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A FAKE PROJECTIVE PLANE VIA 2-ADIC UNIFORMIZATION WITH TORSION

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Abstract. We adapt the theory of non-Archimedean uniformization to construct a smooth surface from a lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ that has nontrivial torsion. It turns out to be a fake projective plane, commensurable with Mumford's fake plane yet distinct from it and the other fake planes that arise from 2-adic uniformization by torsion-free groups. As part of the proof, and of independent interest, we compute the homotopy type of the Berkovich space of our plane.

The original definition of a fake projective plane is a compact complex surface that has the same Betti numbers as $\mathbb{C}P^2$, but is not $\mathbb{C}P^2$. The first example was given by Mumford [15], and all fake planes have recently been classified by Prasad-Yeung [17] and Cartwright-Steger [8]: there are 100 of them up to isomorphism, in 50 complex-conjugate pairs.

Mumford used the theory of 2-adic uniformization, beginning with a well-chosen discrete subgroup of $\mathrm{PGL}_3(\mathbb{Q}_2)$. His construction yields a fake projective plane over $\mathbb{Q}_2$. For this to make sense, we use Mumford's definition of a fake plane X over a general field K, which specializes to the above definition when $K = \mathbb{C}$. Namely: X is a smooth and geometrically connected proper surface over K, such that its base change to $X_{\overline{K}}$ satisfies $P_g = q = 0$, $c_1^2 = 3c_2 = 9$ and has ample canonical class. Here $\overline{K}$ denotes the algebraic closure of K. To get a fake plane in the original sense, one identifies $\overline{\mathbb{Q}}_2$ with $\mathbb{C}$ by some isomorphism.

The machinery used by Mumford required his discrete subgroup of $\mathrm{PGL}_3(\mathbb{Q}_2)$ to be torsion-free, and there are exactly two additional fake planes that can be constructed this way [13]. The purpose of this paper is to show that torsion can be allowed in the construction, leading to a "new" fake plane. Of course, it occurs in the Prasad-Yeung–Cartwright-Steger enumeration; what is new is that there is another fake plane realizable by 2-adic uniformization.

This is interesting for two reasons. First, it shows that uniformization by groups containing torsion is possible and yields varieties with interesting properties. Second, in the 2-adic approach, X is the generic fiber of a flat family over the 2-adic integers $\mathbb{Z}_2$, and the central fiber gives a great deal of geometric information about X that is not available in the Prasad-Yeung approach. For example, Ishida [12] showed that Mumford's fake plane covers an elliptic surface whose singular fibers have specific types, and Keum was able to use this to construct

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another fake plane [14]. The main open problem about fake planes is to construct one by non-transcendental methods. Since 2-adic uniformization yields additional information about the planes that may be constructed using it, we may reasonably hope that it will help solve this problem.

1. Non-Archimedean uniformization. In this section we give background material on non-Archimedean uniformization and recall how this guided Mumford in choosing the torsion-free lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ that uniformizes his fake plane. We call his lattice Σ_M; his notation in [15] was Γ. In the next section we will describe another lattice $\Sigma_L \subseteq \mathrm{PGL}_3(\mathbb{Q}_2)$ and show how to use it to build a fake plane, even though Σ_L contains torsion.

Let R be a complete discrete valuation ring, K its field of fractions and $k = R/\pi R$ the residue field, where $\pi \in R$ is a fixed uniformizer. We assume k is finite with say q elements. We write $\mathcal{B}_K$ for the Bruhat–Tits building of $\mathrm{PGL}_3(K)$. This is a 2-dimensional simplicial complex whose vertices are the homothety classes of rank-three R-submodules of K^3. Two vertices are joined by an edge if (after scaling) one module contains the other with quotient a 1-dimensional k vector space. Three vertices span a triangle if they are pairwise joined by edges. $\mathrm{PGL}_3(K)$ acts on $\mathcal{B}_K$ in the obvious way.

The Drinfeld upper-half plane Ω^2_K over K means the set of closed points of $\mathbb{P}^2_K$, minus those that lie on K-rational lines. It is an admissible open subset of the rigid analytic space $\mathbb{P}^{2,\mathrm{an}}_K$, hence a rigid analytic space itself. We write $\widehat{\Omega}^2_K$ for the 'standard' formal model of Ω^2_K from [16, Prop. 2.4], where it is denoted $\mathcal{P}(\Delta_*)$ with $\Delta_* = \mathcal{B}_K$. This is a formal scheme, flat and locally of finite type over $\mathrm{Spf}\, R$, and equipped with a $\mathrm{PGL}_3(K)$-action. It has the following properties.

- The closed fiber $\widehat{\Omega}^2_{K,0}$ is normal crossing, with each component a non-singular rational surface over k, isomorphic to $\mathbb{P}^2_k$ blown up at all k-rational points.
- The double curves of $\widehat{\Omega}^2_{K,0}$ that lie in one of these components are the exceptional curves of this blowup, which are (-1)-curves, and the proper transforms of k-rational lines of $\mathbb{P}^2_k$, which are $(-q)$-curves. Each double curve has different self-intersection numbers in the two components containing it.
- The dual complex of the closed fiber $\widehat{\Omega}^2_{K,0}$ is $\mathrm{PGL}_3(K)$-equivariantly isomorphic to $\mathcal{B}_K$.

The second property allows us to orient the edges of $\mathcal{B}_K$, a property we will use only in Section 3. An edge corresponds to a curve where two components of $\widehat{\Omega}^2_K$ meet; we orient the edge so that it goes from the component in which the curve has self-intersection -1 to the one in which it has self-intersection $-q$. The mnemonic is that the arrow on the edge can be thought of as a greater-than sign, indicating $-1 > -q$. Obviously $\mathrm{PGL}_3(K)$ respects the orientations of edges. A triangle in $\mathcal{B}_K$ corresponds to an intersection point of 3 components of $\widehat{\Omega}^2_K$, and from the description of the double curves it is easy to see that the edges corresponding to the three incident double curves form an oriented circuit. This induces a cyclic ordering on the set of these double curves. (Everything in this paragraph could alternately be developed in terms of R-submodules of K^3 containing each other.)

Now suppose Γ is a torsion-free lattice in $\mathrm{PGL}_3(K)$; all lattices are uniform, so the coset space is compact [19]. Because Γ is discrete and torsion-free, it acts freely on $\mathcal{B}_K$. By the correspondence between the vertices of $\mathcal{B}_K$ and the components of $\widehat{\Omega}^2_K$, Γ also acts freely on $\widehat{\Omega}^2_K$, and properly discontinuously with respect to the Zariski topology. The quotient $\widehat{\mathfrak{X}}_\Gamma := \widehat{\Omega}^2_K/\Gamma$ is a proper flat formal R-scheme, whose closed fiber $\widehat{\mathfrak{X}}_{\Gamma,0}$ is a normal crossing divisor [16, Thm. 3.1].

Its relative dualizing sheaf $\omega_{\widehat{\mathfrak{X}}_\Gamma/R}$ over R is thus the sheaf of relative log differential 2-forms. Since there are 'enough' double curves on each component one can show that $\omega_{\widehat{\mathfrak{X}}_\Gamma/R}$ is relatively ample ([16, p. 204]). This implies that the formal scheme $\widehat{\mathfrak{X}}_\Gamma$ is algebraizable, that is, isomorphic to the π-adic formal completion of a proper flat R-scheme $\mathfrak{X}_\Gamma$, which is uniquely determined up to isomorphism. The generic fiber $X_\Gamma := \mathfrak{X}_{\Gamma,\eta}$ is then a proper smooth surface over K, and has ample canonical class. See [11, §5.4] for background.

On the other hand, Γ also acts freely and properly discontinuously on Ω^2_K. This allows the construction of the rigid analytic quotient Ω^2_K/Γ, which turns out to be K-isomorphic to the rigid analytic surface X_Γ^{an} got from X_Γ by analytification. In other words, Ω^2_K/Γ is the Raynaud generic fiber of the formal scheme $\hat{\mathfrak{X}}_\Gamma$. In particular, the closed points of X_Γ are in bijection with those of Ω^2_K/Γ.

Now we come to Mumford's construction of his fake plane:

PROPOSITION 1.1 ([15, §1]). *Let N be the number of Γ-orbits on the vertices of $\mathcal{B}_K$, and as usual write $q(X) := \dim \mathrm{H}^1(X, \mathcal{O}_X)$ for the irregularity and $P_g(X) := \dim \mathrm{H}^2(X, \mathcal{O}_X)$ for the geometric genus of $X = X_\Gamma$. Then*

(a) $\chi(\mathcal{O}_X) := 1 - q(X) + P_g(X)$ *is equal to* $\frac{N}{3}(q-1)^2(q+1)$;
(b) $c_1^2(X) = 3c_2(X) = 3N(q-1)^2(q+1)$;
(c) $q(X) = 0$;
(d) *the canonical class K_X is ample.*

Mumford took $R = \mathbb{Z}_2$ (so $q = 2$) and chose a lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ we call Σ_M, which is vertex-transitive (so $N = 1$) and torsion-free (so the machinery applies). Abbreviating X_{Σ_M} to X_M, it follows that X_M is a fake projective plane over $\mathbb{Q}_2$.

