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ON THE HARNACK INEQUALITY FOR PARABOLIC MINIMIZERS IN METRIC MEASURE SPACES

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Abstract. In this note we consider problems related to parabolic partial differential equations in geodesic metric measure spaces, that are equipped with a doubling measure and a Poincaré inequality. We prove a location and scale invariant Harnack inequality for a minimizer of a variational problem related to a doubly non-linear parabolic equation involving the p-Laplacian. Moreover, we prove the sufficiency of the Grigor'yan–Saloff-Coste theorem for general $p > 1$ in geodesic metric spaces. The approach used is strictly variational, and hence we are able to carry out the argument in the metric setting.

1. Introduction. The purpose of this note is to study parabolic minimizers, which in the Euclidean case are related to the doubly non-linear parabolic equation

$$\frac{\partial(|u|^{p-2}u)}{\partial t} - \nabla\cdot(|\nabla u|^{p-2}\nabla u) = 0\,, \tag{1.1}$$

where $1 < p < \infty$. When $p = 2$ we can recover the heat equation from (1.1). A function $u : \Omega \times (0,T) \to \boldsymbol{R}$, where $\Omega \subset \boldsymbol{R}^n$ is a non-empty open set, is called a parabolic quasiminimizer related to the equation (1.1) if it satisfies

$$\begin{aligned} p\int_{\mathrm{supp}(\phi)} |u|^{p-2}u\frac{\partial\phi}{\partial t}\,dx\,dt + \int_{\mathrm{supp}(\phi)} |\nabla u|^p\,dx\,dt \\ \le K\int_{\mathrm{supp}(\phi)} |\nabla(u+\phi)|^p\,dx\,dt \end{aligned}$$

for some $K \ge 1$ and every smooth compactly supported function ϕ in $\Omega \times (0,T)$. More precisely, in the Euclidean setting every weak solution to (1.1) is a parabolic minimizer, i.e., a parabolic quasiminimizer with $K = 1$.

Elliptic quasiminimizers were introduced by Giaquinta and Giusti in [11, 12]. They enable the study of elliptic problems, such as the p-Laplace equation and p-harmonic functions, in metric measure spaces under the doubling property and a Poincaré inequality. We refer, e.g., to [3], [5], [6], [19], [20], and the references in these papers. Following Giaquinta–Giusti, Wieser [31] generalized the notion of quasiminimizers to the parabolic setting in Euclidean spaces. Parabolic quasiminimizers have also been studied by Zhou [32, 33], Gianazza–Vespri [10], Marchi [22], and Wang [30]. The literature for parabolic quasiminimizers is very

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small compared to the elliptic case. In recent papers [18], [24], parabolic quasiminimizers related to the heat equation have been studied in general metric measure spaces. The variational approach taken in these papers opens up a possibility to develop a systematic theory for parabolic problems in this generality.

Our main result is a scale and location invariant Harnack inequality, Theorem 6.6, in geodesic metric measure spaces for a positive parabolic minimizer that is locally bounded away from zero and locally bounded. We assume the measure to be doubling and to support a $(1, p)$-Poincaré inequality. We take a purely variational approach and prove the Harnack inequality without making any reference to the equation (1.1).

In Euclidean spaces, the Harnack inequality for a positive weak solution to the equation (1.1), that is bounded away from zero, was proved in [17]. Their proof is based on Moser's method and on an abstract lemma due to Bombieri and Giusti. The argument in [17] relies on the equation and uses, for instance, the fact that if u is a weak supersolution to (1.1), then u^{-1} is a weak subsolution of the same equation.

Our proof is based on the one in [17]. However, since we deal with parabolic minimizers and upper gradients in the metric setting, changes in the argument are required. To give an example, in the strictly variational setting it is not true that if u is a parabolic superminimizer, then u^{-1} is a parabolic subminimizer. Instead we establish the required estimates separately for both super- and subminimizers.

Grigor'yan [13] and Saloff-Coste [25] observed independently that the doubling property and a Poincaré inequality for the measure are sufficient and necessary conditions for a scale and location invariant parabolic Harnack inequality for solutions to the heat equation ($p = 2$) on Riemannian manifolds. Later, Sturm [29] generalized this result to the setting of Dirichlet spaces.

One motivation for the present note is to show the sufficiency for general $1 < p < \infty$ in geodesic metric measure spaces without invoking Dirichlet spaces or the Cheeger derivative structure for which we refer to [9]. We also refer to a recent paper [2] and to [1] on parabolic Harnack inequalities on metric measure spaces with a local regular Dirichlet form. It would be very interesting to know whether also the necessity holds in this general setting.

Very recently a similar question has been studied for degenerate parabolic quasilinear partial differential equations in the subelliptic case by Caponga, Citti, and Rea [8]. Their motivating example is a class of subelliptic operators associated to a family of Hörmander vector fields and their Carnot–Carathéodory distance. The setup in the present paper cover also Carnot groups and more general Carnot–Carathéodory spaces.

2. Prelimininaries. In this section we briefly recall the basic definitions and collect some results we will need in the sequel. For a more detailed treatment we refer, for instance, to a monograph by A. and J. Björn [4] and to Heinonen [14], and the references therein.

2.1. Metric measure spaces. Standing assumptions in this paper are as follows. By the triplet (X, d, μ) we denote a complete geodesic metric space X, where d is the metric and μ a Borel measure on X. The measure μ is supposed to be *doubling*, i.e., there exists a constant $C_\mu \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty \tag{2.1}$$

for every $r > 0$ and $x \in X$. Here $B(x, r) := \{y \in X \; ; \; d(y, x) < r\}$. We denote $\lambda B = B(x, \lambda r)$ for each $\lambda > 0$. We want to mention in passing that to require the measure of every ball in X to be positive and finite is anything but restrictive; it does not rule out any interesting measures. Equivalently, for any $x \in X$, we have

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C\left(\frac{R}{r}\right)^{q_\mu} \tag{2.2}$$

for all $0 < r \leq R$ with $q_\mu := \log_2 C_\mu$, where $C > 0$ is a constant which depends only on C_μ. The choice $q_\mu = \log_2 C_\mu$ is not necessarily optimal; the exponent q_μ serves as a counterpart in metric measure space to the dimension of a Euclidean space. In addition to the doubling property, we assume that X supports a *weak* $(1, p)$*-Poincaré inequality* (see below). Moreover, the product measure in the space $X \times (0, T)$, $T > 0$, is denoted by $\nu = \mu \otimes \mathcal{L}^1$, where $\mathcal{L}^1$ is the one dimensional Lebesgue measure.

It is worth noting that our abstract setting causes some, perhaps unexpected, difficulties. For instance, in not too pathological metric spaces, it may happen that $B(x_1, r_1) \subset B(x_2, r_2)$ but $B(x_2, 2r_2) \subset B(x_1, 2r_1)$.

We follow Heinonen and Koskela [15] in introducing upper gradients as follows. A Borel function $g : X \to [0, \infty]$ is said to be an *upper gradient* for an extended real-valued function u on X if for all paths $\gamma : [0, l_\gamma] \to X$, we have

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g \, ds \, . \tag{2.3}$$

If (2.3) holds for p-almost every path in the sense of Definition 2.1 in Shanmugalingam [27], we say that g is a p-weak upper gradient of u. From the definition, it follows immediately that if g is a p-weak upper gradient for u, then g is a p-weak upper gradient also for $u - k$, and $|k|g$ for ku, for any $k \in \boldsymbol{R}$.

The p-weak upper gradients were introduced in Koskela–MacManus [21]. They also showed that if $g \in L^p(X)$ is a p-weak upper gradient of u, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of u such that $g_j \to g$ in $L^p(X)$. If u has an upper gradient in $L^p(X)$, then it has a *minimal* p*-weak upper gradient* $g_u \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of u, $g_u \leq g$ a.e. (see Shanmugalingam [28, Corollary 3.7]).

Let Ω be an open subset of X and $1 \leq p < \infty$. Following Shanmugalingam [27] (see also [4, Corollary 2.9]), we define for $u \in L^p(\Omega)$,

$$\|u\|^p_{N^{1,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \|g_u\|^p_{L^p(\Omega)} \, .$$

The *Newtonian space* $N^{1,p}(\Omega)$ $(\subset L^p(\Omega))$ is the quotient space

$$N^{1,p}(\Omega) = \left\{u \in L^p(\Omega) \; ; \; \|u\|_{N^{1,p}(\Omega)} < \infty\right\} / \sim ,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(\Omega)} = 0$. The space $N^{1,p}(\Omega)$ is a Banach space and a lattice (see Shanmugalingam [27]). If $u, v \in N^{1,p}(\Omega)$ and $u = v$ μ-a.e., then $u \sim v$.

