, then $e \succ I - \sum_{i=1}^{r} e_i$. Therefore, there exists a projection e' such that $e' < e$ and $e' \sim I - \sum_{i=1}^{r} e_i$. Since e' satisfies the assumption of the lemma 3 and by the same reason as mentioned above,

$$f(e') = f(e')_1 \text{ or } -f(e')_{-1} \text{ and } f\left(I - \sum_{s=1}^{r} e_i\right) = f\left(I - \sum_{i=1}^{r} e_i\right)_1 \text{ or } -f\left(I - \sum_{i=1}^{r} e_i\right)_{-1}.$$

Now suppose that $f(e) < 0$, then $f(e_i) < 0$, so that $f\left(\sum_{i=0}^{r} e_i\right) = \sum_{i=1}^{r} f(e_i) < 0$.

$$f(I) = I = f\Big(\sum_{i=1}^{r} e_i + I - \sum_{i=1}^{r} e_i\Big) = f\Big(\sum_{i=1}^{r} e_i\Big) + f\Big(I - \sum_{i=1}^{r} e_i\Big)$$

$$= \sum_{m<0} m f\Big(\sum_{i=1}^{r} e_i\Big)_m + f\Big(I - \sum_{i=1}^{r} e_i\Big).$$

Since $f\Big(\sum_{i=1}^{r} e_i\Big)_m \Big(\text{resp. } f\Big(I - \sum_{i=1}^{r} e_i\Big)_n\Big)$ are mutually orthogonal, and $f\Big(\sum_{i=1}^{r} e_i\Big)_m$ and $f\Big(I - \sum_{i=1}^{r} e_i\Big)_n$ $(m, n = 1, 2, \dots)$ are mutually commutative, from the above equality, we have $f\Big(I - \sum_{i=1}^{r} e_i\Big) \neq f\Big(I - \sum_{i=1}^{r} e_i\Big)_1$ and $f\Big(I - \sum_{i=1}^{r} e_i\Big) \neq -f\Big(I - \sum_{i=1}^{r} e_i\Big)_{-1}$. This is a contradiction. Hence $f(e) > 0$ and so $f(e) = f(e)_1$.

Next let e be any projection of M, then it is easily shown that there exists a family of orthogonal projections $(e_i \mid i = 1, 2, \dots, s)$ such that each e_i satisfies the assumption of the lemma 3 and $e = \sum_{i=1}^{s} e_i$. Then we have

$$f(e) = f\Big(\sum_{i=1}^{s} e_i\Big) = \sum_{i=1}^{s} f(e_i) > 0.$$

Finally, by the same method with the last part of the proof of the case (1), we can show that $f(e) = f(e)_1$.

CASE (3). Suppose that M is of type I_∞, type II_∞ or type III. Let e be a projection such that $e \precsim I - e$, then $I - e$ is an infinite projection. Therefore there exist mutually orthogonal projections $(e_i \mid i = 1, 2, 3, \dots, [K] + 1)$ such that $e_i \sim I - e$ $(i = 1, 2, \dots, [K] + 1)$ and $\sum_{i=1}^{[K]+1} e_i \leqq I - e$. Hence e satisfies the assumption of the lemma 3, so that $f(e) = f(e)_1$ or $-f(e)_{-1}$. Now suppose that $f(e) = -f(e)_{-1}$. Then we shall choose mutually orthogonal, equivalent, infinite projections $(p_i \mid i = 1, 2, 3)$ of M such that $p_1 + p_2 + p_3 = I - e$. Then $p_1 \sim I - p_1, p_2 \sim I - p_2$ and $p_1 + p_2 \sim I - p_1 - p_2$. Hence by the lemma 4,

$$f(p_1 + p_2) = f(p_1 + p_2)_1 = f(p_1) + f(p_2) = f(p_1)_1 + f(p_2)_1.$$

By the above equality, $f(p_1)_1$ and $f(p_2)_1$ are mutually orthogonal. On the other hand, $e + p_1 \sim I - e - p_1$ and $e + p_2 \sim I - e - p_2$. Hence

$$f(e + p_1) = f(e + p_1)_1 = f(e) + f(p_1) = -f(e)_{-1} + f(p_1)_1.$$

Therefore

$$f(e)_{-1} \leqq f(p_1)_1.$$

Analogously we have

$$f(e)_{-1} \leqq f(p_2)_1.$$

Hence by the orthogonality of $f(p_1)_1$ and $f(p_2)_1$ we have $f(e)_{-1} = 0$. This contradicts to our assumption. Therefore $f(e) = f(e)_1$.