We have now provided all the background necessary for the construction of "our" fake plane X_L, so the reader could skip to Section 2 immediately. We will use the same ideas, but more work is required because the group Σ_L uniformizing X_L contains torsion.

The rest of this section is preparation for Section 3, which shows that X_L is distinct from the three fake planes that can be obtained by using torsion-free groups in Mumford's construction. (Besides Mumford's example [15], there are exactly two more [13].) We will prove this distinctness by comparing their Berkovich spaces; here is the necessary background, cf. [4], [5].

For a rigid space or an algebraic variety Z over a complete non-Archimedean field, we denote by Z^{Berk} the associated Berkovich space; see [5, 1.6] for the relation between rigid geometry and Berkovich geometry, and [4, 3.4] for Berkovich GAGA. In both cases, the associated Berkovich space Z^{Berk} is uniquely determined, and the functor $Z \mapsto Z^{\mathrm{Berk}}$ is fully faithful.

Let X be a quasi-projective variety over K, and G a finite group acting on X by automorphisms over K. It is well known that the quotient X/G is represented by a quasi-projective variety over K.

LEMMA 1.2. *The quotient X^{Berk}/G, under the canonically induced action of G on X^{Berk}, is represented by a Berkovich K-analytic space, and is naturally isomorphic to $(X/G)^{\mathrm{Berk}}$. Moreover, the underlying topological space of X^{Berk}/G coincides with the topological quotient of the topological space X^{Berk} by G.*

PROOF. We may assume that X is affine, say $X = \operatorname{Spec} A$ where A is a finite type algebra over K. By [4, Remark 3.4.2], we know that X^{Berk} is the set of all multiplicative seminorms $|\cdot|$ on A that extend the valuation $\|\cdot\|$ on K. Set $B = A^G$, the G-invariant part, which is again a finite type algebra over K. Consider $Y = \operatorname{Spec} B$ and the map $\pi : X^{\mathrm{Berk}} \to Y^{\mathrm{Berk}}$ given by the restriction of seminorms.

First we show that π is surjective. Take $y = |\cdot|_y \in Y^{\mathrm{Berk}}$, and let $\mathfrak{q}$ be the kernel of $|\cdot|_y$, which is a prime ideal of B. Since A/B is finite, there exists a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \cap B = \mathfrak{q}$. Let $\mathcal{H}(y)$ be the completion of the residue field $\kappa = \operatorname{Frac}(B/\mathfrak{q})$ by the valuation induced from $|\cdot|_y$. Since $\kappa = \operatorname{Frac}(B/\mathfrak{q}) \hookrightarrow \operatorname{Frac}(A/\mathfrak{p})$ is finite, $\mathcal{H}(y) \hookrightarrow \operatorname{Frac}(A/\mathfrak{p}) \otimes_\kappa \mathcal{H}(y)$ is a finite extension of fields, and hence the valuation $|\cdot|_y$ extends (uniquely) to a valuation on the latter field. We thus have a multiplicative seminorm $x = |\cdot|_x$ on A, which extends $|\cdot|_y$, which shows the surjectivity of π.

For injectivity, suppose $x = |\cdot|_x$ and $x' = |\cdot|_{x'}$ are points of X^{Berk} and satisfy $x|_B = x'|_B$. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be the kernels of $|\cdot|_x$ and $|\cdot|_{x'}$, and $\mathfrak{q}$ the kernel of their common restriction on B. Since $\mathfrak{q} = \mathfrak{p} \cap B = \mathfrak{p}' \cap B$, there exists $g \in G$ such that $g^{-1}(\mathfrak{p}) = \mathfrak{p}'$. Replacing x' by its g-image, we may assume $\mathfrak{p} = \mathfrak{p}'$. Then, by the uniqueness of the extension of valuations, $|\cdot|_x$ and $|\cdot|_{x'}$ coincide on $\operatorname{Frac}(A/\mathfrak{p}) \otimes_\kappa \mathcal{H}(y)$, and hence we have $x = x'$.

Thus the map $X^{\mathrm{Berk}}/G \to Y^{\mathrm{Berk}}$ is set-theoretically bijective. By the construction, it is clearly continuous. Since X^{Berk} is compact and Y^{Berk} is Hausdorff, $X^{\mathrm{Berk}}/G \to Y^{\mathrm{Berk}}$ is a homeomorphism. Hence one can endow X^{Berk}/G with the structure of a Berkovich strictly K-analytic space induced from that of $Y^{\mathrm{Berk}} = (X/G)^{\mathrm{Berk}}$. It is now clear that the resulting K-analytic space X^{Berk}/G gives the quotient of X^{Berk} by G in the category of Berkovich K-analytic spaces. □

Let Γ be a lattice in $\mathrm{PGL}_3(K)$. (One could replace 3 by any n by making trivial changes below.) By Selberg's lemma [2] we know there exists a torsion free normal subgroup $\Gamma_0 \subseteq \Gamma$ of finite index. Set $G = \Gamma/\Gamma_0$. As discussed earlier in this section, the quotient Ω^2_K/Γ_0 is algebraizable, and is of the form $X^{\mathrm{an}}_{\Gamma_0}$ for a smooth projective variety X_{Γ_0} over K, which is obtained as the generic fiber of the algebraization $\mathfrak{X}_{\Gamma_0}$ of the formal scheme $\widehat{\mathfrak{X}}_{\Gamma_0} = \widehat{\Omega}^2_K/\Gamma_0$. The rigid analytic space $X^{\mathrm{an}}_{\Gamma_0}/G$ ($\cong \Omega^2_K/\Gamma$) is then isomorphic to $(X_{\Gamma_0}/G)^{\mathrm{an}}$, hence is algebraized by the projective variety X_{Γ_0}/G. We define X_Γ as X_{Γ_0}/G. It is independent of the choice of Γ_0 because if Γ_0' were another torsion free normal subgroup of Γ of finite index, then both $X_{\Gamma_0}/(\Gamma/\Gamma_0)$ and $X_{\Gamma_0'}/(\Gamma/\Gamma_0')$ are naturally identified with $X_{\Gamma_0\cap\Gamma_0'}/\big(\Gamma/(\Gamma_0\cap\Gamma_0')\big)$.

We call a retraction τ, from a topological space X to a subspace Z, a *deformation retraction* if there is a homotopy from id_X to τ that is constant on Z. (Usually the homotopy, rather than its time 1 map, is called the deformation retraction. So we are slightly abusing standard terminology.) We say that X *deformation-retracts* to Z if such a τ exists. In this case it is standard that X and Z are homotopy equivalent.

LEMMA 1.3. *Let Γ be a lattice in* $\mathrm{PGL}_3(K)$*, and X_Γ the projective variety over K obtained as above. Then, for any finite field extension K'/K, $X_\Gamma^{\mathrm{Berk}} \otimes_K K'$ deformation-retracts to the quotient of the geometric realization of $\mathcal{B}_K$ by Γ.*

In particular, the homotopy type of $X_\Gamma^{\mathrm{Berk}} \otimes_K K'$ is the same as that of the topological space $\mathcal{B}_K/\Gamma$.

PROOF. For the proof, we use the $\mathrm{PGL}_3(K)$-equivariant deformation retraction $\tau\colon \Omega_K^{2,\mathrm{Berk}} \otimes_K K' \to \mathcal{B}_K$ constructed by Berkovich; see below for details. Since the quotient map $\Omega_K^{2,\mathrm{Berk}} \otimes_K K' \to \big(\Omega_K^{2,\mathrm{Berk}} \otimes_K K'\big)/\Gamma_0$ is a topological covering map, we have the induced deformation retraction $X_{\Gamma_0}^{\mathrm{Berk}} \otimes_K K' = \big(\Omega_K^{2,\mathrm{Berk}} \otimes_K K'\big)/\Gamma_0 \to \mathcal{B}_K/\Gamma_0$. The lemma follows from this and Lemma 1.2. □

The rest of this section is devoted to a description of the deformation retraction $\tau\colon \Omega_K^{2,\mathrm{Berk}} \otimes_K K' \to \mathcal{B}_K$ for any finite extension K'/K, cf. [6] and [7, §5] (see also [9, XI, §3]). Following convention, for a seminorm $x = |\cdot|_x$ on a Banach algebra A and an element $f \in A$, we often write $|f(x)|$ in place of $|f|_x$.

Set $A = K[T_0, T_1, T_2]$, and denote by A_d for $d \geq 0$ the K-vector subspace of A consisting of homogeneous polynomials of degree d. By Goldman-Iwahori [10], the geometric realization of $\mathcal{B}_K$ is naturally identified with the quotient of the space $\mathcal{N}$ of norms on the vector space A_1 over K, by the obvious scaling action of $\mathbb{R}_{>0}$. Here the topology on $\mathcal{N}$ is the weakest one such that all real valued functions of the form $|\cdot| \mapsto |f|$, for $f \in A_1$, are continuous.