However, if $u \in N^{1,p}(\Omega)$, then $u \sim v$ if and only if $u = v$ outside a set of zero Sobolev p-capacity [27].

A function u belongs to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(V)$ for all bounded open sets V with $\overline{V} \subset \Omega$, the latter space being defined by considering V as a metric space with the metric d and the measure μ restricted to it.

Newtonian spaces share many properties of the classical Sobolev spaces. For example, if $u, v \in N^{1,p}_{\mathrm{loc}}(\Omega)$, then $g_u = g_v$ a.e. in $\{x \in \Omega\ ;\ u(x) = v(x)\}$, in particular $g_{\min\{u,c\}} = g_u \chi_{\{u \neq c\}}$ for $c \in \boldsymbol{R}$.

REMARK 2.4. Note that as a consequence of the definition, the functions in $N^{1,p}(\Omega)$ are absolutely continuous on p-almost every path. This means that $u \circ \gamma$ is absolutely continuous on $[0, \mathrm{length}(\gamma)]$ for p-almost every rectifiable arc-length parametrized path γ in Ω. This in turn implies that for each of these paths we have $|(u \circ \gamma)'(s)| \leq g(\gamma(s))$ for almost every $s \in [0, \mathrm{length}(\gamma)]$. We refer to [4, Theorem 1.56 and Lemma 2.14].

We shall also need a Newtonian space with zero boundary values. For a measurable set $E \subset X$, let

$$N^{1,p}_0(E) = \{f|_E\ ;\ f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus E\}.$$

This space equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space.

We say that X supports a *weak* $(1, p)$*-Poincaré inequality* if there exist constants $C_p > 0$ and $\Lambda \geq 1$ such that for all balls $B(x_0, r) \subset X$, all integrable functions u on X and all upper gradients g of u,

$$\fint_{B(x_0,r)} |u - u_B|\, d\mu \leq C_p r \left(\fint_{B(x_0,\Lambda r)} g^p\, d\mu \right)^{1/p}, \tag{2.5}$$

where

$$u_B := \fint_{B(x_0,r)} u\, d\mu := \frac{1}{\mu(B(x_0, r))} \int_{B(x_0,r)} u\, d\mu .$$

If the metric measure space X has not "enough" rectifiable paths, it may happen that the continuous embedding $N^{1,p} \to L^p$, given by the identity map, is onto. If X has no nonconstant rectifiable paths, then $g_u \equiv 0$ is the minimal p-weak upper gradient of every function, and $N^{1,p}(X) = L^p(X)$ isometrically. The fact that the Newtonian space is not simply $L^p(X)$ is implied, for instance, by assuming that X supports a weak $(1, p)$-Poincaré inequality.

2.2. Parabolic setting. Our set-up is the following. Let $\Omega \subset X$ be an open set, and $0 < T < \infty$. We write $\Omega_T := \Omega_{(0,T)} := \Omega \times (0, T)$ for a space-time cylinder, and $z = (x, t)$ is a point in Ω_T. We denote by $L^p(0, T; N^{1,p}(\Omega))$ the parabolic space of functions $u : \Omega_T \to \boldsymbol{R}$ such that, for a.e. $t \in (0, T)$, $x \mapsto u(x, t)$ belongs to $N^{1,p}(\Omega)$ and

$$\int_0^T \|u\|^p_{N^{1,p}(\Omega)}\, dt < \infty,$$

and similarly for $L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$. Here we have defined

$$g_u(x, t) := g_{u(\cdot,t)}(x)$$

at ν-almost every $(x, t) \in \Omega \times (0, T)$.

The following calculus rules will be used throughout the text. Assume $u, v \in L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$. Then for almost every t and μ-almost every x

$$
\begin{aligned}
g_{u+v} &\le g_u + g_v\,,\\
g_{uv} &\le |u| g_v + |v| g_u\,.
\end{aligned}
$$

In particular, if c is a constant, then $g_{cu} = |c| g_u$. For the proof at each time level, see [4]. This proof guarantees that g_{u+v} and g_{uv} are defined at almost every t and μ-almost every x. The definition of the parabolic minimal p-weak upper gradient then implies the result. Note that the above does not claim that uv is in the parabolic Newtonian space, even if u and v are.

In the Euclidean case it can be shown that stating that a function $u : \Omega \times (0, T) \to \boldsymbol{R}$, $u \in L^2_{\mathrm{loc}}(0, T; W^{1,2}_{\mathrm{loc}}(\Omega))$ is a weak solution to the doubly nonlinear parabolic equation (1.1), is equivalent to stating that u is fulfills the variational problem

$$
\begin{aligned}
p \int_{\mathrm{supp}(\phi)} |u|^{p-2} u \frac{\partial \phi}{\partial t}\, dx\, dt &+ \int_{\mathrm{supp}(\phi)} |\nabla u|^p\, dx\, dt\\
&\le \int_{\mathrm{supp}(\phi)} |\nabla u + \nabla \phi|^p\, dx\, dt
\end{aligned}
$$

for every $\phi \in C_0^\infty(\Omega \times (0, T))$. Since partial derivatives cannot be defined in a general metric space, there is little sense in trying to define the weak formulation of the equation (1.1) in the metric setting. The variational approach on the other hand only considers integrals with absolute values of partial derivatives and an inequality – as opposed to demanding a strict equation with gradients. This opens up the possibility to extend the definition of a parabolic minimizer related to the doubly nonlinear equation to metric measure spaces in the following way:

DEFINITION 2.6. We say that a function $u \in L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$ is a *parabolic minimizer* if the inequality

$$
p \int_{\mathrm{supp}(\phi)} |u|^{p-2} u \frac{\partial \phi}{\partial t}\, d\nu + \int_{\mathrm{supp}(\phi)} g_u^p\, d\nu \le \int_{\mathrm{supp}(\phi)} g_{u+\phi}^p\, d\nu \tag{2.7}
$$

holds for all $\phi \in \mathrm{Lip}_0(\Omega_T) = \{f \in \mathrm{Lip}(\Omega_T);\ \mathrm{supp}(f) \subset \Omega_T\}$. If (2.7) holds for all nonnegative $\phi \in \mathrm{Lip}_0(\Omega_T)$ a function $u \in L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$ is a *parabolic superminimizer*; and a *parabolic subminimizer* if (2.7) holds for all nonpositive $\phi \in \mathrm{Lip}_0(\Omega_T)$.

Observe that here parabolic minimizers are scale invariant but not translation invariant.

Let $0 < \alpha \le 1$, let parameters r and T be positive, and $t_0 \in \boldsymbol{R}$. A space-time cylinder in $X \times \boldsymbol{R}$ is denoted by

$$
Q_{\alpha r}(x, t) = B(x, \alpha r) \times (t - T(\alpha r)^p, t + T(\alpha r)^p)\,.
$$

It will also be of use to define positive and negative space-time cylinders as

$$\alpha Q^+(x,t) = B(x,\alpha r) \times \left(t + T\left(\frac{1-\alpha}{2}\right)^p r^p, t + T\left(\frac{1+\alpha}{2}\right)^p r^p\right),$$
$$\alpha Q^-(x,t) = B(x,\alpha r) \times \left(t - T\left(\frac{1+\alpha}{2}\right)^p r^p, t - T\left(\frac{1-\alpha}{2}\right)^p r^p\right).$$

Using these, we write

$$Q_{\alpha r} = Q_{\alpha r}(x_0,t_0), \quad \alpha Q^+ = \alpha Q^+(x_0,t_0), \quad \alpha Q^- = \alpha Q^-(x_0,t_0)\,.$$

Above r is chosen according to (x_0, t_0) and T in such a way that $Q_r \subset \Omega_T$. Our goal in this note is to prove the following Harnack inequality using Moser's argument and energy methods:

Suppose $1 < p < \infty$ and assume that the measure μ in a geodesic metric space X is doubling with doubling constant C_μ, and supports a weak $(1, p)$-Poincaré inequality with constants C_p and Λ. Then a parabolic Harnack inequality is valid as follows. Let $u > 0$ be a parabolic minimizer in $Q_r \subset \Omega_T$, locally bounded and locally bounded away from zero. Let $0 < \delta < 1$. We have

$$\operatorname*{ess\,sup}_{\delta Q^-} u \le C \operatorname*{ess\,inf}_{\delta Q^+} u\,, \tag{2.8}$$

where $C = C(C_\mu, C_p, \Lambda, \delta, p, T)$.

Note that the constant in the Harnack estimate does not depend on r and so is scale invariant, as long as r is such that $Q_r \subset \Omega_T$. The parameter T controls only the relative proportions of the spatial and time faces of Q_r.