Next let e be a projection such that $e \succ I - e$, then $f(I-e) = f(I-e)_1$, so that $f(e) = I - f(I-e)_1 = f(e)_1$. Therefore for any projection e, $f(e) = f(e)_1$.

This completes the proof of the lemma.

LEMMA 6. *All unitary elements of M_u are expressible in the form* exp $(ih)(h \in M_s)$.

PROOF. At first let u be a unitary element such that $\| I - u \| < 1$, then it is well known that there exists $\log u = -\sum_{n=1}^{\infty} \frac{(I-u)^n}{n}$ and $u = \exp i(-i \log u)$ and moreover we can easily show that $-i \log u \in M_s$. Next let u be a general unitary element and A be a self-adjoint maximal abelian subalgebra of M which contains u. Then A is considered to be composed of all continuous functions on a compact Haudorff space Ω, we shall denote the value of an element a of A at a point λ $(\in \Omega)$ by $a(\lambda)$, and put $G = \{\lambda \mid |I - u(\lambda)| < 1 - \delta\}$ $(\delta > 0$ and $1 - \delta > 0)$. Then G is an open set of Ω. Moreover since Ω is a Stonean space, the closure $\overline{G}$ of G is open and closed. Therefore the characteristic function $e(\lambda)$ of $\overline{G}$ is a projection e of A, and moreover $\| e - ue \| \leqq 1 - \delta$.

Hence putting $ih = -\sum_{n=1}^{\infty} \frac{(e-ue)^n}{n}$, we have $ue = e + \sum_{n=1}^{\infty} \frac{(ih)^n}{n!}$ and

$$(I-e) + ue = I + \sum_{n=1}^{\infty} \frac{(ih)^n}{n!} = \exp(ih) \tag{6}$$

Next we shall define a function $\theta(\lambda)$ on $\Omega - \overline{G}$ as follows: $u(\lambda) = \exp i\,\theta(\lambda)$ and $0 \leqq \theta(\lambda) < 2\pi$.

Then since $|1 - u(\lambda)| \geqq 1 - \delta > 0$ on $\Omega - \overline{G}$, $\theta(\lambda)$ is a continuous function on $\Omega - \overline{G}$, we shall extend $\theta(\lambda)$ to a continuous function $\theta_1(\lambda)$ on Ω as follows: $\theta_1(\lambda) = \theta(\lambda)$ on $\Omega - \overline{G}$ and $\theta_1(\lambda) = 0$ on $\overline{G}$.

It is clear that

$$\begin{aligned}(\exp(i\,\theta_1))(\lambda) &= (\exp(i\,\theta_1))(\lambda) = (\exp(i\,\theta))(\lambda) = u(\lambda) && \text{on } \Omega - \overline{G} \\ &= 1 && \text{on } \overline{G}.\end{aligned}$$

Hence

$$\exp(i\theta_1) = e + u(I-e) \tag{7}$$

By (6) and (7), we have

$$\begin{aligned}(\exp(i\,\theta_1))(\exp(ih)) &= \{e + u(I-e)\}\{(I-e) + ue\} \\ &= ue + u(1-e) = u = \exp(i(\theta_1 + h)).\end{aligned}$$

This completes the proof of the lemma 6.

LEMMA 7. *If $f(I) = I$ (resp. $= -I$), the mapping f (resp. $-f$) preserves the power structure of normal elements, that is, $f(a^n) = f(a)^n$ (a normal).*

PROOF. We shall prove the lemma for the case $f(I) = I$.