If $\boldsymbol{L} = \{L_0, L_1, L_2\}$ is a K-basis of A_1, then we write $\widetilde{\Lambda}(\boldsymbol{L})$ for the subset of $\mathcal{N}$ consisting of all norms $|\cdot|$ that have $\boldsymbol{L}$ as an orthogonal basis (i.e., $|\sum_{i=0}^2 a_i L_i| = \max_i \|a_i\| \cdot |L_i|$). The image of $\widetilde{\Lambda}(\boldsymbol{L})$ in $\mathcal{B}_K = \mathcal{N}/(\mathbb{R}_{>0})$ is the apartment $\Lambda(\boldsymbol{L})$ corresponding to $\boldsymbol{L}$. The fact that $\mathcal{B}_K$ is covered by apartments reflects the fact that any K-vector space norm has an orthogonal basis. Let $\widetilde{\Delta}$ be the subset of $\widetilde{\Lambda}(\boldsymbol{L})$ defined by

$$\widetilde{\Delta} = \big\{|\cdot| \in \widetilde{\Lambda}(\boldsymbol{L}) \;\big|\; |L_0| \geq |L_1| \geq |L_2| \geq |\pi L_0|\big\}\,.$$

We define Δ as $\widetilde{\Delta}/(\mathbb{R}_{>0})$, which is the chamber in the apartment $\Lambda = \Lambda(\boldsymbol{L})$ whose three vertices $|\cdot|_0$ and $|\cdot|_1$ and $|\cdot|_2$ are characterized by

$$\begin{aligned} &|L_0|_0 = 1, \quad |L_1|_0 = 1 \quad &&\text{and } |L_2|_0 = 1\;; \\ &|L_0|_1 = 1, \quad |L_1|_1 = 1 \quad &&\text{and } |L_2|_1 = \|\pi\|\,; \\ &|L_0|_2 = 1, \quad |L_1|_2 = \|\pi\| \quad &&\text{and } |L_2|_2 = \|\pi\|\,. \end{aligned}$$

These vertices correspond respectively to the R-modules

$$\begin{aligned} M_0 &= R\,L_0 + R\,L_1 + R\,L_2 \\ M_1 &= R\,L_0 + R\,L_1 + \pi^{-1} R\,L_2 \\ M_2 &= R\,L_0 + \pi^{-1} R\,L_1 + \pi^{-1} R\,L_2\,. \end{aligned}$$

The correspondence is that $|\cdot|_i$ is the π-adic norm with unit ball M_i, i.e., for non-zero $L \in A_1$, $|L|_i = \|\pi\|^k$, where $k = \max\{l \mid L \in \pi^l M_i\}$. Notice that, in terms of the inhomogeneous coordinates $t_1 = L_1/L_0$, $t_2 = L_2/L_0$, the chamber Δ is described as

$$\Delta = \bigl\{x = |\cdot|_x \in \Lambda \bigm| 1 \geq |t_1|_x \geq |t_2|_x \geq \|\pi\|\bigr\}\,.$$

(For this to make sense, one must observe that $|t_i|_x$ is invariant under scaling $|\cdot|_x$ by elements of $\mathbb{R}_{>0}$.)

Let us construct the retraction map $\tau\colon \Omega_K^{2,\mathrm{Berk}} \otimes_K K' \to \mathcal{B}_K$. Recall that the affine space $\mathbb{A}_{K'}^{3,\mathrm{Berk}}$ is the set of all multiplicative seminorms on $A_{K'} = K'[T_0, T_1, T_2]$ that extend the valuation $\|\cdot\|$ on K'. It is endowed with the weakest topology with respect to which all real valued functions on $A_{K'}$ of the form $|\cdot| \mapsto |f|$ for $f \in A_{K'}$ are continuous. Let $0 = |\cdot|_0 \in \mathbb{A}_{K'}^{3,\mathrm{Berk}}$ be the point corresponding to the origin, i.e., the seminorm that vanishes on the ideal (T_0, T_1, T_2). The projective plane $\mathbb{P}_{K'}^{2,\mathrm{Berk}}$ is, as a topological space, the quotient of $\mathbb{A}_{K'}^{3,\mathrm{Berk}} \setminus \{0\}$ divided by the equivalence relation $\sim$ defined as follows: For $x, y \in \mathbb{A}_{K'}^{3,\mathrm{Berk}} \setminus \{0\}$, $x \sim y$ if there exists a positive real number $\lambda > 0$ such that, for any $d \geq 0$ and $f \in A_{K',d}$, we have $|f(x)| = \lambda^d |f(y)|$. Now, let $\widetilde{\Omega}$ be the pull-back of $\Omega_K^{2,\mathrm{Berk}} \otimes_K K'$ by the projection map $\mathbb{A}_{K'}^{3,\mathrm{Berk}} \setminus \{0\} \to \mathbb{P}_{K'}^{2,\mathrm{Berk}}$. Then $\widetilde{\Omega}$ is the subspace of $\mathbb{A}_{K'}^{3,\mathrm{Berk}}$ consisting of seminorms $x = |\cdot|_x$ whose restriction to A_1 gives a norm on A_1. Indeed, the restriction of $x = |\cdot|_x$ to A_1 fails to be a norm if and only if there exists a non-zero K-rational linear form $L \in A_1$ such that $|L(x)| = 0$. That is, if and only if x lies outside $\Omega_K^{2,\mathrm{Berk}} \otimes_K K'$. By restricting seminorms on $A_{K'}$ to seminorms on A, we therefore have a continuous map $\widetilde{\tau}\colon \widetilde{\Omega} \to \mathcal{N}$. This is the promised retraction $\tau\colon \Omega_K^{2,\mathrm{Berk}} \otimes_K K' \to \mathcal{B}_K$.

We next construct an inclusion $j\colon \mathcal{B}_K \hookrightarrow \Omega_K^{2,\mathrm{Berk}} \otimes_K K'$. For any $|\cdot| \in \mathcal{N}$, take an orthogonal basis $\boldsymbol{L} = \{L_0, L_1, L_2\}$ of A_1, i.e., one with the property $|\sum_{i=0}^{2} a_i L_i| = \max \|a_i\| \cdot |L_i|$. Then one can extend $|\cdot|$ to a multiplicative seminorm on $A = K[L_0, L_1, L_2]$ in an obvious way. Namely, for $f = \sum_\nu a_\nu L^\nu$, written with the multi-index $\nu = (\nu_0, \nu_1, \nu_2)$, we set $|f| = \max_\nu \|a_\nu\| \cdot |L|^\nu$. This gives $\mathcal{N} \to \widetilde{\Omega}$, and thus the continuous map $j\colon \mathcal{B}_K \to \Omega_K^{2,\mathrm{Berk}} \otimes_K K'$ by passage to the quotients. It is clear that we have $\tau \circ j = \mathrm{id}_{\mathcal{B}_K}$. Regarding $\mathcal{B}_K$ as a subspace of $\Omega_K^{2,\mathrm{Berk}} \otimes_K K'$ via j, the map τ of the previous paragraph is a retraction. The next step is to show that it is a deformation retraction.

We need to construct a homotopy from the identity map on $\Omega_K^{2,\mathrm{Berk}} \otimes_K K'$ to $j \circ \tau$. This is done by first constructing a natural homotopy $\Phi\colon \tau^{-1}(\Delta) \times [0,1] \to \tau^{-1}(\Delta)$ from $\mathrm{id}_{\tau^{-1}(\Delta)}$ to $j \circ \tau$, where Δ is the chamber constructed above from $\{T_0, T_1, T_2\}$. Then one extends the homotopy to all of $\Omega_K^{2,\mathrm{Berk}} \otimes_K K'$ by demanding $\mathrm{PGL}_3(K)$-equivariance. The subspace $\tau^{-1}(\Delta) \subseteq \Omega_K^{2,\mathrm{Berk}} \otimes_K K'$ is an affinoid subdomain, of which the affinoid algebra is of the form $\mathcal{A}_{K'} = \mathcal{A} \otimes_K K'$ with $\mathcal{A}$ a strictly K-affinoid algebra. The explicit description of (a formal model of) $\mathcal{A}$ is given in [15, p. 234]; in our setting, the formula there should be read with $T_0 = l_2$, $T_1 = l_1$, and $T_2 = l_0$. In particular, $\tau^{-1}(\Delta)$ is an affinoid subdomain of the Berkovich spectrum $Y = \mathcal{M}(\mathcal{B})$ of $\mathcal{B} = K'\langle t_1, \frac{t_2}{t_1}, \frac{\pi}{t_2}\rangle$.

Now it turns out that our situation is exactly the one in [7, Lemma 5.6], where $X = \mathcal{M}(K')$ (= point) and

$$S = \left\{(r_0, r_1, r_2) \in [0,1]^3 \;\middle|\; r_0 r_1 r_2 = \|\pi\|\right\}.$$

We have a bijection $\Delta \xrightarrow{\sim} S$ by $x \mapsto \left(|t_1(x)|, \frac{|t_2(x)|}{|t_1(x)|}, \frac{\|\pi\|}{|t_2(x)|}\right)$. Furthermore, under this identification, our map τ coincides with the map τ there, restricted to our $\tau^{-1}(\Delta)$. Also, our map j coincides with Berkovich's θ. Hence, to construct the homotopy Φ, we can apply the argument in Step 2 of the proof of [7, Lemma 5.6]. Let us briefly sketch the construction to facilitate the reader's own checking.