2.3. Sobolev-Poincaré inequalities. We shall need the Sobolev inequality for functions with zero boundary values; if $f \in N_0^{1,p}(B(x_0,R))$, then there exists a constant $C > 0$ only depending on p, C_μ, and the constants C_p and Λ in the Poincaré inequality, such that

$$\left(\fint_{B(x_0,R)} |f|^\kappa \, d\mu\right)^{1/\kappa} \le CR \left(\fint_{B(x_0,R)} g_f^p \, d\mu\right)^{1/p}, \tag{2.9}$$

where

$$\kappa = \begin{cases} pq_\mu/(q_\mu - p), & \text{if } \; 1 < p < q_\mu\,, \\ \infty, & \text{if } \; p \ge q_\mu\,. \end{cases}$$

For this result we refer to [20]. The following weighted version of the Poincaré inequality will also be needed.

LEMMA 2.10. *Let $f \in N^{1,p}(B(x_0,R))$, and*

$$\phi(x) = \left(1 - \frac{d(x,x_0)}{R}\right)_+^\theta,$$

where $\theta > 0$. Then there exists a positive constant $C = C(C_\mu, C_p, p, \theta)$ such that

$$\fint_{B(x_0,r)} |f - f_\phi|^p \phi \, d\mu \le Cr^p \fint_{B(x_0,r)} g_f^p \phi \, d\mu \tag{2.11}$$

for all $0 < r < R$, *where*

$$f_\phi = \frac{\int_{B(x_0,r)} f\phi \, d\mu}{\int_{B(x_0,r)} \phi \, d\mu} .$$

SKETCH OF PROOF. The main idea in the proof, for which we refer to Saloff-Coste [26, Theorem 5.3.4], is to connect two points in the ball $B(x_0, r)$ with a certain finite chain of balls. For this chain we need to assume that our space X is geodesic. □

REMARK 2.12. Lemma 2.10 will be used later in the proof of Lemma 5.2. We stress, however, that apart from Lemma 2.10, all other estimates prior to Lemma 5.2 are valid without X being geodesic.

2.4. Bombieri's and Giusti's abstract lemma. A delicate step in the proof based on Moser's work is to use a parabolic version of the John–Nirenberg inequality, i.e., exponential integrability of BMO functions. To avoid the use of the parabolic BMO class, the parabolic John–Nirenberg theorem is replaced with an abstract lemma due to Bombieri and Giusti [7]. Consult [26] or [17] for the proof.

LEMMA 2.13. *Let ν be a Borel measure and consider a collection of bounded measurable sets U_α, $0 < \alpha \le 1$, with $U_{\alpha'} \subset U_\alpha$ if $\alpha' \le \alpha$.*

Fix $0 < \delta < 1$, let θ, γ, and A be positive constants, and $0 < q \le \infty$. Moreover, if $q < \infty$, we assume that

$$\nu(U_1) \le A\nu(U_\delta)$$

holds. Let f be a positive measurable function on U_1 such that for every $0 < s \le \min(1, q/2)$ we have

$$\left(\fint_{U_{\alpha'}} f^q \, d\nu \right)^{1/q} \le \left(\frac{A}{(\alpha - \alpha')^\theta} \fint_{U_\alpha} f^s \, d\nu \right)^{1/s}$$

for every α, α' such that $0 < \delta \le \alpha' < \alpha \le 1$. Assume further that f satisfies

$$\nu\left(\{x \in U_1;\ \log f > \lambda\}\right) \le \frac{A\nu(U_\delta)}{\lambda^\gamma}$$

for all $\lambda > 0$. Then

$$\left(\fint_{U_\delta} f^q \, d\nu \right)^{1/q} \le C ,$$

where $C = C(q, \delta, \theta, \gamma, A)$.

3. Reverse Hölder inequalities for parabolic superminimizers. In this section, we prove an energy estimate for parabolic superminimizers. After this, using the energy estimate we prove a reverse Hölder inequality for negative powers of parabolic superminimizers.

Establishing energy estimates for parabolic superminimizers is based on substituting a suitably chosen test function into the inequality (2.7), and then performing partial integration to extract the desired inequality from it. While doing this, we take the time derivative of u^{p-1}, even though u is not assumed to have sufficient time regularity for this. Therefore, the reader should consider the time derivation of u as being formal. Justifications for the formal treatment will be given in Remark 3.6.

LEMMA 3.1. *Let $u > 0$ be a parabolic superminimizer, locally bounded away from zero, and $0 < \varepsilon \neq p-1$. Then*

$$\begin{aligned} \operatorname*{ess\,sup}_{0<t<T} \int_\Omega u^{p-1-\varepsilon}\phi^p \, d\mu + \int_{\mathrm{supp}(\phi)} g_u^p u^{-1-\varepsilon}\phi^p \, d\nu \\ \leq C_1 \int_{\mathrm{supp}(\phi)} u^{p-1-\varepsilon} g_\phi^p \, d\nu + C_2 \int_{\mathrm{supp}(\phi)} u^{p-1-\varepsilon} |(\phi^p)_t| \, d\nu \end{aligned}$$

for every $\phi \in \mathrm{Lip}_0(\Omega_T)$, $0 \leq \phi \leq 1$, where

$$C_1 = \left(\frac{p}{\varepsilon}\right)^p \left(1 + \frac{\varepsilon|p-1-\varepsilon|}{p(p-1)}\right), \qquad C_2 = \left(1 + \frac{p(p-1)}{\varepsilon|p-1-\varepsilon|}\right).$$

PROOF. Assume $\varepsilon > 0$, $\varepsilon \neq p-1$. Let ϕ be a function $0 \leq \phi \leq 1$, $\phi \in \mathrm{Lip}_0(\Omega_T)$. Since ϕ has compact support, we can choose $0 < t_1 < t_2 < T$ such that $\phi(x,t) = 0$ μ-almost everywhere when $t \notin (t_1, t_2)$. Since by assumption u is locally bounded away from zero, we may assume a positive constant $\alpha > 0$ such that after denoting $v = \alpha u$ we have $1 - \varepsilon\phi^p v^{-\varepsilon-1} > 0$ ν-almost everywhere in the support of ϕ. It then follows that ν-almost everywhere in the support of ϕ, we have

$$g_{v+\phi^p v^{-\varepsilon}} \leq (1 - \varepsilon\phi^p v^{-\varepsilon-1}) g_v + p v^{-\varepsilon}\phi^{p-1} g_\phi \,. \tag{3.2}$$

That (3.2) does indeed hold can be seen in the following way: Let t be such that $v(\cdot,t) \in N^{1,p}(\Omega)$. Consider any arc-length parametrization γ of a rectifiable path on which $v(\cdot,t)$ is absolutely continuous. Since $\phi(\cdot,t)$ is Lipschitz-continuous, it is absolutely continuous on γ. Define $h : [0, \mathrm{length}(\gamma)] \to [0,\infty)$ by

$$h(s) = v(\gamma(s),t) + \phi(\gamma(s),t)^p v(\gamma(s),t)^{-\varepsilon} \,.$$

Then h is absolutely continuous, and so we have

$$\begin{aligned} h'(s) = (1 - \varepsilon\phi(\gamma(s),t)^p v(\gamma(s),t)^{-\varepsilon-1}) \frac{\partial v(\gamma(s),t)}{\partial s} \\ + p\phi(\gamma(s),t)^{p-1} v(\gamma(s),t)^{-\varepsilon} \frac{\partial \phi(\gamma(s),t)}{\partial s} \end{aligned}$$

for almost every $s \in [0, \mathrm{length}(\gamma)]$ with respect to the Lebesgue measure. We know that

$$\left|\frac{\partial v(\gamma(s),t)}{\partial s}\right| \leq g_v(\gamma(s),t)\,, \qquad \left|\frac{\partial \phi(\gamma(s),t)}{\partial s}\right| \leq g_\phi(\gamma(s),t)$$

for almost every $s \in [0, \mathrm{length}(\gamma)]$. Hence

$$\begin{aligned} \left|\frac{\partial (v + \phi^p v^{-\varepsilon})(\gamma(s),t)}{\partial s}\right| &= |h'(s)| \\ &\leq (1 - \varepsilon\phi(\gamma(s),t)^p v(\gamma(s),t)^{-\varepsilon-1}) g_v(\gamma(s),t) \\ &\quad + p\phi(\gamma(s),t)^{p-1} v(\gamma(s),t)^{-\varepsilon} g_\phi(\gamma(s),t) \end{aligned}$$

for almost every $s \in [0, \mathrm{length}(\gamma)]$. The fact that this holds for p-almost every rectifiable path γ now implies (3.2). Using the convexity of the mapping $t \mapsto t^p$ we have

$$
\begin{aligned}
g^p_{v+\phi^p v^{-\varepsilon}} &\le \left((1-\varepsilon\phi^p v^{-\varepsilon-1})g_v + \varepsilon\phi^p v^{-\varepsilon-1}\frac{pv}{\varepsilon\phi}g_\phi\right)^p \\
&\le (1-\varepsilon\phi^p v^{-\varepsilon-1})g_v^p + p^p\varepsilon^{1-p}v^{p-\varepsilon-1}g_\phi^p\,.
\end{aligned}
\tag{3.3}
$$