Let e_1 and e_2 be mutually orthogonal projections of M, then by the lemma 6, $f(e_1)$, $f(e_2)$ and $f(e_1)+f(e_2)$ are projections, so that $f(e_1)$ and $f(e_2)$ are mutually orthogonal. Let $(e_i \mid i=1,2,\ldots,m)$ be mutually orthogonal projections and let $(\alpha_i \mid i=1,2,\ldots,m)$ be complex numbers, then $(f(e_i) \mid i=1,2,\ldots,m)$ are mutually orthogonal projections of N. We have

$$f\left(\left(\sum_{i=1}^{m}\alpha_i e_i\right)^n\right)=f\left(\sum_{i=1}^{m}\alpha^n e_i\right)=\sum_{i=1}^{m}\alpha_i^n f(e_i)$$
$$=\left(\sum_{i=1}^{m}\alpha_i f(e_i)\right)^n=\left(f\left(\sum_{i=1}^{m}\alpha_i e_i\right)\right)^n \qquad \text{for any positive integer } n.$$

On the other hand, since M is an AW^*-algebra, the above elements $\Sigma\alpha_i e_i$ are everywhere dense in all normal elements of M. Therefore for any normal element a of M, there exists a sequence $\left\{\sum_{i=1}^{m_k}\alpha_{i,k}e_{i,k}\right\}$ such that

unif. $\lim \sum_{i=1}^{m_k}\alpha_{i,k}\, e_{i,k}=a$, so that we have

$$f(a^n)=\text{unif.}\lim_{k\to\infty} f\left(\left(\sum_{i=1}^{m_k}\alpha_{i,k}e_{i,k}\right)^n\right)=\text{unif.}\lim_{k\to\infty} f\left(\sum_{i=1}^{m_k}\alpha_{i,k}e_{i,k}\right)^n=f(a)^n.$$

Hence f preserves the power structure of normal elements. This completes the proof of the lemma.

PROOF OF THE THEOREM. Suppose that $f(I)=I$, then by the lemma 7, f preserves the power structure of normal elements. Hence for any element $h\in M_s$,

$$\rho(\exp(ith))=(\exp(itf(h))=\sum_{n=0}^{\infty}\frac{(itf(h))^n}{n!}=\sum_{n=0}^{\infty}\frac{(it)^n f(h)^n}{n!}$$
$$=\sum_{n=0}^{\infty} f\left(\frac{(it)^n h^n}{n!}\right)=f(\exp\,(ith)).$$

Moreover, by the lemma 6, any unitary element u is expressed by the form $u=\exp(ih)$, so that

$$\rho(u)=f(u) \qquad \text{for any } u\in M_u.$$

If u and v belong to M_u,

$$\rho(u)\rho(v)=f(u)f(v)=\rho(uv)=f(uv).$$

Since any element of a C^*-algebra with the identity is expressed by a finite linear combination of unitary elements, we obtain

$$f(ab)=f(a)f(b) \qquad \text{for } a,b\in M,$$
$$f(a^*)=f(a)^* \qquad \text{for } a\in M.$$

Moreover since $\rho(M_u)=N_u$ belongs to $f(M)$, $f(M)=N$. By the above consideration, we can conclude that f is a linear $*$-isomorphism of M onto N and the group isomorphism ρ is uniquely extended to the linear $*$-isomorphism f.

With a slight modification of the above proof, we can show that if $f(I) = -I$, ρ is uniquely extended to a conjugate linear $*$-isomorphism of M onto N. This completes the proof of the theorem.

REFERENCES

[1] H. H. DYE, The unitary structure in finite rings of operators, Duke Math. Journ., vol. 20(1953), pp. 55–69.

[2] N. DUNFORD AND I. E. SEGAL, Semigroups of operators and the Weierstrass theorem, Bull. Amer. Math. Soc., vol. 52(1946), pp. 911–914.

[3] I. KAPLANSKY, Projections in Banach algebras, Ann. of Math., vol. 53(1951), pp. 233–249.

[4] W. MAAK, Fastperiodische Funktionen, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Berlin, 1950.

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