Consider the K'-affinoid torus $\mathbb{T} = \left\{x \in \mathbb{A}^{1,\mathrm{Berk}}_{K'} \mid |u(x)| = 1\right\}$ corresponding to the affinoid algebra $K'\langle u, u^{-1}\rangle$, and let G be the kernel of the multiplication map $\mathbb{T}^3 \to \mathbb{T}$. The K'-analytic group G acts on Y, and moreover on $\tau^{-1}(\Delta)$, diagonally with respect to the coordinates $t_1, \frac{t_2}{t_1}, \frac{\pi}{t_2}$ of $\mathcal{A}_{K'}$. Notice that this action respects the fibers of $\tau\colon \tau^{-1}(\Delta) \to \Delta$. For any $0 \le r \le 1$, let G_r be the subgroup of G consisting of x with $|(u_i - 1)(x)| \le r$ for all $i = 1, 2, 3$. Clearly, G_0 is the unit group $\{1\}$, and so we set $g_0 = 1$. If $0 < r < 1$, then G_r is a polydisc (of radius r), and hence it has the maximal point g_r; if $r = 1$, we have $G_1 = G$, and hence has the maximal point g_1 (cf. [4, p. 101] for the notion of 'maximal point'). The map $[0,1] \to G$ mapping $r \mapsto g_r$ is continuous, which then gives rise to the desired homotopy

$$\Phi\colon \tau^{-1}(\Delta) \times [0,1] \to \tau^{-1}(\Delta), \quad (y, r) \mapsto y_r := g_r * y,$$

where $g_r * y$ denotes the '$*$-multiplication' defined in [4, §5.2]. (See [4, 6.1.3] for an explicit description of y_r.)

2. Construction of the fake plane. We fix $R = \mathbb{Z}_2$ throughout the rest of the paper and suppress the subscript $K = \mathbb{Q}_2$ from Ω^2, $\widehat{\Omega}^2$ and $\mathcal{B}$.

We recall the following construction from [1]. Let $\mathcal{O}$ be the ring of algebraic integers in $\mathbb{Q}(\sqrt{-7})$, Γ_L be the unitary group of the standard Hermitian lattice $\mathcal{O}\left[\frac{1}{2}\right]^3$, and $P\Gamma_L$ its quotient by scalars. To get a lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ we fix an embedding $\mathcal{O} \to \mathbb{Z}_2$. This identifies $P\Gamma_L$ with a lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$, indeed one of the two densest-possible lattices.

(The two meanings of "lattice" are a common difficulty in this subject. We use "Hermitian lattice" for a free module equipped with a Hermitian form, and "lattice" for a finite-covolume discrete subgroup. Also, in [1] we defined Γ_L as the isometry group of $L\left[\frac{1}{2}\right] := L \otimes_{\mathcal{O}} \mathcal{O}\left[\frac{1}{2}\right]$ for a more-complicated Hermitian $\mathcal{O}$-lattice L. But $L\left[\frac{1}{2}\right] = \mathcal{O}\left[\frac{1}{2}\right]^3$. So the definitions are equivalent.)

We write $\lambda, \bar{\lambda}$ for $(-1 \pm \sqrt{-7})/2$. These are the two primes lying over 2, and we choose the notation so that λ is a uniformizer of $\mathbb{Z}_2$ and $\bar{\lambda}$ is a unit. Defining θ as $\lambda - \bar{\lambda} = \sqrt{-7}$, we obtain an induced inner product on $\mathcal{O}\left[\frac{1}{2}\right]^3 / \theta\mathcal{O}\left[\frac{1}{2}\right]^3 \cong \mathbb{F}_7^3$. This pairing is symmetric and nondegenerate, yielding a natural map from Γ_L to the 3-dimensional orthogonal group over $\mathbb{F}_7$. This descends to a homomorphism $P\Gamma_L \to \mathrm{PO}_3(7) \cong \mathrm{PGL}_2(7)$. We write Φ_L for the kernel.

LEMMA 2.1. *$\Phi_L \subseteq \mathrm{PGL}_3(\mathbb{Q}_2)$ is torsion-free.*

PROOF. We adapt Siegel's proof [18, §39] that the kernel of $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/N)$ is torsion-free for any $N > 2$. Suppose given some nontrivial $y \in \Phi_L$ with finite order, which we may suppose is a rational prime p. Choose some lift $x \in \Gamma_L$ of it, so x^p is a scalar σ. By $y \in \Phi_L$, the image of x in $\mathrm{O}_3(7)$ is a scalar, which is to say that $x \equiv \pm I \bmod \theta$. We claim: for any $n \geq 1$, x is congruent mod θ^n to some scalar $\sigma_n \in \mathcal{O}[\frac{1}{2}]$. It follows easily that x itself is a scalar, contrary to the hypothesis $y \neq 1$.

We prove the claim if $p \neq 7$ and then show how to adapt the argument if $p = 7$. We established the claim for $n = 1$ in the previous paragraph, with $\sigma_1 = \pm 1$. For the inductive step suppose $n \geq 1$ and $x \equiv \sigma_n I \bmod \theta^n$, so $x = \sigma_n I + \theta^n T$ for some endomorphism T of $\mathcal{O}[\frac{1}{2}]^3$. We must show that T is congruent mod θ to some scalar. Reducing $x^p = \sigma$ modulo θ^n and θ^{n+1} shows that θ^n divides $\sigma - \sigma_n^p$ and that $\sigma_n^p I + p\sigma_n^{p-1}\theta^n T \equiv \sigma I \bmod \theta^{n+1}$. Rearranging shows that $p\sigma_n^{p-1}T$ is the scalar $(\sigma - \sigma_n^p)/\theta^n$, modulo θ. Since σ_n and $p \neq 7$ are invertible mod θ, this shows that T is also a scalar mod θ. So σ is a scalar mod θ^{n+1}, finishing the induction.

If $p = 7 = -\theta^2$ then we write $x = \sigma_n I + \theta^n T$ as before, but reduce $x^7 = \sigma$ modulo θ^{n+2} and θ^{n+3} rather than modulo θ^n and θ^{n+1}. This shows that θ^{n+2} divides $\sigma - \sigma_n^7$ and that $\sigma_n^7 I + 7\sigma_n^6\theta^n T \equiv \sigma I \bmod \theta^{n+3}$. The rest of the argument is the same. □

LEMMA 2.2. *$P\Gamma_L \to \mathrm{PGL}_2(7)$ is surjective.*

PROOF. We showed in [1, Thm. 3.2] that $P\Gamma_L$ has two orbits on vertices of $\mathcal{B}$, with stabilizers $L_3(2) := \mathrm{PSL}_3(\mathbb{F}_2) \cong \mathrm{PSL}_2(\mathbb{F}_7)$ and the symmetric group S_4. Fix a vertex v of the first type. By Lemma 2.1, Φ_L is torsion-free, so the map $P\Gamma_L \to \mathrm{PGL}_2(7)$ is injective on this $L_3(2)$. Its image must be the unique copy of this group in $\mathrm{PGL}_2(7)$, namely $\mathrm{PSL}_2(7)$. Next, the fourteen subgroups S_4 of $L_3(2)$ are the $P\Gamma_L$-stabilizers of the neighbors of v. These are all conjugate in $P\Gamma_L$, but not in $L_3(2)$. Therefore the image of $P\Gamma_L$ in $\mathrm{PGL}_2(7)$ must be strictly larger than $\mathrm{PSL}_2(7)$, hence equal to $\mathrm{PGL}_2(7)$. □

Since Φ_L is torsion-free, non-Archimedean uniformization yields a $\mathbb{Z}_2$-scheme $\mathfrak{X}_{\Phi_L}$. We will write $\mathfrak{W}_L$ for it and W_L for its generic fiber. We fix a Sylow 2-subgroup of $\mathrm{PGL}_2(7)$, which is a dihedral group D_{16} of order 16, and write Σ_L for its preimage in $P\Gamma_L$. (Σ is meant to suggest Sylow.) Because Φ_L is normal in Σ_L, the quotient group D_{16} acts on $\widehat{\Omega}^2/\Phi_L$, hence on $\mathfrak{W}_L$ by the uniqueness of algebraization. (Indeed all of $P\Gamma_L/\Phi_L = \mathrm{PGL}_2(7)$ acts.) Because $\mathfrak{W}_L$ is projective and flat over $\mathbb{Z}_2$, the quotient $\mathfrak{W}_L/D_{16}$ is also projective and flat over $\mathbb{Z}_2$. We write X_L for its generic fiber W_L/D_{16}. This is our fake projective plane, proven to be such in Theorem 2.4 below.