Assume $0 < \tau_1 < \tau_2 < T$. For a small enough $h > 0$, denote

$$
\chi^h_{[\tau_1,\tau_2]}(t) = \begin{cases} (t-\tau_1)/h\,, & \tau_1 < t < \tau_1 + h\,,\\ 1\,, & \tau_1 + h \le t \le \tau_2 - h\,,\\ (\tau_2 - t)/h\,, & \tau_2 - h < t < \tau_2\,,\\ 0\,, & \text{otherwise}\,. \end{cases}
$$

Integrating by parts, we find

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_\Omega v^{p-1}(\phi^p v^{-\varepsilon}\chi^h_{[\tau_1,\tau_2]})_t\,d\mu\,dt &= \frac{(p-1)}{p-1-\varepsilon} \\
\cdot\left(\int_{\tau_1}^{\tau_2}\int_\Omega v^{p-1-\varepsilon}\phi^p(\chi^h_{[\tau_1,\tau_2]})_t\,d\mu\,dt \right.&+ \left.\int_{\tau_1}^{\tau_2}\int_\Omega v^{p-1-\varepsilon}(\phi^p)_t\chi^h_{[\tau_1,\tau_2]}\,d\mu\,dt\right).
\end{aligned}
$$

After taking the limit $h \to 0$ in the expression above, and using Lebesgue's differentiation theorem, we have

$$
\begin{aligned}
\lim_{h\to 0}\int_{\tau_1}^{\tau_2}\int_\Omega v^{p-1}(\phi^p v^{-\varepsilon}\chi^h_{[\tau_1,\tau_2]})_t\,d\mu\,dt &= \frac{(p-1)}{p-1-\varepsilon} \\
\cdot\left(-\left[\int_{\Omega\times\{t\}} v^{p-1-\varepsilon}\phi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} \right.&+ \left.\int_{\tau_1}^{\tau_2}\int_\Omega v^{p-1-\varepsilon}(\phi^p)_t\,d\mu\,dt\right).
\end{aligned}
$$

As u is a positive parabolic superminimizer related to the doubly nonlinear equation, also v is a parabolic superminimizer. Moreover, by Remark 3.6 below, $\phi^p v^{-\varepsilon}\chi^h_{[\tau_1,\tau_2]}$ is a nonnegative admissible test function. Hence by the definition of a parabolic superminimizer and (3.3) we have

$$
\begin{aligned}
&\frac{p(p-1)}{p-1-\varepsilon}\left(-\left[\int_{\Omega\times\{t\}} v^{p-1-\varepsilon}\phi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_\Omega v^{p-1-\varepsilon}(\phi^p)_t\,d\mu\,dt\right) \\
&\quad\le \lim_{h\to 0}\left(-\int_{\mathrm{supp}(\phi^p\chi^h_{[\tau_1,\tau_2]})} g_v^p\,d\nu + \int_{\mathrm{supp}(\phi^p\chi^h_{[\tau_1,\tau_2]})} g^p_{v+\phi^p v^{-\varepsilon}\chi^h_{[\tau_1,\tau_2]}}\,d\nu\right) \\
&\quad\le -\varepsilon\int_{\mathrm{supp}(\phi^p\chi_{[\tau_1,\tau_2]})}\phi^p v^{-\varepsilon-1}g_v^p\,d\nu + p^p\varepsilon^{1-p}\int_{\mathrm{supp}(\phi^p\chi_{[\tau_1,\tau_2]})} v^{p-1-\varepsilon}g_\phi^p\,d\nu\,.
\end{aligned}
$$

On one hand, setting $\tau_1 = t_1$, and $\tau_2 = t_2$, we obtain

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega \phi^p v^{-\varepsilon-1}g_v^p\,d\mu\,dt \le& \frac{p(p-1)}{\varepsilon|p-1-\varepsilon|}\int_{t_1}^{t_2}\int_\Omega v^{p-1-\varepsilon}|(\phi^p)_t|\,d\mu\,dt \\
&+ \left(\frac{p}{\varepsilon}\right)^p\int_{t_1}^{t_2}\int_\Omega v^{p-1-\varepsilon}g_\phi^p\,d\mu\,dt\,.
\end{aligned}
\tag{3.4}
$$

On the other hand, if $\varepsilon < p-1$, set $\tau_1 = t$ and $\tau_2 = t_2$. If $\varepsilon > p-1$, set $\tau_1 = t_1$ and $\tau_2 = t$. We obtain

$$\begin{aligned}
&\frac{p(p-1)}{\varepsilon|p-1-\varepsilon|}\int_\Omega v^{p-1-\varepsilon}(x,t)\phi^p(x,t)\,d\mu \\
&\qquad \le \frac{p(p-1)}{\varepsilon|p-1-\varepsilon|}\int_{t_1}^{t_2}\int_\Omega v^{p-1-\varepsilon}|(\phi^p)_t|\,d\mu\,dt \\
&\qquad\quad + \left(\frac{p}{\varepsilon}\right)^p\int_{t_1}^{t_2}\int_\Omega v^{p-1-\varepsilon}g_\phi^p\,d\mu\,dt\,.
\end{aligned} \tag{3.5}$$

This holds for almost every $t \in (t_1, t_2)$. Dividing (3.5) by $p(p-1)/\varepsilon|p-1-\varepsilon|$, and adding the resulting expression to (3.4) yields the desired estimate for v, since the constants in the inequality do not depend on $t \in (t_1, t_2)$ and ϕ, g_ϕ vanish outside the support of ϕ. The proof is completed by dividing the resulting expression sidewise with the constant $\alpha^{p-1-\varepsilon}$. □

REMARK 3.6. We now give justifications for the formal treatment above. By a change of variable, it is straightforward to see that for a nonnegative parabolic super- or subminimizer v and for an admissible test function ψ, for any small enough s, we have

$$\begin{aligned}
&p\int_{\mathrm{supp}(\psi)} v^{p-1}(x,t-s)\psi_t\,d\nu + \int_{\mathrm{supp}(\psi)} g^p_{v(x,t-s)}\,d\nu \\
&\qquad \le \int_{\mathrm{supp}(\psi)} g^p_{v(x,t-s)+\psi(x,t)}\,d\nu\,.
\end{aligned}$$

We multiply this inequality sidewise with a standard mollifier with respect to the time variable s, and then integrate both sides with respect to s. After using Fubini's theorem on the left side, this yields

$$\begin{aligned}
&p\int_{\mathrm{supp}(\psi)} (v^{p-1})_\sigma\psi_t\,d\nu + \left(\int_{\mathrm{supp}(\psi)} g_v^p\,d\nu\right)_\sigma \\
&\qquad \le \left(\int_{\mathrm{supp}(\psi)} g^p_{v(x,t-s)+\psi(x,t)}\,d\nu\right)_\sigma\,.
\end{aligned} \tag{3.7}$$

Here we have used the notation

$$(v^{p-1})_\sigma(x,t) = \int_{\boldsymbol{R}} \theta_\sigma(s)v^{p-1}(x,t-s)\,ds\,,$$

where θ is the standard mollifier and $\sigma > 0$ is assumed to be small enough so that everything stays in the time cylinder. To be precise, in the proof of Lemma 3.1 we then choose the test function

$$\psi = \phi^p((v^{p-1})_\sigma)^{-\varepsilon/(p-1)}\chi^h_{[\tau_1,\tau_2]} \tag{3.8}$$

with $\phi \in \mathrm{Lip}_0(\Omega_{(t_1,t_2)})$. The test function ψ now has compact support and belongs to the space $L^p(0,T;N^{1,p}(\Omega))$. By Lemma 2.7 in [23], easily adaptable for minimizers related to the doubly nonlinear equation, ψ can be plugged into the inequality (3.7). Similarly to the formal proof above, partial integration is then performed to write the expression in a form where $(v^{p-1})_\sigma$ is not differentiated with respect to time. Once this is done we can take the limits $\sigma \to 0$ and $h \to 0$, which leads us back to the inequality above (3.4). For details on justifying the convergence of the upper gradient terms in (3.7) as $\sigma \to 0$, we refer the reader to [24].

We prove next a reverse Hölder type inequality for negative powers of parabolic superminimizers. The first step of the proof consists of combining Sobolev's inequality with the energy esimate of Lemma 3.1. Then, because the energy estimate is homogeneous in powers, the obtained inequalities can be combined as in Moser's iteration to complete the proof.

