SUBMANIFOLDS WITH CONSTANT SCALAR CURVATURE IN A UNIT SPHERE

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Abstract. We study the submanifolds in the unit sphere S^{n+p} with constant scalar curvature and parallel normalized mean curvature vector field. In this case, we can generalize the work of the second author about hypersurfaces in [8] to submanifolds in a unit sphere.

1. Introduction and theorems. Let $R^{n+p}(c)$ be an $(n + p)$-dimensional Riemannian manifold with constant sectional curvature c, we also call it a space form. When $c = 1$, $R^{n+p}(c) = S^{n+p}$ is the $(n+p)$-dimensional unit sphere space; when $c = 0$, $R^{n+p}(c) = E^{n+p}$ is the $(n + p)$-dimensional Euclidean space. Let M be an n-dimensional compact submanifold in $R^{n+p}(c)$, and $e_1, \ldots, e_n$ a local orthonormal frame of tangent vector fields on M, $e_{n+1}, \ldots, e_{n+p}$ a local orthonormal frame of normal vector fields on M, $\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+p}$ its dual coframe field, then the second fundamental form and the mean curvature vector of M are

$$(1.1) \qquad A = \sum_{i,j,\alpha} h_{ij}^{\alpha} \omega_i \otimes \omega_j \otimes e_{\alpha}, \quad \mathbf{H} = \sum_{\alpha} H^{\alpha} e_{\alpha} = \frac{1}{n} \sum_{i,\alpha} h_{ii}^{\alpha} e_{\alpha}.$$

When $p = 1$, i.e., M is a hypersurface in $R^{n+1}(c)$, there are many well-known results for hypersurfaces with constant scalar curvature (see [6], [8], [9], [3], [4], [5] ect.).

In [6], Cheng and Yau have proved some well-known results by introducing a self-adjoint differential operator $\square$ on hypersurface defined by

$$(1.2) \qquad \square f = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij},$$

where $f \in C^2(M)$, (f_{ij}) is its Hessian.

In [8], the second author proved the following rigidity theorems by the study of the above operator $\square$ and some estimates:

THEOREM 1.1 ([8]). *Let M ($n \geq 3$) be an n-dimensional compact hypersurface with constant normalized scalar curvature r in the $(n + 1)$-dimensional unit sphere $S^{n+1}(1)$. If*
　(1) $\bar{r} \equiv r - 1 \geq 0$,

(2) *the square norm $|A|^2$ of the second fundamental form of M satisfies*

$$(1.3) \qquad n\bar{r} \leq |A|^2 \leq \frac{n}{(n-2)(n\bar{r}+2)}[n(n-1)\bar{r}^2 + 4(n-1)\bar{r} + n],$$

then either $|A|^2 = n\bar{r}$ and M is totally umbilical, or

$$(1.4) \qquad |A|^2 = \frac{n}{(n-2)(n\bar{r}+2)}[n(n-1)\bar{r}^2 + 4(n-1)\bar{r} + n],$$

and $M = S^1(\sqrt{1-a^2}) \times S^{n-1}(a), a = \sqrt{(n-2)/(n(\bar{r}+1))}$.

THEOREM 1.2 ([8]). *Let M ($n \geq 3$) be an n-dimensional compact hypersurface with constant normalized scalar curvature r in the $(n+1)$-dimensional Euclidean space E^{n+1}. If the square norm $|A|^2$ of the second fundamental form of M satisfies*

$$(1.5) \qquad nr \leq |A|^2 \leq \frac{n(n-1)}{(n-2)}r,$$

then $|A|^2 \equiv nr$ and M is the n-dimensional round sphere $S^n(a)$ for $a = \sqrt{1/r}$.