The reader familiar with Mumford's construction [15] will recognize that our constructions parallel his. (X_L and X_M are even commensurable, by [1, Theorem 3.3].) Mumford considered the projective unitary group Γ_M of $M\left[\frac{1}{2}\right]$, where M is a different Hermitian $\mathcal{O}$-lattice. He found that its action on $M\left[\frac{1}{2}\right]/\theta M\left[\frac{1}{2}\right]$ induces a surjection $P\Gamma_M \to \mathrm{PSL}_2(7)$. The subgroup Σ_M of $P\Gamma_M$ corresponding to a Sylow 2-subgroup $D_8 \subseteq \mathrm{PSL}_2(7)$ uniformizes his fake plane. His Σ_M is torsion-free, while our Σ_L contains finite subgroups D_8. The following lemma is the key that allows the construction of "our" fake plane to work despite this torsion.

LEMMA 2.3. *D_{16} acts freely on the closed points of W_L. In particular, $W_L \to X_L$ is étale and X_L is smooth.*

We remark that D_{16} has horrible stabilizers in the central fiber of $\mathfrak{W}_L$, such as components with pointwise stabilizer $(\mathbb{Z}/2)^2$.

PROOF. Recall from Section 1 that the closed points of W_L are in bijection with the Φ_L-orbits on the points of Ω^2. The freeness of D_{16}'s action on this set of orbits is equivalent to the freeness of Σ_L's action on Ω^2. Since Φ_L is a torsion-free uniform lattice, it is normal hyperbolic in the sense of [16, §1], so it acts freely on Ω^2. An infinite-order element of Σ_L cannot have fixed points in Ω^2, because some power of it is a nontrivial element of Φ_L. So only the torsion elements of Σ_L could have fixed points.

Because Φ_L is torsion-free, the map $\Sigma_L \to \Sigma_L/\Phi_L = D_{16}$ preserves the orders of torsion elements. Therefore every torsion element of Σ_L has 2-power order. To show that none of them have fixed points in Ω^2, it suffices to show that no involution in Σ_L has a fixed point.

In fact, no involution in $\mathrm{PGL}_3(\mathbb{Q}_2)$ has a fixed point in Ω^2. To see this, suppose $g \in \mathrm{GL}_3(\mathbb{Q}_2)$ represents an involution of $\mathrm{PGL}_3(\mathbb{Q}_2)$. Its square is a scalar, so its eigenvalues are $\pm\alpha$ for some $\alpha \in \overline{\mathbb{Q}_2}$ with $\alpha^2 \in \mathbb{Q}_2$. Since $\det g = \pm\alpha^3$ also lies in $\mathbb{Q}_2$, we have $\alpha \in \mathbb{Q}_2$. So g's eigenspaces are defined over $\mathbb{Q}_2$. Now, Ω^2 was defined as the set of closed points of $\mathbb{P}^2_{\mathbb{Q}_2}$, minus those lying on lines defined over $\mathbb{Q}_2$. Since g's eigenspaces are defined over $\mathbb{Q}_2$, g has no fixed points in Ω^2. □

THEOREM 2.4. *X_L is a fake projective plane.*

PROOF. First we count sixteen Φ_L-orbits on vertices of $\mathcal{B}$: the $P\Gamma_L$-orbit of vertices with stabilizer $L_3(2)$ splits into $[\mathrm{PGL}_2(7) : L_3(2)] = 2$ orbits under Φ_L, and the $P\Gamma_L$-orbit of vertices with stabilizer S_4 splits into $[\mathrm{PGL}_2(7) : S_4] = 14$ orbits under Φ_L.

So Proposition 1.1 shows that $\chi(W_L) = 16$, $q(W_L) = 0$, $c_1^2(W_L) = 3c_2(W_L) = 144$, and that W_L has ample canonical class. We now use three times the fact that $W_L \to X_L$ is étale. First, since the degree is 16, we have $\chi(X_L) = 1$ and $c_1^2(X_L) = 3c_2(X_L) = 9$. Second, X_L has the same Kodaira dimension as W_L (e.g. [3, Chap. I, (7.4)]), hence has general type. Third, since W_L has irregularity 0, the following lemma shows that X_L also has irregularity 0. From the definition of χ it follows that $P_g(X_L) = 0$, completing the proof. □

LEMMA 2.5. *Let X and Y be algebraic varieties over a field K, and $f : Y \to X$ a finite flat morphism of degree not divisible by the characteristic of K. Let $q > 0$ be a positive integer. Then, if $\mathrm{H}^q(Y, \mathcal{O}_Y) = 0$, we have $\mathrm{H}^q(X, \mathcal{O}_X) = 0$.*

PROOF. By flatness, $f_*\mathcal{O}_Y$ is locally free on X. Then the trace map $\mathrm{tr}_{Y/X}\colon f_*\mathcal{O}_Y \to \mathcal{O}_X$, divided by the degree of f, gives a splitting of the inclusion $\mathcal{O}_X \hookrightarrow f_*\mathcal{O}_Y$. Since $\mathcal{O}_X$ is a direct summand of $f_*\mathcal{O}_Y$, the lemma follows immediately. □

3. Distinctness from other fake planes. Our final result is the following:

THEOREM 3.1. *The fake plane X_L is not isomorphic over $\overline{\mathbb{Q}}_2$ to any fake plane uniformized by a torsion-free subgroup of* $\mathrm{PGL}_3(\mathbb{Q}_2)$.

PROOF. Suppose X is a fake projective plane uniformized by a torsion-free subgroup $P\Gamma$ of $\mathrm{PGL}_3(\mathbb{Q}_2)$. Although we don't need it, we remark that there are three possibilities: Mumford's example and two due to Ishida–Kato [13]. By Lemma 1.3, the Berkovich space X^{Berk} has the homotopy type of $\mathcal{B}/P\Gamma$. Since $\mathcal{B}$ is contractible and Γ acts freely (by the absence of torsion), the fundamental group of X^{Berk} is isomorphic to $P\Gamma$. Furthermore, Lemma 1.3 assures us that the base extension $X^{\mathrm{Berk}} \otimes_{\mathbb{Q}_2} K'$ has the same homotopy type, for any finite extension K' of $\mathbb{Q}_2$.

Repeating the argument shows that $X_L^{\mathrm{Berk}} \otimes_{\mathbb{Q}_2} K'$ is homotopy equivalent to $\mathcal{B}/\Sigma_L$, for any finite extension K' of $\mathbb{Q}_2$. And Lemma 3.4 below shows that $\mathcal{B}/\Sigma_L$ has the homotopy type of the standard presentation complex of $\mathbb{Z}/42$. That is, a circle with a disk attached by wrapping its boundary 42 times around the circle. It follows that $X_L^{\mathrm{Berk}} \otimes_{\mathbb{Q}_2} K'$ has fundamental group $\mathbb{Z}/42$.

If X and X_L were isomorphic over $\overline{\mathbb{Q}}_2$ then they would be isomorphic over some finite extension K' of $\mathbb{Q}_2$. Then the isomorphism $X \otimes_{\mathbb{Q}_2} K' \cong X_L \otimes_{\mathbb{Q}_2} K'$ would imply $X^{\mathrm{Berk}} \otimes_{\mathbb{Q}_2} K' \cong X_L^{\mathrm{Berk}} \otimes_{\mathbb{Q}_2} K'$. But this is impossible since the left side has infinite fundamental group and the right side has fundamental group $\mathbb{Z}/42$. □

It remains to state and prove Lemma 3.4, describing the homotopy type of $\mathcal{B}/\Sigma_L$. The rest of this section is devoted to this. The key is to understand the central fiber of $\mathfrak{W}_L/D_{16}$, which in turn requires understanding the central fiber of $\mathfrak{W}_L$. Recall that the central fiber of $\widehat{\Omega}^2$ is a normal crossing divisor with properties described in Section 1.

The central fiber of $\mathfrak{W}_L$ is normal crossing because it is the quotient of the central fiber of $\widehat{\Omega}^2$ by the group Φ_L acting freely. To describe it we need to enumerate its components, double curves and triple points. Our description in the next lemma refers to the set $\mathcal{E}$ of "elements of the finite projective geometry", meaning the seven points and seven lines of the projective plane over $\mathbb{F}_2$. We regard these as the vertices of a graph, with two elements incident if one corresponds to a point and the other to a line containing it. The symbols e, f will always refer to elements of $\mathcal{E}$, and the symbols p, p', p'' (resp. l, l', l'') will always refer to points (resp. lines) of this finite geometry. The automorphism group of the graph is $\mathrm{PGL}_2(7) \cong \mathrm{PSL}_2(7) \rtimes (\mathbb{Z}/2) \cong \mathrm{GL}_3(2) \rtimes (\mathbb{Z}/2)$. Classically, the elements of $\mathrm{PGL}_2(7)$ not in $\mathrm{PSL}_2(7)$ are called "correlations"; they exchange points and lines.

LEMMA 3.2. *$\mathfrak{W}_{L,0}$ has* 16 *components,* 112 *double curves and* 112 *triple points. In more detail,*

(i) *We may label* $\mathfrak{W}_{L,0}$*'s components* Π, Π^* *and* C_e *with* $e \in \mathcal{E}$*, such that* $\mathrm{PGL}_2(7)$ *permutes the* C_e*'s the same way it acts on* $\mathcal{E}$*. Furthermore, correlations exchange* Π *and* Π^*.