LEMMA 3.9. *Let $u > 0$ be a parabolic superminimizer in $Q_r \subset \Omega_T$, locally bounded away from zero and let $0 < \delta < 1$. Then there exist constants $C = C(C_\mu, C_p, \Lambda, p, \delta, T)$ and $\theta = \theta(C_\mu, p)$ such that*

$$\operatorname*{ess\,inf}_{Q_{\alpha' r}} u \geq \left(\frac{C}{(\alpha - \alpha')^\theta}\right)^{-1/q} \left(\fint_{Q_{\alpha r}} u^{-q}\, d\nu\right)^{-1/q}$$

for every $0 < \delta \leq \alpha' < \alpha \leq 1$ and for all $0 < q \leq p$.

PROOF. Let us fix α', α such that $0 < \delta \leq \alpha' < \alpha \leq 1$, and divide the interval (α', α) as follows: $\alpha_0 = \alpha$, $\alpha_\infty = \alpha'$, and

$$\alpha_j = \alpha - (\alpha - \alpha')(1 - \gamma^{-j}),$$

where $\gamma = 2 - p/\kappa = 1 + (\kappa - p)/\kappa > 1$. We set

$$Q_j = Q_{\alpha_j r} = B_j \times T_j = B(x_0, \alpha_j r) \times (t_0 - T(\alpha_j r)^p, t_0 + T(\alpha_j r)^p),$$

and choose the sequence of test-functions $\{\phi_j\}_{j=0}^\infty$ so that $\operatorname{supp}(\phi_j) \subset Q_j$, $0 \leq \phi_j \leq 1$ on Q_j, and $\phi_j = 1$ in Q_{j+1}. Moreover, let each ϕ_j be such that

$$g_{\phi_j} \leq \frac{4\gamma^j}{(\alpha - \alpha')r} \quad \text{and} \quad |(\phi_j)_t| \leq \frac{1}{T}\left(\frac{4\gamma^j}{(\alpha - \alpha')r}\right)^p.$$

Assume $\varepsilon > 0$, $\varepsilon \neq p - 1$. We have

$$g^p_{u^{(p-1-\varepsilon)/p}\phi_j} \leq 2^{p-1} u^{p-1-\varepsilon} g^p_{\phi_j} + 2^{p-1}\left(\frac{|p - 1 - \varepsilon|}{p}\right)^p u^{-\varepsilon-1} g_u^p \phi_j^p.$$

Using Hölder's inequality brings us to the estimate

$$\begin{aligned}
&\fint_{T_{j+1}} \fint_{B_{j+1}} u^{(p-1-\varepsilon)\gamma}\, d\mu dt \\
&\leq \fint_{T_{j+1}} \left(\fint_{B_{j+1}} u^{(p-1-\varepsilon)} \phi_j^p\, d\mu\right)^{(\kappa-p)/\kappa} \left(\fint_{B_{j+1}} (u^{(p-1-\varepsilon)/p}\phi_j)^\kappa\, d\mu\right)^{p/\kappa} dt \\
&\leq \frac{|T_j|\mu(B_j)}{|T_{j+1}|\mu(B_{j+1})} \left(\operatorname*{ess\,sup}_{T_j} \fint_{B_j} u^{(p-1-\varepsilon)} \phi_j^p\, d\mu\right)^{(\kappa-p)/\kappa} \\
&\qquad \cdot \fint_{T_j} \left(\fint_{B_j} (u^{(p-1-\varepsilon)/p}\phi_j)^\kappa\, d\mu\right)^{p/\kappa} dt.
\end{aligned}$$

Observe that $|T_j| = 2T(\alpha_j r)^p$ and $\alpha_{j+1} \geq \min\{\delta, (1+\gamma)^{-1}\}\alpha_j$. Thus the multiplicative constant on the right-hand side is bounded by a constant independent of j, r, T, α', and α. We estimate the last term in the preceding inequality by Sobolev's inequality (see (2.9)). We find

$$\begin{aligned}\left(\fint_{B_j} (u^{(p-1-\varepsilon)/p}\phi_j)^\kappa \, d\mu\right)^{p/\kappa} &\leq Cr^p \fint_{B_j} g^p_{u^{(p-1-\varepsilon)/p}\phi_j} \, d\mu \\ &\leq Cr^p \fint_{B_j} \left(u^{p-1-\varepsilon} g^p_{\phi_j} + \left(\frac{|p-1-\varepsilon|}{p}\right)^p u^{-\varepsilon-1} g_u^p \phi_j^p\right) d\mu\,,\end{aligned}$$

where $C = C(C_\mu, C_p, \Lambda, p)$. Since $\varepsilon > 0$, $\varepsilon \neq p-1$, we may use Lemma 3.1 to obtain

$$\begin{aligned}\fint_{T_{j+1}} \fint_{B_{j+1}} u^{(p-1-\varepsilon)\gamma} \, d\mu \, dt &\leq C\left(\operatorname*{ess\,sup}_{T_j} \fint_{B_j} u^{p-1-\varepsilon}\phi_j^p \, d\mu\right)^{(\kappa-p)/\kappa} \\ &\quad \cdot \frac{C}{T\delta^p} \int_{T_j} \fint_{B_j} \left(u^{p-1-\varepsilon} g^p_{\phi_j} + \left(\frac{|p-1-\varepsilon|}{p}\right)^p u^{-\varepsilon-1} g_u^p \phi_j^p\right) d\mu \, dt \\ &\leq C\left(\int_{T_j} \fint_{B_j} u^{p-1-\varepsilon}\left(C_1 g^p_{\phi_j} + C_2 |(\phi_j^p)_t|\right) d\mu \, dt\right)^{(\kappa-p)/\kappa} \\ &\quad \cdot \frac{C}{T\delta^p} \int_{T_j} \fint_{B_j} \left(u^{p-1-\varepsilon} g^p_{\phi_j} + |p-1-\varepsilon|^p u^{p-1-\varepsilon}(C_1 g^p_{\phi_j} + C_2|(\phi_j^p)_t|)\right) d\mu \, dt \\ &\leq C(1+|p-1-\varepsilon|^p)\left(\frac{\gamma^{jp}}{(\alpha-\alpha')^p} \fint_{T_j} \fint_{B_j} u^{p-1-\varepsilon} \, d\mu \, dt\right)^\gamma,\end{aligned}$$

where $C = C(\varepsilon, C_\mu, C_p, \Lambda, p, \delta, T)$ is uniformly bounded for every ε, except in the neighborhood of $\varepsilon = 0$. For each $j = 0, 1, \ldots$, we can now use the above estimate with $\varepsilon_j \geq 2p-1$ chosen in such a way that $p-1-\varepsilon_j = -p\gamma^j$, to write

$$\begin{aligned}&\left(\fint_{Q_{j+1}} u^{-p\gamma^{j+1}} \, d\nu\right)^{-1/p\gamma^{j+1}} \\ &\qquad \geq (C2p^p\gamma^{pj})^{-1/p\gamma^{j+1}} \left(\frac{\gamma^{jp}}{(\alpha-\alpha')^p}\right)^{-1/p\gamma^j} \left(\fint_{Q_j} u^{-p\gamma^j} \, d\nu\right)^{-1/p\gamma^j},\end{aligned} \tag{3.10}$$

where $C = C(C_\mu, C_p, \Lambda, p, \delta, T)$. By iterating this, since $\gamma > 1$, we find that

$$\begin{aligned}\operatorname*{ess\,inf}_{Q_\infty} u &\geq (Cp)^{\sum_{j=1}^\infty -1/\gamma^j} \gamma^{\sum_{j=0}^\infty -(1+\gamma)j/\gamma^j} \left(\frac{1}{(\alpha-\alpha')}\right)^{\sum_{j=0}^\infty -1/\gamma^j} \left(\fint_{Q_0} u^{-p} \, d\nu\right)^{-1/p} \\ &= \left(\frac{C}{(\alpha-\alpha')}\right)^{-\gamma/(\gamma-1)} \left(\fint_{Q_0} u^{-p} \, d\nu\right)^{-1/p},\end{aligned}$$

where the constant $C = C(C_\mu, C_p, \Lambda, p, \delta, T)$ is positive and finite. The proof is now completed for any $0 < q \leq p$ by using a result from real analysis (see [16, Theorem 3.38]). □

We also prove a reverse Hölder inequality for positive powers of parabolic superminimizers.

LEMMA 3.11. *Let $u > 0$ be a parabolic superminimizer in $Q_r \subset \Omega_T$ which is locally bounded away from zero, and $0 < \delta < 1$. Then there exist constants $0 < C = C(C_\mu, C_p, \Lambda, p, q, \delta, T)$ and $\theta = \theta(C_\mu, p)$ such that*

$$\left(\fint_{Q_{\alpha' r}} u^q \, d\nu\right)^{1/q} \leq \left(\frac{C}{(\alpha - \alpha')^\theta}\right)^{1/s} \left(\fint_{Q_{\alpha r}} u^s \, d\nu\right)^{1/s}$$

for all $0 < \delta \leq \alpha' < \alpha \leq 1$ and for all $0 < s < q < (p-1)(2 - p/\kappa)$ and κ is as in (2.3).