In this paper, we study submanifolds of the unit sphere with constant scalar curvatures and parallel normalized mean curvature vector field, and we discover that above Theorem 1.1 about hypersurfaces still hold for submanifolds. In fact, we prove

THEOREM 1.3. *Let M ($n \geq 4$) be an n-dimensional compact submanifold with constant normalized scalar curvature r in the $(n+p)$-dimensional unit sphere $S^{n+p}(1)$. If*

(1) $\bar{r} \equiv r - 1 \geq 0$,

(2) $\nabla^\perp e_{n+1} = 0$, *where e_{n+1} is a unit normal vector field, which is parallel to* **H**,

(3) *the square norm $|A|^2$ of the second fundamental form of M satisfies*

$$(1.6) \qquad n\bar{r} \leq |A|^2 \leq \frac{n}{(n-2)(n\bar{r}+2)}[n(n-1)\bar{r}^2 + 4(n-1)\bar{r} + n],$$

then either $|A|^2 = n\bar{r}$ and M is totally umbilical, or

$$(1.7) \qquad |A|^2 = \frac{n}{(n-2)(n\bar{r}+2)}[n(n-1)\bar{r}^2 + 4(n-1)\bar{r} + n],$$

and $M = S^1(\sqrt{1-a^2}) \times S^{n-1}(a) \hookrightarrow S^{n+1} \hookrightarrow S^{n+p}, a = \sqrt{(n-2)/(n(\bar{r}+1))}$.

REMARK 1.1. Hou and Cheng got some partial results of Theorem 1.3 in [7] and [1], and Cheng also proved a general result on complete submanifolds with constant scalar curvature and parallel normalized mean curvature vector field in Euclidean spaces in [2].

2. Preliminaries and lemmas. Let M be an n-dimensional compact submanifold in the $(n+p)$-dimensional space form $R^{n+p}(c)$. We shall make use of the following convention on the range of indices:

$$1 \le B, C, D \le n+p, \quad 1 \le i, j, k \le n, \quad n+1 \le \alpha, \beta, \gamma \le n+p.$$

We choose a local orthonormal frame field $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}\}$ along M, where $\{e_i\}_{i=1,2,\ldots,n}$ are tangent to M and $\{e_\alpha\}_{\alpha=n+1,n+2,\ldots,n+p}$ are normal to M. Let $\{\omega_B\}$ be the corresponding dual coframe, and $\{\omega_{BC}\}$ the connection 1-forms on $R^{n+p}(c)$. With restricting on M, the second fundamental form, the curvature tensor and the normal curvature tensor can be given by

$$(2.1) \qquad \omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \quad A = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

$$(2.2) \qquad d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

$$(2.3) \qquad d\omega_{\alpha\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} = -\frac{1}{2} \sum_{k,l} R_{\alpha\beta kl}^\perp \omega_k \wedge \omega_l,$$

and the mean curvature by $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha$, where $H^\alpha = \frac{1}{n} \sum_i h_{ii}^\alpha$.

The covariant derivatives of the second fundamental form are given by

$$(2.4) \qquad \sum_k h_{ij,k}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{ki}^\alpha \omega_{kj} + \sum_k h_{kj}^\alpha \omega_{ki} + \sum_\beta h_{ij}^\beta \omega_{\beta\alpha},$$

$$(2.5) \qquad \begin{aligned} \sum_l h_{ij,kl}^\alpha \omega_l &= dh_{ij,k}^\alpha + \sum_l h_{lj,k}^\alpha \omega_{li} + \sum_l h_{ij,l}^\alpha \omega_{lk} + \sum_l h_{il,k}^\alpha \omega_{lj} \\ &\quad + \sum_\beta h_{ij,k}^\beta \omega_{\beta\alpha}. \end{aligned}$$

We can define trace-free linear maps $\phi_\alpha \colon T_x M \to T_x M$ by

$$\langle \phi^\alpha X, Y \rangle = \langle A^\alpha X, Y \rangle - H^\alpha \langle X, Y \rangle,$$

where $x \in M$, A^α is the shape operator of e_α, $A^\alpha(e_i) = -\sum_j \langle \overline{\nabla}_{e_i} e_\alpha, e_j \rangle e_j = \sum_j h_{ij}^\alpha e_j$, and ϕ is a bilinear map $\phi \colon T_x M \times T_x M \to T_x^\perp M$ defined by

$$(2.6) \qquad \phi(X, Y) = \sum_{\alpha=n+1}^{n+p} \langle \phi^\alpha X, Y \rangle e_\alpha.$$

It's easy to check that $|\phi|^2 = |A|^2 - nH^2$, where $H^2 = \sum_\alpha (H^\alpha)^2$.