(ii) Π *and* Π^* *are disjoint.*

(iii) $D_e := C_e \cap \Pi$ *and* $D_e^* := C_e \cap \Pi^*$ *are irreducible curves.*

(iv) *If* $e, f \in \mathcal{E}$ *are incident then each of* $P_{ef} := \Pi \cap C_e \cap C_f$ *and* $P_{ef}^* := \Pi^* \cap C_e \cap C_f$ *is a single point.*

(v) *If* e *and* f *are distinct non-incident elements of* $\mathcal{E}$*, then* $C_e \cap C_f = \emptyset$*.*

(vi) *If* $e, f \in \mathcal{E}$ *are incident then* $C_e \cap C_f$ *has two components. One, which we call* D_{ef}*, has self-intersection* -1 *in* C_e *and* -2 *in* C_f*. The other, called* D_{fe}*, has these numbers reversed.*

(vii) *The singular locus of each* C_e *is a curve of three components. For each* $f \in \mathcal{E}$ *incident to* e*, exactly one of these components meets* C_f*; we call it* E_{ef}*.*

(viii) *If* $e, f \in \mathcal{E}$ *are incident then* $Q_{ef} := E_{ef} \cap C_f$ *is a single point.*

(ix) *Each* C_e *has two triple-self-intersection points. At such a triple point the incident double curves are* E_{ef_1}, E_{ef_2} *and* E_{ef_3} *where* f_1, f_2, f_3 *are the elements of the geometry incident to* e*. We may label these triple points* R_{eo}*, where* o *is a cyclic ordering on* $\{f_1, f_2, f_3\}$*, such that* $\mathrm{PGL}_2(7)$ *permutes them the same way it permutes the ordered pairs* (e, o)*.*

The components fall into two $\mathrm{PGL}_2(7)$*-orbits:*

(1) $\{\Pi, \Pi^*\}$

(2) *the fourteen* C_e*'s.*

The double curves fall into four $\mathrm{PGL}_2(7)$*-orbits:*

(3) *the seven* D_p*'s and seven* D_l^**'s*

(4) *the seven* D_l*'s and seven* D_p^**'s*

(5) *the forty-two* D_{ef}*'s*

(6) *the forty-two* E_{ef}*'s.*

The triple points fall into three $\mathrm{PGL}_2(7)$*-orbits:*

(7) *the twenty-one* P_{ef}*'s and twenty-one* P_{ef}^**'s*

(8) *the forty-two* Q_{ef}*'s*

(9) *the twenty-eight* R_{eo}*'s.*

Note that $P_{ef} = P_{fe}$ and $P_{ef}^* = P_{fe}^*$, unlike all other cases involving double subscripts.

PROOF. We will pass between vertices of $\mathcal{B}$ and components of $\widehat{\Omega}^2$ without comment whenever it is convenient. By [1, Thm. 3.2], $P\Gamma_L$ acts on the vertices of $\mathcal{B}$ with two orbits, having stabilizers $L_3(2)$ and S_4. Write $\tilde{\Pi}$ for a component of $\widehat{\Omega}^2$ with stabilizer $L_3(2)$. Recall that Φ_L is the kernel of a surjection $P\Gamma_L \to \mathrm{PGL}_2(7)$. Since Φ_L is torsion-free, $L_3(2)$ must inject into $\mathrm{PGL}_2(7)$, so its $P\Gamma_L$-orbit splits into two Φ_L-orbits. We write Π for the Φ_L-orbit containing $\tilde{\Pi}$, and Π^* for the other Φ_L-orbit. We use the same notation for the corresponding components of $\mathfrak{W}_{L,0}$. The same argument shows that the $P\Gamma_L$-orbit with stabilizer S_4 splits into $[\mathrm{PGL}_2(7) : S_4] = 14$ orbits under Φ_L. There is only one conjugacy class of S_4's in $\mathrm{PGL}_2(7)$, represented by the stabilizer of a point of $\mathcal{E}$. Therefore $\mathrm{PGL}_2(7)$'s action on these components of $\mathfrak{W}_{L,0}$ must correspond to its action on $\mathcal{E}$. We have proven (i). We will call the components other than Π, Π^* the *side components*; this reflects our mental image of $\mathfrak{W}_{L,0}$: Π above, Π^* below, and the other components around the sides.

By the explicit description of $P\Gamma_L$ in the proof of [1, Thm. 3.2], each of $\tilde{\Pi}$'s neighbors in $\mathcal{B}$ has $P\Gamma_L$-stabilizer S_4, hence is inequivalent to $\tilde{\Pi}$. Therefore the union of the $P\Gamma_L$-translates of $\tilde{\Pi}$ is the disjoint union of its components. Since Φ_L permutes these components freely, it follows that Π and Π^* are disjoint, proving (ii). It also follows that $\tilde{\Pi}$ maps isomorphically to Π.

Therefore Π is a copy of $\mathbb{P}^2_{\mathbb{F}_2}$ blown up at its seven $\mathbb{F}_2$-points. The curves along which it meets other components are the seven exceptional divisors and the strict transforms of the seven $\mathbb{F}_2$-rational lines. Suppose $e \in \mathcal{E}$ corresponds to one of these curves. The $L_3(2)$-stabilizer of e preserves exactly one element of $\mathcal{E}$, namely e itself. Therefore it preserves exactly one side component, namely C_e. So C_e must be the side component that meets Π along the chosen curve. In this way the 14 side components account for all the double curves lying in Π, proving that each $C_e \cap \Pi$ is irreducible. By symmetry the same holds for $C_e \cap \Pi^*$. This proves (iii), and then (iv) is immediate.

A simple counting argument shows that $\mathfrak{W}_{L,0}$ has 112 double curves and 112 triple points. We have already named the 28 double curves (D_e and D_e^*) that lie in Π or Π^*, leaving 84. We observe that if two side components meet then their intersection consists of an even number of components. This is because for any distinct $e, f \in \mathcal{E}$, there is some $g \in \mathrm{PGL}_2(7)$ exchanging them. So if a component of $C_e \cap C_f$ has self-intersection -1 in C_e and -2 in C_f, then its g-image has these self-intersection numbers reversed, and therefore cannot be the same curve.

If e, f are incident then $C_e \cap C_f$ contains P_{ef} and is therefore nonempty. By the previous paragraph it has evenly many components. Because there are 21 unordered incident pairs e, f, this accounts for either 42 or 84 of the 84 remaining double curves, according to whether $C_e \cap C_f$ has 2 or 4 components. We will see soon that they account for exactly 42 of them. For now all we need is that they account for at least 42.

We claim next that if e and f are a point and a nonincident line, then $C_e \cap C_f = \emptyset$. This is because such pairs $\{e, f\}$ form a $\mathrm{PGL}_2(7)$-orbit of size 28. If $C_e \cap C_f \neq \emptyset$ then the argument from the previous paragraph shows that such intersections account for at least 56 double curves, while at most 42 remain unaccounted for. The same argument shows $C_e \cap C_f = \emptyset$ if e, f are distinct lines or distinct points. This proves (v).

Consider one of the $112 - 42 = 70$ triple points outside $\Pi \cup \Pi^*$, and the three (local) components of $\mathfrak{W}_{L,0}$ there. Two of these have the same type (i.e., they both correspond to points or both to lines). Since these components meet, the previous paragraph shows that they must coincide. It follows that each side-component has at least one curve of self-intersection. We saw above that if e, f are incident then $C_e \cap C_f$ has either two or four components, and in the latter case these intersections account for all double curves not in $\Pi \cup \Pi^*$. Therefore this case is impossible, proving (vi). Now (ii)–(vi) show that every one of the $112 - 28 - 42 = 42$ remaining double curves is a self-intersection curve of a side component. So each side component contains $42/14 = 3$ such curves, proving the first part of (vii).

Next we claim that there exist incident e, f such that there is a triple point where two of the (local) components are C_e and the third is C_f. To see this choose any incident e, f and recall from (iv) that $C_e \cap C_f = D_{ef} \cup D_{fe}$ meets $\Pi \cup \Pi^*$ exactly twice. So it must contain some other triple point. By our understanding of double curves the third component there must be C_e or C_f. After exchanging the names e and f if necessary, this proves the claim. It follows by $\mathrm{PGL}_2(7)$ symmetry that for any ordered pair (e, f) with e, f incident, there is such a triple point. In fact there is exactly one such triple point, since there are 42 ordered incident pairs e, f and only 70 triple points outside $\Pi \cup \Pi^*$. It follows from this uniqueness that exactly one of the three self-intersection curves of C_e meets C_f, and it does so at a single point. This proves the second half of (vii) and all of (viii).