PROOF. Assume $0 < s < q < (p-1)(2-p/\kappa)$, where κ is as in the Sobolev–Poincaré inequality. Then there exists a k such that $s\gamma^{k-1} \leq q \leq s\gamma^k$. Let ρ_0 be such that $0 < \rho_0 \leq s$ and $q = \gamma^k \rho_0$. Now for each $j = 0, \ldots, k-1$, there exists a $0 < \varepsilon_j < p-1$ such that $p - 1 - \varepsilon_j = \rho_0 \gamma^j$. By the first part of the proof of the previous lemma, we have

$$\begin{aligned}
&\left(\fint_{Q_{j+1}} u^{\rho_0 \gamma^{j+1}} \, d\nu\right)^{1/\rho_0 \gamma^{j+1}} \\
&\qquad \leq (C 2 p^p \gamma^{pj})^{1/\rho_0 \gamma^{j+1}} \left(\frac{\gamma^{pj}}{(\alpha - \alpha')^p}\right)^{1/\rho_0 \gamma^j} \left(\fint_{Q_j} u^{\rho_0 \gamma^j} \, d\nu\right)^{1/\rho_0 \gamma^j},
\end{aligned}$$

where $C = C(C_\mu, C_p, \Lambda, p, q, \delta, T)$. Iterating this estimate for $j = 0, \ldots, k-1$ yields

$$\left(\fint_{Q_{\alpha' Q}} u^q \, d\nu\right)^{1/q} \leq \left(\frac{C}{(\alpha - \alpha')^{\gamma^*}}\right)^{1/\rho_0} \left(\fint_{\alpha Q} u^{\rho_0} \, d\nu\right)^{1/\rho_0}, \tag{3.12}$$

where $C = C(C_\mu, C_p, \Lambda, p, q, \delta, T)$ blows up as q tends to $(p-1)(2 - p/\kappa)$ and

$$\gamma^* = \frac{p\gamma}{\gamma - 1}(1 - \gamma^{-k}) \leq \frac{p\gamma}{\gamma - 1}.$$

Using Hölder's inequality on the right-hand side of (3.12), setting $\theta = p\gamma/(\gamma - 1)$ and using the fact that $s/\gamma \leq \rho_0 \leq s$ completes the proof. □

4. Reverse Hölder inequalities for parabolic subminimizers. In this section we prove estimates analogous to those in Section 3, but this time for parabolic subminimizers. This is done essentially identically to what was done for superminimizers, but with a slight change in the test function we use. Then we utilize the obtained energy estimate to prove a reverse Hölder inequality for positive powers of parabolic subminimizers.

LEMMA 4.1. *Let $u > 0$ be a locally bounded parabolic subminimizer and let $\varepsilon \geq 1$. Then*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<T} \int_\Omega u^{p-1+\varepsilon} \phi^p \, d\mu + \int_{\operatorname{supp}(\phi)} u^{\varepsilon - 1} g_u \phi^p \, d\nu \\
&\qquad \leq C_1 \int_{\operatorname{supp}(\phi)} u^{p-1+\varepsilon} g_\phi^p \, d\nu + C_2 \int_{\operatorname{supp}(\phi)} u^{p-1+\varepsilon} |(\phi^p)_t| \, d\nu
\end{aligned}$$

for every $\phi \in \operatorname{Lip}_0(\Omega_T)$, $0 \leq \phi \leq 1$, where

$$C_1 = \left(\frac{p}{\varepsilon}\right)^p \left(1 + \frac{\varepsilon|p-1+\varepsilon|}{p(p-1)}\right), \quad C_2 = \left(1 + \frac{p(p-1)}{\varepsilon|p-1+\varepsilon|}\right).$$

PROOF. Let $0 \le \phi \le 1$, $\phi \in \mathrm{Lip}_0(\Omega_T)$, for some $0 < t_1 < t_2 < T$. Let $\varepsilon > 0$. Since by assumption u is locally bounded, we can take a constant $\alpha > 0$ such that after denoting $v = \alpha u$, we have $1 - \varepsilon\phi^p v^{\varepsilon-1} > 0$ almost everywhere in the support of ϕ. Since u is a subminimizer, also v is a subminimizer and we can plug $-\phi(x,t)^p v(x,t)^\varepsilon \chi^h_{[\tau_1,\tau_2]}$ as a test function into the inequality (2.7). The rest of the proof is now completely analogous to the proof of Lemma 3.1. □

We prove a reverse Hölder type inequality for positive powers of parabolic subminimizers. Again, the proof consists of combining the energy estimate of Lemma 4.1 with Moser's iteration to obtain the inequality.

LEMMA 4.2. *Let $u > 0$ be a parabolic subminimizer in $Q_r \subset \Omega_T$ which is locally bounded and let $0 < \delta < 1$. Then there exist constants $C = C(C_\mu, C_p, \Lambda, p, \delta, T)$ and $\theta = \theta(C_\mu, p)$ such that the inequality*

$$\operatorname*{ess\,sup}_{Q_{\alpha' r}} u \le \left(\frac{C}{(\alpha-\alpha')^\theta}\right)^{1/q} \left(\fint_{Q_{\alpha r}} u^q \, d\nu\right)^{1/q}$$

holds for every $0 < \delta \le \alpha' < \alpha \le 1$ and for all $0 < q \le p$.

PROOF. The steps of the proof are analogous to the proof of Lemma 3.9. The difference is that here we use Lemma 4.1 and the observation that for each γ^j, $j = 0, 1, \dots$, there exists a $\varepsilon_j \ge 1$ such that $p - 1 + \varepsilon_j = p\gamma^j$. □

5. Measure estimates for parabolic superminimizers. The following logarithmic energy estimate will also be important to our argument. Regarding the time derivation of u^{p-1}, the proof presented below is again formal. Justifications for this can be given as in Remark 3.6; we use the test function as in (3.8), but with $\varepsilon = p - 1$.

LEMMA 5.1. *Let $u > 0$ be a parabolic superminimizer, locally bounded away from zero. Then the inequality*

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_\Omega g^p_{\log u}\phi^p \, d\mu dt - p\left[\int_{\Omega\times\{t\}} \log u \phi^p \, d\mu\right]_{t=\tau_1}^{\tau_2} \\ \le \frac{p^p}{(p-1)^p}\int_{\tau_1}^{\tau_2}\int_\Omega g^p_\phi \, d\mu dt + p\int_{\tau_1}^{\tau_2}\int_\Omega |\log u||(\phi^p)_t| \, d\mu dt\end{aligned}$$

holds for every $\phi \in \mathrm{Lip}_0(\Omega_T)$, such that $0 \le \phi \le 1$ and almost every $0 < \tau_1 < \tau_2 < T$.

PROOF. Let $\phi \in \mathrm{Lip}_0(\Omega_T)$ be such that $0 \le \phi \le 1$. As in the preceding lemma, since both the definition of a parabolic superminimizer and the final weak Harnack inequality are scalable properties, we may assume that u has been scaled in such a way that $1 - (p-1)\phi^p u^{-p} > 0$ almost everywhere in the support of ϕ, and by using the convexity of the mapping $t \mapsto t^p$, we find

$$
\begin{aligned}
g^p_{u+\phi^p u^{1-p}} &\le \left((1-(p-1)\phi^p u^{-p})g_u + p\phi^p(p-1)u^{-p}\frac{u}{\phi(p-1)}g_\phi\right)^p \\
&\le (1-(p-1)\phi^p u^{-p})g_u^p + p^p(p-1)^{1-p}g_\phi^p\,.
\end{aligned}
$$

Let $\chi^h_{[\tau_1,\tau_2]}$ be defined as in Lemma 3.1. Integrating by parts, we obtain

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1}(\phi^p u^{1-p}\chi^h_{[\tau_1,\tau_2]})_t\,d\mu\,dt \\
&\qquad = (p-1)\int_{\tau_1}^{\tau_2}\int_\Omega\Big[(\log u)\phi^p(\chi^h_{[\tau_1,\tau_2]})_t + (\log u)(\phi^p)_t\chi^h_{[\tau_1,\tau_2]}\Big]\,d\mu\,dt\,.
\end{aligned}
$$

Taking the limit $h\to 0$, we obtain by Lebesgue's theorem of differentiation

$$
\begin{aligned}
&\lim_{h\to 0}\int_{\tau_1}^{\tau_2}\int_\Omega u^{p-1}(\phi^p u^{1-p}\chi^h_{[\tau_1,\tau_2]})_t\,d\mu\,dt \\
&\qquad = (p-1)\left(-\left[\int_{\Omega\times\{t\}}(\log u)\phi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_\Omega(\log u)(\phi^p)_t\,d\mu\,dt\right).
\end{aligned}
$$