The Gauss equation is

$$(2.7) \qquad R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha).$$

In particular

$$(2.8) \qquad n(n-1)(r-c) = n^2 H^2 - |A|^2 \,,$$

where $R = n(n-1)r$ is the scalar curvature of M. We call r the normalized scalar curvature (see [8], [9], [10]).

The Codazzi equation and the Ricci equation are (see [10])

$$(2.9) \qquad h_{ij,k}^\alpha = h_{ik,j}^\alpha \,,$$

$$(2.10) \qquad R_{\alpha\beta ij}^\perp = \sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta) \,.$$

If the normalized scalar curvature r is a constant with $r \geq c$, then $n^2 H^2 - |A|^2 \geq 0$. We can choose a unit normal vector field e_{n+1} which is parallel to $\mathbf{H}$. Hence we have (see [10])

$$(2.11) \qquad H^{n+1} = H \,, \quad H^\alpha = 0 \ (n+2 \leq \alpha \leq n+p) \,,$$

$$(2.12) \qquad \phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij} \,, \quad \phi_{ij}^\alpha = h_{ij}^\alpha, \ n+2 \leq \alpha \leq n+p \,.$$

We introduce a differential operator $\square$ by defining

$$(2.13) \qquad \square f = \sum_{i,j=1}^n (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij} \,,$$

where $f \in C^2(M)$, (f_{ij}) is its Hessian.

By use of Codazzi equation, we easily know that the operator $\square$ is self-adjoint. That is, we have

$$(2.14) \qquad \int_M \square f \, dv = 0 \,, \quad f \in C^2(M) \,.$$

The following lemma was proved in [8] and [9] for hypersurfaces in a space form. In fact, it's still true for submanifolds in a space form.

LEMMA 2.1.   *Assume that the normalized scalar curvature r is a constant and $r \geq c$, then*

$$(2.15) \qquad |\nabla A|^2 \geq n^2 |\nabla H|^2 \,.$$

PROOF.   From the Gauss equation (2.8), we get $n^4 |\nabla(H^2)|^2 = |\nabla(|A|^2)|^2$ and

$$|\nabla(|A|^2)|^2 = 4\sum_k \left(\sum_{i,j,\alpha} h_{ij}^\alpha h_{ij,k}^\alpha \right)^2 \leq 4\left(\sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \right)\left(\sum_{i,j,k,\alpha} (h_{ij,k}^\alpha)^2 \right) = 4|A|^2 |\nabla A|^2 \,.$$

Hence

$$(2.16) \qquad 4n^2 H^2 (n^2 |\nabla H|^2) = n^4 |\nabla(H^2)|^2 = |\nabla(|A|^2)|^2 \leq 4|A|^2 |\nabla A|^2 \,.$$

If $r \geq c$, then $n^2 H^2 \geq |A|^2$, we get (2.15).   □

We need the following inequalities in the proof of our Theorem 1.3:

LEMMA 2.2 ([12]). *Let $B : R^n \to R^n$ be a symmetric linear map such that $\mathrm{tr}B = 0$, then*

$$(2.17) \qquad -\frac{n-2}{\sqrt{n(n-1)}}|B|^3 \leq \mathrm{tr}B^3 \leq \frac{n-2}{\sqrt{n(n-1)}}|B|^3 \,,$$

where $|B|^2 = \mathrm{tr}B^2$, and the equality holds if and only if at least $(n-1)$ eigenvalues of B are equal.

LEMMA 2.3 ([13]). *Let $C, B : R^n \to R^n$ be symmetric linear maps such that $[C, B] = 0$ and $\mathrm{tr}C = \mathrm{tr}B = 0$, then*

$$(2.18) \qquad -\frac{n-2}{\sqrt{n(n-1)}}|C|^2|B| \leq \mathrm{tr}(C^2 B) \leq \frac{n-2}{\sqrt{n(n-1)}}|C|^2|B| \,.$$

LEMMA 2.4 ([11]). *Let $B^1, B^2, \ldots, B^m$ be symmetric $(n \times n)$-matrices. Set $S_{\alpha\beta} = \mathrm{tr}(B^\alpha B^\beta)$, $S_\alpha = S_{\alpha\alpha}$, $S = \sum_\alpha S_\alpha$, then*

$$(2.19) \qquad \sum_{\alpha,\beta}|B^\alpha B^\beta - B^\beta B^\alpha|^2 + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2}\left(\sum_\alpha S_\alpha\right)^2 \,,$$

where $|B|^2 = \mathrm{tr}B^t B$.