The remaining $112-42-42 = 28$ triple points must all be triple-self-intersections of the C_e's, so each C_e contains two of them. Now fix e and write τ and τ' for these self-intersection points. Obviously the only double-curves that can pass through τ or τ' are E_{ef_1}, E_{ef_2} and E_{ef_3}. The $S_4 \subseteq \mathrm{PGL}_2(7)$ fixing e contains an element of order 3 cyclically permuting f_1, f_2 and f_3, and fixing each of τ, τ' (since its order is 3). It follows that each of τ, τ' lies in all three of the E_{ef_i}. Recall from Section 1 that each triple point of $\widehat{\Omega}^2_K$ determines a cyclic ordering on the three components of $\widehat{\Omega}^2_K$ that pass throuch it. Therefore each of τ, τ' determines a cyclic ordering on $\{E_{ef_1}, E_{ef_2}, E_{ef_3}\}$, hence on $\{f_1, f_2, f_3\}$. Since S_4 acts on $\{f_1, f_2, f_3\}$ as S_3, both cyclic ordering occur, and it follows that τ, τ' induce the two possible cyclic orderings. This proves (ix).

The orbits listed in (1)–(9) merely summarize some of the information given in (i)–(ix). □

Translating the lemma into the dual-complex language gives a complete description of the dual complex of $\mathfrak{W}_{L,0}$:

(1) Its vertices are Π, Π^* and the C_e with $e \in \mathcal{E}$.

(2) For each p, there is an edge D_p from Π to C_p and an edge D^*_p from C_p to Π^*.

(3) For each l there is an edge D^*_l from Π^* to C_l and an edge D_l from C_l to Π.

(4) For each ordered pair (e, f) with e, f incident, there is an edge D_{ef} from C_e to C_f and an edge E_{ef} from C_e to itself.

(5) For each point p and line l that are incident, there is a 2-cell $P_{pl} = P_{lp}$ with its boundary attached along the loop $D_p.D_{pl}.D_l$, and a 2-cell $P^*_{pl} = P^*_{lp}$ with its boundary attached along the loop $D^*_l.D_{lp}.D^*_p$.

(6) For each ordered pair (e, f) with e, f incident, there is a 2-cell Q_{ef} with its boundary attached along the loop $D_{ef}.D_{fe}.E_{ef}$.

(7) For each e, there are 2-cells R_{eo} and $R_{eo'}$ where o, o' are the two cyclic orderings on $\{f_1, f_2, f_3\}$. Their boundaries are attached along the loops $E_{ef_1}.E_{ef_2}.E_{ef_3}$ and $E_{ef_3}.E_{ef_2}.E_{ef_1}$.

Really we are interested in the complex $\mathcal{B}/\Sigma_L$, which is the same as the quotient of the complex just described by the dihedral group D_{16}. It is easy to see that if an element of D_{16} fixes setwise one of the cells just listed, then it fixes it pointwise. Therefore $\mathcal{B}/\Sigma_L$ is a CW complex with one cell for each D_{16}-orbit of cells of $\mathcal{B}/\Phi_L$. To tabulate these orbits we note that D_{16} contains correlations, so Π and Π^* are equivalent, and every C_l is equivalent to some C_p. Next, the subgroup D_8 sending points to points, and lines to lines, is the flag stabilizer in $L_3(2)$. So it acts on the points (resp. lines) with orbits of sizes 1, 2 and 4. We write p, p', p'' (resp. l, l', l'') for representatives of these orbits. Since D_{16} normalizes D_8, the correlations in it exchange the orbit of points of size 1, resp. 2, resp. 4 with the orbit of lines of the same size. That is, p, resp. p', resp. p'' is D_{16}-equivalent to l, resp. l', resp. l''.

LEMMA 3.3. *$\mathcal{B}/\Sigma_L$ is the CW complex with four vertices $\overline{\Pi}$, $\overline{C}_p$, $\overline{C}_{p'}$ and $\overline{C}_{p''}$, and higher-dimensional cells as follows. Its* 18 *edges are*

	$\overline{D}_p$	$\overline{D}_{p'}$	$\overline{D}_{p''}$	$\overline{D}_{pp}$	$\overline{D}_{pp'}$	$\overline{D}_{p'p}$	$\overline{D}_{p'p''}$	$\overline{D}_{p''p'}$	$\overline{D}_{p''p''}$
from	$\overline{\Pi}$	$\overline{\Pi}$	$\overline{\Pi}$	$\overline{C}_p$	$\overline{C}_p$	$\overline{C}_{p'}$	$\overline{C}_{p'}$	$\overline{C}_{p''}$	$\overline{C}_{p''}$
to	$\overline{C}_p$	$\overline{C}_{p'}$	$\overline{C}_{p''}$	itself	$\overline{C}_{p'}$	$\overline{C}_p$	$\overline{C}_{p''}$	$\overline{C}_{p'}$	itself
	$\overline{D}^*_p$	$\overline{D}^*_{p'}$	$\overline{D}^*_{p''}$	$\overline{E}_{pp}$	$\overline{E}_{pp'}$	$\overline{E}_{p'p}$	$\overline{E}_{p'p''}$	$\overline{E}_{p''p'}$	$\overline{E}_{p''p''}$
from	$\overline{C}_p$	$\overline{C}_{p'}$	$\overline{C}_{p''}$	$\overline{C}_p$	$\overline{C}_p$	$\overline{C}_{p'}$	$\overline{C}_{p'}$	$\overline{C}_{p''}$	$\overline{C}_{p''}$
to	$\overline{\Pi}$	$\overline{\Pi}$	$\overline{\Pi}$	itself	itself	itself	itself	itself	itself

Its 15 *two-cells and their boundaries are*

$$
\begin{array}{ll@{\qquad}ll}
\overline{P}_{pp} & : \overline{D}_p.\overline{D}_{pp}.\overline{D}^*_p & \overline{P}_{pp'} & : \overline{D}_p.\overline{D}_{pp'}.\overline{D}^*_{p'} \\
\overline{P}_{p'p} & : \overline{D}_{p'}.\overline{D}_{p'p}.\overline{D}^*_p & \overline{P}_{p'p''} & : \overline{D}_{p'}.\overline{D}_{p'p''}.\overline{D}^*_{p''} \\
\overline{P}_{p''p'} & : \overline{D}_{p''}.\overline{D}_{p''p'}.\overline{D}^*_{p'} & \overline{P}_{p''p''} & : \overline{D}_{p''}.\overline{D}_{p''p''}.\overline{D}^*_{p''} \\
\overline{Q}_{pp} & : \overline{D}_{pp}.\overline{D}_{pp}.\overline{E}_{pp} & \overline{Q}_{pp'} & : \overline{D}_{pp'}.\overline{D}_{p'p}.\overline{E}_{pp'} \\
\overline{Q}_{p'p} & : \overline{D}_{p'p}.\overline{D}_{pp'}.\overline{E}_{p'p} & \overline{Q}_{p'p''} & : \overline{D}_{p'p''}.\overline{D}_{p''p'}.\overline{E}_{p'p''} \\
\overline{Q}_{p''p'} & : \overline{D}_{p''p'}.\overline{D}_{p'p''}.\overline{E}_{p''p'} & \overline{Q}_{p''p''} & : \overline{D}_{p''p''}.\overline{D}_{p''p''}.\overline{E}_{p''p''} \\
\overline{R}_p & : \overline{E}_{pp}.\overline{E}^2_{pp'} & \overline{R}_{p'} & : \overline{E}_{p'p}.\overline{E}^2_{p'p''} \\
\overline{R}_{p''} & : \overline{E}_{p''p'}.\overline{E}^2_{p''p''} & &
\end{array}
$$

PROOF. The remarks above show that the D_{16}-orbits on vertices of $\mathcal{B}/\Phi_L$ have representatives Π, C_p, $C_{p'}$, $C_{p''}$. We add a bar to indicate their images, the vertices of $\mathcal{B}/\Sigma_L$.

By the presence of correlations, the edges D_e and D^*_e with e a line are D_{16}-equivalent to edges D^*_f and D_f with f a point. Therefore orbit representatives for the D_{16}-action on the 28 edges listed under (2) and (3) are D_p, $D_{p'}$, $D_{p''}$, D^*_p, $D^*_{p'}$, $D^*_{p''}$. We add a bar to indicate their images in $\mathcal{B}/\Sigma_L$.

Again using the presence of correlations, the D_{16}-orbits of ordered pairs (e, f) with e and f incident are in bijection with the D_8-orbits of such pairs in which e is a point. These D_8-orbits are represented by

$$(p,l),\ (p,l'),\ (p',l),\ (p',l''),\ (p'',l') \text{ and } (p'',l''), \tag{3.1}$$

which therefore index the six D_{16}-orbits on the 42 edges D_{ef} (resp. E_{ef}). The edges D_{pl}, $D_{pl'}$, $D_{p'l}$, $D_{p'l''}$, $D_{p''l'}$ and $D_{p''l''}$ go from C_p to C_l, C_p to $C_{l'}$, $C_{p'}$ to C_l, etc. Therefore their images go from $\overline{C}_p$ to itself, $\overline{C}_p$ to $\overline{C}_{p'}$, $\overline{C}_{p'}$ to $\overline{C}_p$, etc. We call the images $\overline{D}_{pp}$, $\overline{D}_{pp'}$, $\overline{D}_{p'p}$, etc. The edges E_{pl}, $E_{pl'}$, $E_{p'l}$, $E_{p'l''}$, $E_{p''l'}$ and $E_{p''l''}$ are loops based at C_p, C_p, $C_{p'}$, $C_{p'}$, $C_{p''}$ and $C_{p''}$. We call their images $\overline{E}_{pp}$, $\overline{E}_{pp'}$, $\overline{E}_{p'p}$, etc.