As u is a parabolic superminimizer and $\phi^p u^{1-p}\chi^h_{[\tau_1,\tau_2]}$ is a nonnegative admissible test-function, we obtain

$$
\begin{aligned}
&p(p-1)\left(-\left[\int_{\Omega\times\{t\}}\log u\phi^p\,d\mu\right]_{t=\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_\Omega(\log u)(\phi^p)_t\,d\mu\,dt\right) \\
&\qquad \le \lim_{h\leftarrow 0}\left(-\int_{\operatorname{supp}(\phi^p\chi^h_{[\tau_1,\tau_2]})} g_u^p\,d\nu + \int_{\operatorname{supp}(\phi^p\chi^h_{[\tau_1,\tau_2]})} g^p_{u+\phi^p u^{1-p}\chi^h_{[\tau_1,\tau_2]}}\,d\nu\right) \\
&\qquad \le -(p-1)\int_{\tau_1}^{\tau_2}\int_\Omega\phi^p g^p_{\log u}\,d\mu\,dt + p^p(p-1)^{1-p}\int_{\tau_1}^{\tau_2}\int_\Omega g_\phi^p\,d\mu\,dt\,.
\end{aligned}
$$

Rearranging terms completes the proof. □

Next, using the logarithmic energy estimate, we establish monotonicity in time of the weighted integral of $\log u$. This in turn enables us to estimate the measure of the level sets of $\log u$ around a time level t_0.

LEMMA 5.2. *Let $u>0$ be a parabolic superminimizer in $Q_r\subset\Omega_T$ and assume u is locally bounded away from zero. Let $0<\alpha<1$. Define*

$$
\phi(x) = \left(1-2\frac{d(x,x_0)}{(1+\alpha)r}\right)_+\,,
$$

where $0<\alpha<1$ and $(x,t)\in Q_r$. Let

$$
\beta = \frac{1}{N}\int_{B(x_0,r)}\log u(x,t_0)\phi^p(x)\,d\mu\,,
$$

where

$$
N = \int_{B(x_0,r)}\phi^p(x)\,d\mu\,.
$$

Then there exist positive constants $C = C(C_\mu, C_p, p, \alpha)$ and $C' = C'(C_\mu, p, \alpha)$ such that

$$\nu\left(\{(x,t) \in Q_{\alpha r};\ t \le t_0,\ \log u(x,t) > \lambda + \beta + C'\}\right) \le C\frac{\nu(Q_{\alpha r})}{\lambda^{p-1}}$$

and

$$\nu\left(\{(x,t) \in Q_{\alpha r};\ t \ge t_0,\ \log u(x,t) < -\lambda + \beta - C'\}\right) \le C\frac{\nu(Q_{\alpha r})}{\lambda^{p-1}}$$

for every $\lambda > 0$.

PROOF. From the definition of ϕ, it readily follows that $0 \le \phi \le 1$, $g_\phi \le (\alpha r)^{-1}$, and for every $t \in [t_0 - T(\alpha r)^p, t_0 + T(\alpha r)^p]$,

$$\left(\frac{1-\alpha}{2}\right)^p \mu(B(x_0,\alpha r)) \le N \le \mu(B(x_0,r))\,. \tag{5.3}$$

We write

$$v(x,t) = \log u(x,t) - \beta \qquad \text{and} \qquad V(t) = \frac{1}{N}\int_{B(x_0,r)} v(x,t)\phi^p(x)\,d\mu\,,$$

and find that $V(t_0) = 0$. Let $0 \le \xi(t) \le 1$ be a smooth function such that $\operatorname{supp}(\xi) \subset (t_0 - Tr^p, t_0 + Tr^p)$, and $\xi(t) = 1$ for all $t \in [t_0 - T(\alpha r)^p, t_0 + T(\alpha r)^p]$.

Write $\psi(x,t) = \phi(x)\xi(t)$. Since u is a positive superminimizer bounded away from zero, we can use Lemma 5.1 with ψ as a test function. We obtain for $t_0 - T(\alpha r)^p < t_1 < t_2 < t_0 + T(\alpha r)^p$, since on this interval $\xi(t) = 1$,

$$\begin{aligned}\int_{t_1}^{t_2}\int_{B(x_0,r)} g_v^p\phi^p\,d\mu dt - p\,[NV(t)]_{t=t_1}^{t_2} &\le \frac{p^p}{(p-1)^p}\int_{t_1}^{t_2}\int_{B(x_0,r)} g_\phi^p\,d\mu dt \\ &\le \frac{Cp^p}{(p-1)^p}(t_2 - t_1)\frac{\mu(B(x_0,r))}{(\alpha r)^p}\,,\end{aligned}$$

where $C = C(p)$. On the other hand, from the weighted Poincaré inequality (2.11), we have

$$\begin{aligned}&\left(\frac{1-\alpha}{2}\right)^p\int_{t_1}^{t_2}\int_{B(x_0,\alpha r)} |v - V(t)|^p\,d\mu dt \\ &\qquad \le \int_{t_1}^{t_2}\int_{B(x_0,\alpha r)} |v - V(t)|^p\phi^p\,d\mu dt \le Cr^p\int_{t_1}^{t_2}\int_{B(x_0,r)} g_v^p\phi^p\,d\mu dt\,,\end{aligned}$$

where $C = C(C_\mu, C_p, p, \alpha)$. By combining these we find

$$\begin{aligned}&\frac{(1-\alpha)^p}{CNr^p}\int_{t_1}^{t_2}\int_{B(x_0,\alpha r)} |v - V(t)|^p\,d\mu dt + V(t_1) - V(t_2) \\ &\qquad \le \frac{C(t_2 - t_1)\mu(B(x_0,r))}{N(\alpha r)^p} \le \left(\frac{2}{1-\alpha}\right)^p\frac{C(t_2 - t_1)\mu(B(x_0,r))}{\mu(B(x_0,\alpha r))(\alpha r)^p} \\ &\qquad \le C'\frac{(t_2 - t_1)}{(\alpha r)^p}\,,\end{aligned}$$

where $C' = C'(C_\mu, C_p, p, \alpha)$. We denote

$$w(x,t) = v(x,t) + \frac{C'(t - t_0)}{(\alpha r)^p}$$

and

$$W(t) = V(t) + \frac{C'(t-t_0)}{(\alpha r)^p},$$

and restate the preceding inequality as

$$\frac{(1-\alpha)^p}{CNr^p}\int_{t_1}^{t_2}\int_{B(x_0,\alpha r)} |w - W(t)|^p \, d\mu dt + W(t_1) - W(t_2) \le 0\,.$$

This implies that $W(t_1) \le W(t_2)$ whenever $t_0 - T(\alpha r)^p \le t_1 < t_2 \le t_0 + T(\alpha r)^p$, i.e., the function W is increasing, thus differentiable for almost every $t \in (t_0 - T(\alpha r)^p, t_0 + T(\alpha r)^p)$. As a consequence, we obtain

$$\frac{(1-\alpha)^p}{CNr^p}\int_{B(x_0,\alpha r)} |w - W(t)|^p \, d\mu - W'(t) \le 0 \tag{5.4}$$

for almost every $t_0 - T(\alpha r)^p < t < t_0 + T(\alpha r)^p$. Let us denote

$$E_\lambda(t) = \{x \in B(x_0, \alpha r)\ ;\ w(x,t) > \lambda\}\,,$$
$$E_\lambda^- = \{(x,t) \in Q_{\alpha r}\ ;\ t < t_0,\ w(x,t) > \lambda\}\,.$$

For every $t_0 - T(\alpha r)^p < t < t_0$ and $\lambda > 0$, since $W(t) \le W(t_0) = 0$, we have

$$(\lambda - W(t))^p \mu(E_\lambda^-(t)) \le \int_{B(x_0,\alpha r)} |w - W(t)|^p \, d\mu\,.$$