In [10], the second author calculated the Laplacian of $|\phi|^2$ for submanifolds in a unit sphere (also see [14], [15]):

LEMMA 2.5 ([10]). *In the same notations as above, we have*

$$
\frac{1}{2}\triangle(|A|^2) = |\nabla A|^2 + \sum_{i,j,\alpha} nH_{,ij}^\alpha h_{ij}^\alpha + n|\phi|^2 + \sum_{i,j,k,\alpha,\beta} nH^\alpha h_{ij}^\alpha h_{jk}^\beta h_{ki}^\beta
$$

$$(2.20)$$

$$
- \sum_{i,j,k,l}\left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha\right)^2 - \sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^\perp)^2 \,.
$$

3. Proof of Theorem 1.3. Let $\bar{r} = r - 1$. Then, for $n^2 H^2 - |A|^2 = n(n-1)\bar{r}$, we have

$$
n^2\triangle(H^2) = 2n^2 H\triangle H + 2n^2|\nabla H|^2 = \triangle|A|^2 .
$$

If $\nabla^\perp e_{n+1} = 0$, then $H_{,i}^\alpha = H_{,i}\delta_{\alpha n+1}$ and $H_{,ij}^\alpha = H_{,ij}\delta_{\alpha n+1}$. From Lemma 2.5, we get

$$
\square(nH) = (nH\delta_{ij} - h_{ij})(nH)_{,ij} = n^2 H\triangle H - nh_{ij}H_{,ij}
$$

$$
= \frac{1}{2}\triangle(|A|^2) - n^2|\nabla H|^2 - nh_{ij}H_{,ij}
$$

$$(3.1) \qquad = (|\nabla A|^2 - n^2|\nabla H|^2) + n|\phi|^2 + \sum_{i,j,k,\alpha,\beta} nH^\alpha h_{ij}^\alpha h_{jk}^\beta h_{ki}^\beta
$$

$$
- \sum_{i,j,k,l}\left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha\right)^2 - \sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^\perp)^2 \,.
$$

By use of Lemmas 2.2, 2.3 and (2.12), we have

$$\sum_{i,j,k,\alpha,\beta} H^\alpha h_{ij}^\alpha h_{jk}^\beta h_{ki}^\beta = \sum_{i,j,k,\beta} H h_{ij}^{n+1} h_{jk}^\beta h_{ki}^\beta$$

$$= \sum_{i,j,k} H h_{ij}^{n+1} h_{jk}^{n+1} h_{ki}^{n+1} + \sum_{\beta=n+2}^{n+p} \sum_{i,j,k} H h_{ij}^{n+1} \phi_{jk}^\beta \phi_{ki}^\beta$$

$$= H \cdot \text{tr}(\phi^{n+1} + HI)^3 + \sum_{\beta=n+2}^{n+p} H \phi_{ij}^{n+1} \phi_{jk}^\beta \phi_{ki}^\beta + \sum_{\beta=n+2}^{n+p} H^2 |\phi^\beta|^2$$

$$= H \cdot \text{tr}(\phi^{n+1})^3 + 3H^2 |\phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p} H^2 |\phi^\beta|^2$$

(3.2)

$$+ \sum_{\beta=n+2}^{n+p} \sum_{i,j,k} H \phi_{ij}^{n+1} \phi_{jk}^\beta \phi_{ki}^\beta$$

$$\geq -\frac{n-2}{\sqrt{n(n-1)}} |H| \cdot |\phi^{n+1}|^3 + 2H^2 |\phi^{n+1}|^2 + H^2 |\phi|^2 + nH^4$$

$$- \frac{n-2}{\sqrt{n(n-1)}} \sum_{\beta=n+2}^{n+p} |H| \cdot |\phi^{n+1}| |\phi^\beta|^2$$

$$= 2H^2 |\phi^{n+1}|^2 + H^2 |\phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}} |H| \cdot |\phi^{n+1}| |\phi|^2 .$$