The two-cells P_{pl} meet Π but not Π^*, while the P^*_{pl} meet Π^* but not Π. Therefore the D_{16}-orbits on these cells are in bijection with the D_8-orbits on the P_{pl}. As in the previous paragraph, orbit representatives are P_{pl}, $P_{pl'}$, $P_{p'l}$, $P_{p'l''}$, $P_{p''l'}$ and $P_{p''l''}$. We call their images $\overline{P}_{pp}$, $\overline{P}_{pp'}$, etc., and their attaching maps are easy to work out. For example, the boundary of $P_{pl'}$ is given above as $D_p.D_{pl}.D_l$. The images of the first two terms are $\overline{D}_p$ and $\overline{D}_{pp}$, and D_l is D_{16}-equivalent to D^*_p, so the image of the third term is $\overline{D}^*_p$. Therefore the boundary of the disk $\overline{P}_{pl'}$ is attached along $\overline{D}_p.\overline{D}_{pp}.\overline{D}^*_p$.

In the same way, D_{16}-orbit representatives on the 2-cells Q_{ef} are Q_{pl}, $Q_{pl'}$, $Q_{p'l}$, $Q_{p'l''}$, $Q_{p''l'}$ and $Q_{p''l''}$. We indicate their images in a similar way to the other images: we add a bar and convert subscript l's to p's. As an example we work out the boundary of $\overline{Q}_{p'p''}$, using the boundary of $Q_{p'l''}$ given above as $D_{p'l''}.D_{l''p'}.E_{p'l''}$. The images of the first and third terms are $\overline{D}_{p'p''}$ and $\overline{E}_{p'p''}$. For the image of the second term, we apply a correlation sending l'' to p''. So the ordered pair (l'', p') is D_{16}-equivalent to some ordered pair (p'', m) where m is a line incident to p'' and D_8-equivalent to l'. This is D_8-equivalent to some pair from (3.1), and (p'', l') is the only possibility. Therefore $D_{l''p'}$ is D_{16}-equivalent to $D_{p''l'}$, so the boundary of $\overline{Q}_{p'p''}$ is $\overline{D}_{p'p''}.\overline{D}_{p''p'}.\overline{E}_{p'p''}$. The other cases are essentially the same.

For the D_{16}-orbits on the 2-cells R_{eo} we note that each of p, p', p'' is fixed by an element of D_8 that exchanges two of the three lines incident to that point. It follows that the D_{16}-orbit representatives on these 2-cells are R_{po}, $R_{p'o'}$ and $R_{p''o''}$, where o (resp. o', o'') is a fixed cyclic ordering on the three lines incident to p (resp. p', p''). We write $\overline{R}_p$, $\overline{R}_{p'}$ and $\overline{R}_{p''}$ for their images. Their boundary maps can be worked out using the following. The three lines through p are l, l', and another line which is D_8-equivalent to l'. The three pairs (p', m), with m a line through p', are D_8-equivalent to (p', l), (p', l'') and (p', l''). The three pairs (p'', m) with m a line through p'', are D_8-equivalent to (p'', l'), (p'', l'') and (p'', l''). It follows that the boundaries of $\overline{R}_p$, $\overline{R}_{p'}$ and $\overline{R}_{p''}$ are attached along the stated loops. □

LEMMA 3.4. *$\mathcal{B}/\Sigma_L$ is homotopy-equivalent to the standard presentation complex of $\mathbb{Z}/42$. In particular, its fundamental group is $\mathbb{Z}/42$.*

PROOF. To simplify matters we build up the 2-complex in several stages, suppressing the bars from the names of cells to lighten the notation. First we define K_1 as the 1-complex with the 4 vertices and the edges D_p, $D_{p'}$, $D_{p''}$, D^*_p, $D^*_{p'}$, $D^*_{p''}$. We collapse the last three edges to points, leaving a rose with three petals, which we will call K_2. If the boundary of a 2-cell to be attached later involves one of the collapsed edges then we will also collapse that portion of the 2-cell's boundary.

We define K_3 by attaching to K_2 the edges

$$D_{pp}, D_{pp'}, D_{p'p}, D_{p'p''}, D_{p''p'}, D_{p''p''} \tag{3.2}$$

(which are loops in K_2) and the 2-cells P_{**} having the same subscripts. We may deformation-retract K_3 back to K_2 because each of the newly-adjoined edges is involved in exactly one of the 2-cells. In particular, the loops (3.2) are homotopic rel basepoint to the inverses of D_p, D_p, $D_{p'}$, $D_{p'}$, $D_{p''}$ and $D_{p''}$.

We define K_4 by attaching to K_2 the edges

$$E_{pp}, E_{pp'}, E_{p'p}, E_{p'p''}, E_{p''p'}, E_{p''p''} \tag{3.3}$$

and the 2-cells Q_{**} having the same subscripts. Just as for K_3, we may deformation-retract K_4 back to K_2. The loops (3.3) are homotopic rel basepoint to D_p^2, $D_{p'}D_p$, $D_pD_{p'}$, $D_{p''}D_{p'}$, $D_{p'}D_{p''}$ and $D_{p''}^2$.

Finally we define K_5 by attaching the cells R_p, $R_{p'}$, $R_{p''}$ to K_2. $\mathcal{B}/\Sigma_L$ is homotopy-equivalent to this, hence to the rose with three petals D_p, $D_{p'}$, $D_{p''}$ with three disks attached, along the curves $D_p^2\big(D_{p'}D_p\big)^2$, $D_pD_{p'}\big(D_{p''}D_{p'}\big)^2$ and $D_{p'}D_{p''}\big(D_{p''}D_{p''}\big)^2$. Regarding these as relators defining $\pi_1(\mathcal{B}/\Sigma_L)$, the third one allows us to eliminate $D_{p'}$ and replace it by $D_{p''}^{-5}$. Then the second one allows us to eliminate D_p and replace it by $D_{p''}^{13}$. The remaining relation then reads $D_{p''}^{42} = 1$. $\square$

REFERENCES

[1] D. ALLCOCK AND F. KATO, The densest lattices in $\mathrm{PGL}_3(\mathbb{Q}_2)$, Adv. Math. 242 (2013), 102–115.

[2] R. ALPERIN, An elementary account of Selberg's Lemma, L'Enseignement Mathématique 33 (1987), 269–273.

[3] W. BARTH, W, C. PETERS AND A. VAN DE VEN, Compact Complex Surfaces, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[4] V. G. BERKOVICH, Spectral theory and analytic geometry over non-Archimedean fields, Mathematical Surveys and Monographs, 33, American Mathematical Society, Providence, RI, 1990.

[5] V. G. BERKOVICH, Étale cohomology for non-Archimedean analytic spaces, Inst. Hautes Études Sci. Publ. Math. No. 78 (1993), 5–161.

[6] V. G. BERKOVICH, The automorphism group of the Drinfeld half-plane, C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), no. 9, 1127–1132.

[7] V. G. BERKOVICH, Smooth p-adic analytic spaces are locally contractible, Invent. Math. 137 (1999), no. 1, 1–84.

[8] D. CARTWRIGHT AND T. STEGER, Enumeration of the 50 fake projective planes, C. R. Math. Acad. Sci. Paris 348 (2010), no. 1-2, 11–13.

[9] J.-F. DAT, S. ORLIK AND M. RAPOPORT, Period domains over finite and p-adic fields, Cambridge Tracts in Mathematics, 183. Cambridge University Press, Cambridge, 2010.

[10] O. GOLDMAN AND N. IWAHORI, The space of $\mathfrak{p}$-adic norms, Acta Math. 109 (1963), 137–177.

[11] A. GROTHENDIECK AND J. DIEUDONNÉ, Elements de Géométrie Algébrique III, Publ. Math. IHES 11 (1961), 17 (1963).

[12] M.-N. ISHIDA, An elliptic surface covered by Mumford's fake projective plane, Tohoku Math. J. (2) 40 (1988), no. 3, 367–396.

[13] M.-N. ISHIDA AND F. KATO, The strong rigidity theorem for non-Archimedean uniformization, Tohoku Math. J. (2) 50 (1998), no. 4, 537–555.

[14] J. KEUM, A fake projective plane with an order 7 automorphism, Topology 45 (2006), no. 5, 919–927.

[15] D. MUMFORD, An algebraic surface with K ample, $(K^2) = 9$, $p_g = q = 0$, Amer. J. Math. 101 (1979), no. 1, 233–244.

[16] G. A. MUSTAFIN, Nonarchimedean uniformization, Math. USSR Sbornik, 34 (1978), 187–214.

[17] G. PRASAD AND S.-K. YEUNG, Fake projective planes, Invent. Math. 168 (2007), no. 2, 321–370; Addendum in Invent. Math. 182 (2010), no. 1, 213–227.

[18] C. SIEGEL, Symplectic Geometry, American J. Math. 65 (1943) 1–86.

[19] T. TAMAGAWA, On discrete subgroups of p-adic algebraic groups, 1965 Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963) 11–17, Harper & Row, New York.

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