Hence we have

$$\frac{(1-\alpha)^p}{CNr^p}\mu(E_\lambda^-(t)) - \frac{W'(t)}{(\lambda - W(t))^p} \le 0$$

for almost every $t_0 - T(\alpha r)^p < t < t_0$. This yields, after integrating over the interval $(t_0 - T(\alpha r)^p, t_0)$,

$$\frac{\nu(E_\lambda^-)}{Nr^p} \le \frac{C}{(1-\alpha)^p}[(\lambda - W(t))^{-(p-1)}]_{t=t_0-T(\alpha r)^p}^{t_0} \le \frac{C}{(1-\alpha)^p\lambda^{p-1}}\,,$$

where $C = C(C_\mu, C_p, p, \alpha)$. Together with (5.3), this implies

$$\nu\left(\{(x,t) \in Q_{\alpha r}\ ;\ t \le t_0,\ \log u(x,t) > \lambda + \beta + C'\}\right) \le C\frac{\nu(Q_{\alpha r})}{\lambda^{p-1}}\,,$$

where $C = C(C_\mu, C_p, p, \alpha)$. Denote then

$$E_\lambda^+(t) = \{x \in B(x_0, \alpha r)\ ;\ w(x,t) < -\lambda\}\,,$$
$$E_\lambda^+ = \{(x,t) \in Q_{\alpha r}\ ;\ t > t_0,\ w(x,t) < -\lambda\}\,.$$

Similarly to the case of E_λ^-, using the monotonicity of $W(t)$, we obtain

$$(\lambda + W(t))^p \mu(E_\lambda^+(t)) \le \int_{B(x_0,\alpha r)} |w - W(t)|^p \, d\mu$$

for every $t_0 < t < t_0 + T(\alpha r)^p$. This together with (5.4) leads to

$$\frac{(1-\alpha)^p\mu(E_\lambda^+(t))}{CNr^p} - \frac{W'(t)}{(\lambda + W(t))^p} \le 0$$

for almost every $t_0 < t < t_0 + T(\alpha r)^p$. Integration over the interval $(t_0, t_0 + T(\alpha r)^p)$ gives now

$$\frac{\nu(E_\lambda^+)}{Nr^p} \le -\frac{C}{(1-\alpha)^p}[(\lambda + W(t))^{-(p-1)}]_{t=t_0}^{t_0+T(\alpha r)^p} \le \frac{C}{(1-\alpha)^p\lambda^{p-1}}\,,$$

and thus after using (5.3) we may conclude

$$\nu(\{(x,t) \in Q_{\alpha r}\ ;\ t \ge t_0,\ \log u < -\lambda + \beta - C'\}) \le C\frac{\nu(Q_{\alpha r})}{\lambda^{p-1}}\,.$$

Again $C = C(C_\mu, C_p, p, \alpha)$. □

6. Harnack's inequality for parabolic minimizers. Having established a logarithmic measure estimate for superminimizers around a time level t_0, we have the prerequisites to use Lemma 2.13. This way for parabolic superminimizers we can glue the reverse Hölder inequality for negative powers together with the reverse Hölder inequality for positive powers. We obtain a weak form of the Harnack inequality for parabolic superminimizers locally bounded away from zero. This result is in some sense finer than the final Harnack inequality since we only assume the superminimizing property, and hence it is of interest in itself. Observe in the following how, from applying Lemma 2.13 separately on both sides of the time level t_0, a waiting time inevitably appears between the negative and positive time segments.

LEMMA 6.1. *Let $u > 0$ be a parabolic superminimizer in $Q_r \subset \Omega_T$ which is bounded away from zero. Then*

$$\left(\fint_{\delta Q^-} u^q\, d\nu\right)^{1/q} \le C \operatorname*{ess\,inf}_{\delta Q^+} u\,,$$

where $0 < \delta < 1$ and $0 < q < (p-1)(2 - p/\kappa)$. Here $C = C(C_\mu, C_p, \Lambda, p, q, \delta, T)$.

PROOF. Assume $0 < \delta < 1$. Let β and C' be as in Lemma 5.2. By Lemma 3.9 there exists a positive constant $C = C(C_\mu, C_p, \Lambda, p, \delta, T)$, such that for every $0 < s \le p$ and $0 < \delta \le \alpha' < \alpha < 1$, we have

$$\begin{aligned}\left(\operatorname*{ess\,sup}_{\alpha' Q^+} u^{-1}e^{\beta - C'}\right)^{-1} &= \operatorname*{ess\,inf}_{\alpha' Q^+} ue^{-\beta + C'}\\ &\ge C\left(\frac{1}{(\alpha-\alpha')^\theta}\fint_{\alpha Q^+}(ue^{-\beta+C'})^{-s}\, d\nu\right)^{-1/s}.\end{aligned}\tag{6.2}$$

By Lemma 5.2 applied to $\{Q_{(3+\delta)r/4}\ ;\ t \ge t_0\}$, we have

$$\begin{aligned}&\nu\left(\{(x,t) \in \frac{1+\delta}{2}Q^+\ ;\ \log(u^{-1}e^{\beta - C'}) > \lambda\}\right)\\ &\quad\le \nu\big(\{(x,t) \in Q_{(3+\delta)r/4}\ ;\ t \ge t_0,\ \log(u^{-1}e^{\beta-C'}) > \lambda\}\big)\\ &\quad\le C\frac{\nu(Q_{(3+\delta)r/4})}{\lambda^{p-1}} \le C\frac{\nu(\delta Q^+)}{\lambda^{p-1}}\end{aligned}\tag{6.3}$$

for every $\lambda > 0$. In the last step of the above inequality, we used the doubling property of μ, and so $C = C(C_\mu, C_p, p, \delta)$. From (6.2) and (6.3), we now see that the conditions of Lemma 2.13, with $(1+\delta)/2Q^+$ in place of U_1, are met. Hence

$$\operatorname*{ess\,sup}_{\delta Q^+} u^{-1} e^{\beta - C'} \leq C \,, \tag{6.4}$$

where $C = C(C_\mu, C_p, \Lambda, p, \delta, T)$. From Lemma 3.11 we know there exists a positive constant $C = C(C_\mu, C_p, \Lambda, p, q, \delta, T)$ for which

$$\left(\fint_{\alpha' Q^-} (u e^{-\beta - C'})^q \, d\nu \right)^{1/q} \leq \left(\frac{C}{(\alpha - \alpha')^\theta} \right)^{1/s} \left(\fint_{\alpha Q^-} (u e^{-\beta - C'})^s \, d\nu \right)^{1/s}$$

for every $0 \leq \delta < \alpha' < \alpha \leq 1$ and for all $0 < s < q < (p-1)(2 - p/\kappa)$. Moreover for δQ^-, since u is a positive superminimizer bounded away from zero, we can use Lemma 5.2 to get

$$\nu\left(\left\{ (x,t) \in \frac{1+\delta}{2} Q^- \; ; \; \log(u e^{-\beta - C'}) > \lambda \right\} \right) \leq C \frac{\nu(\delta Q^-)}{\lambda^{p-1}} \,.$$

Therefore, by Lemma 2.13 we have

$$\left(\fint_{\delta Q^-} (u e^{-\beta - C'})^q \, d\nu \right)^{1/q} \leq C \,, \tag{6.5}$$

where $C = C(C_\mu, C_p, \Lambda, p, q, \delta, T)$. Multiplying (6.5) with (6.4) gives the result

$$\left(\fint_{\delta Q^-} u^q \, d\mu dt \right)^{1/q} \leq C \operatorname*{ess\,inf}_{\delta Q^+} u$$

for every $0 < q < (p-1)(2 - p/\kappa)$, where $C = C(C_\mu, C_p, \Lambda, p, q, \delta, T)$. □

We end this paper by completing the proof of Harnack's inequality for parabolic minimizers. This is the first point at which we make use of the fact that a minimizer is both a sub- and superminimizer.

THEOREM 6.6. *Suppose* $1 < p < \infty$ *and assume that the measure* μ *in a geodesic metric space* X *is doubling with doubling constant* C_μ, *and the space supports a weak* $(1, p)$-*Poincaré inequality with constants* C_p *and* Λ. *Then a parabolic Harnack inequality is valid as follows: Let* $u > 0$ *be a parabolic minimizer in* $Q_r \subset \Omega_T$ *which is locally bounded away from zero, and locally bounded. Let* $0 < \delta < 1$. *Then*

$$\operatorname*{ess\,sup}_{\delta Q^-} u \leq C \operatorname*{ess\,inf}_{\delta Q^+} u \,,$$

where $0 < C < \infty$ *and* $C = C(C_\mu, C_p, \Lambda, p, \delta, T)$.

PROOF. By assumption, u is both a parabolic sub- and superminimizer. Hence we may combine Lemma 4.2 with Lemma 6.1 to obtain

$$\operatorname*{ess\,sup}_{\delta Q^-} u \leq \left(\frac{C}{((1+\delta)/2 - \delta)^\theta} \right)^{1/(p-1)} \left(\fint_{(1+\delta)/2Q^-} u^{p-1} \, d\nu \right)^{1/(p-1)}$$

$$\le C \operatorname*{ess\,inf}_{(1+\delta)/2Q^+} u \le C \operatorname*{ess\,inf}_{\delta Q^+} u\,,$$

where $\theta = \theta(C_\mu, p)$ and so $C = C(C_\mu, C_p, \Lambda, p, \delta, T)$. □

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