Using the Lemma 2.4, we get

$$\sum_{i,j,k,l} \left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha \right)^2 + \sum_{\alpha,\beta,i,j} (R_{\alpha\beta ij}^\perp)^2$$

$$= \sum_{\alpha,\beta} [\text{tr}(A^\alpha A^\beta)]^2 + \sum_{\alpha \neq n+1, \beta \neq n+1, i, j} (R_{\alpha\beta ij}^\perp)^2$$

$$\leq [\text{tr}(A^{n+1} A^{n+1})]^2 + 2 \sum_{\beta \neq n+1} (\text{tr } A^{n+1} A^\beta)^2 + \frac{3}{2} \left[\sum_{\beta \neq n+1} |\phi^\beta|^2 \right]^2$$

(3.3)

$$= |\phi^{n+1}|^4 + 2nH^2 |\phi^{n+1}|^2 + n^2 H^4 + 2 \sum_{\beta \neq n+1} (\text{tr } \phi^{n+1} \phi^\beta)^2$$

$$+ \frac{3}{2} (|\phi|^2 - |\phi^{n+1}|^2)^2$$

$$\leq \frac{5}{2} |\phi^{n+1}|^4 + 2nH^2 |\phi^{n+1}|^2 + n^2 H^4 + 2|\phi^{n+1}|^2 (|\phi|^2 - |\phi^{n+1}|^2)$$

$$+ \frac{3}{2} |\phi|^4 - 3|\phi|^2 |\phi^{n+1}|^2$$

$$= \frac{1}{2} |\phi^{n+1}|^4 + 2nH^2 |\phi^{n+1}|^2 + n^2 H^4 - |\phi|^2 |\phi^{n+1}|^2 + \frac{3}{2} |\phi|^4 .$$

From Lemma 2.1, $|\nabla A|^2 \geq n^2 |\nabla H|^2$, and we have, by putting (3.2) and (3.3) into (3.1),

$$\square(nH)$$

$$\geq n|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \cdot |\phi^{n+1}||\phi|^2 + nH^2|\phi|^2$$

$$- \frac{1}{2}|\phi^{n+1}|^4 + |\phi|^2|\phi^{n+1}|^2 - \frac{3}{2}|\phi|^4$$

(3.4)

$$= |\phi|^2 \left[n - \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \cdot |\phi| + nH^2 - |\phi|^2 \right]$$

$$+ (|\phi| - |\phi^{n+1}|) \left[\frac{n(n-2)}{\sqrt{n(n-1)}}|H| \cdot |\phi|^2 - \frac{1}{2}(|\phi| - |\phi^{n+1}|)(|\phi| + |\phi^{n+1}|)^2 \right].$$

We will show the following key claim:

(3.5)
$$(|\phi| - |\phi^{n+1}|)(|\phi| + |\phi^{n+1}|)^2 \leq \frac{32}{27}|\phi|^3.$$

(1) If $|\phi| = 0$, then $|\phi^{n+1}| = 0$, and (3.5) holds.

(2) If $|\phi| \neq 0$, define $x = |\phi^{n+1}|/|\phi|$, so $x \in [0, 1]$. We have by a direct calculation

(3.6)
$$\max_{x\in[0,1]} (1 - x)(1 + x)^2 = \frac{32}{27},$$

so (3.5) still holds, and we complete the proof of the claim.

As $n^2H^2 \geq |A|^2 = |\phi|^2 + nH^2$, $n(n-1)H^2 \geq |\phi|^2$, then

(3.7)
$$\frac{n(n-2)}{\sqrt{n(n-1)}}|H| \cdot |\phi|^2 - \frac{1}{2}(|\phi| - |\phi^{n+1}|)(|\phi| + |\phi^{n+1}|)^2 \geq \left(\frac{n-2}{n-1} - \frac{16}{27} \right)|\phi|^3.$$

If $n \geq 4$, $(n-2)/(n-1) - 16/27 > 0$, so we have

$$\square(nH) \geq |\phi|^2 \left[n - \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \cdot |\phi| + nH^2 - |\phi|^2 \right]$$

(3.8)

$$= |\phi|^2 \left[|\phi| + \frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}|H| + \sqrt{n + \frac{n^3H^2}{4(n-1)}} \right]$$

$$\cdot \left[-|\phi| - \frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}|H| + \sqrt{n + \frac{n^3H^2}{4(n-1)}} \right].$$

It is checked directly that our assumption (1.6), i.e.,

(3.9)
$$|A|^2 \leq \frac{n}{(n-2)(n\bar{r}+2)}[n(n-1)\bar{r}^2 + 4(n-1)\bar{r} + n],$$

is equivalent to

$$(3.10) \quad \frac{(n-2)^2}{4(n-1)^2}[n(n-1)\bar{r} + |A|^2] \cdot [|A|^2 + n(n-1)\bar{r} + 4(n-1)]$$
$$\leq \left[n + \frac{n^2}{2}\bar{r} - \frac{n-2}{2(n-1)}|A|^2\right]^2.$$

Noting (3.9) implies $n + (n^2/2)\bar{r} - (n-2)|A|^2/(2(n-1)) > 0$, we know (3.10) is equivalent to

$$(3.11) \quad \frac{n-2}{2(n-1)}\sqrt{[n(n-1)\bar{r} + |A|^2] \cdot [|A|^2 + n(n-1)\bar{r} + 4(n-1)]}$$
$$\leq n + \frac{n^2}{2}\bar{r} - \frac{n-2}{2(n-1)}|A|^2.$$

We note that (3.11) is equivalent to

$$(3.12) \quad |\phi| \leq -\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}|H| + \sqrt{n + \frac{n^3 H^2}{4(n-1)}}.$$

Therefore, the right-hand side of (3.8) is non-negative. From (2.14) we also have $\int_M \square(nH)dv = 0$.

Thus either $|A|^2 = n\bar{r}$, that is $|\phi| = 0$ or

$$(3.13) \quad |A|^2 = \frac{n}{(n-2)(n\bar{r}+2)}[n(n-1)\bar{r}^2 + 4(n-1)\bar{r} + n],$$

and if (3.13) holds, we get $|\phi| = |\phi^{n+1}|$.

Let N_1 be the sub-bundle spanned by $\{e_{n+2}, \ldots, e_{n+p}\}$. Then, with the assumption $\nabla^\perp e_{n+1} = 0$, it's easy to check that N_1 is parallel in the normal bundle. If $|\phi| = 0$ or $|\phi| = |\phi^{n+1}|$, we can always get that $|\phi^\alpha| = 0$ for each $n + 2 \leq \alpha \leq n + p$, that means M is totally geodesic with respect to N_1. So from [15, Theorem 1] we know that M lies in S^{n+1}, which is an $(n + 1)$-dimensional totally geodesic submanifold in S^{n+p}. Using Theorem 1.1, we complete the proof of Theorem 1.3.

REMARK 3.1.   If $n = 3$, we have from the first inequality of (3.4)

$$(3.14) \quad \begin{aligned} \square(nH) &\geq |\phi|^2\left[3 + 3H^2 - \frac{3}{2}|\phi|^2 - \frac{3}{\sqrt{6}}|H| \cdot |\phi^{n+1}| + \frac{1}{2}|\phi^{n+1}|^2\right] \\ &= |\phi|^2\left[3 + \frac{9}{4}H^2 - \frac{3}{2}|\phi|^2 + \frac{1}{2}(|\phi^{n+1}| - \sqrt{\frac{3}{2}}|H|)^2\right] \\ &\geq |\phi|^2\left[3 + \frac{9}{4}H^2 - \frac{3}{2}|\phi|^2\right]. \end{aligned}$$

Hence if $|\phi|^2 \leq 2 + \frac{3}{2}H^2$, which is equivalent to

$$(3.15) \quad |A|^2 \leq 4 + 6\bar{r},$$

we get that M is totally umbilical. We also note that (1.6) is equivalent to

$$(3.16) \qquad n\bar{r} \le |A|^2 \le \frac{n(n-1)}{n-2}\bar{r} + \frac{2(n-1)}{n-2} + \frac{n-2}{n\bar{r}+2}.$$

Thus, when $n = 3$, (3.16) is

$$(3.17) \qquad |A|^2 \le 4 + 6\bar{r} + \frac{1}{3\bar{r}+2}.$$

Thus (3.15) is stronger than (3.17).

REMARK 3.2. When $n = 3$, our technique here is not effective to prove Theorem 1.3, so it is an interesting problem to know that Theorem 1.3 holds or not for $n = 3$.

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