HESSIAN MANIFOLDS
OF NONPOSITIVE CONSTANT HESSIAN SECTIONAL
CURVATURE

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Abstract. We classify the maximal Hessian manifolds of constant Hessaian sectional curvature nonpositive.

1. Introduction. Hessian geometry is a close relative of Kählerian geometry, and connected with many interesting fields such as affine differential geometry and information geometry. It was founded and deeply studied by J. L. Koszul, H. Shima and others. A systematic introduction to Hessian geometry is given in the book [5]. In [4], Shima defined Hessian sectional curvatures and studied spaces of constant Hessian sectional curvature. In this paper, we give a characterization of certain important examples of such spaces as the standard models (Theorem 4.1 in the case of constant Hessian sectional curvature zero, and Theorem 5.2 in the case of constant Hessian sectional curvature negative).

Throughout this paper, let M be an n-dimensional connected manifold, ∇ an affine connection of torsion-free, and g a Riemannian metric on M. All the objects are assumed to be smooth. We denote by $\Gamma(E)$ the set of sections of a vector bundle $E \to M$. For example, $\Gamma(TM^{(p,q)})$ means the set of tensor fields of type (p, q) on M. The torsion tensor field and the curvature tensor tensor field of ∇ are denoted by $T^{\nabla} \in \Gamma(TM^{(1,2)})$ and $R^{\nabla} \in \Gamma(TM^{(1,3)})$, respectively.

DEFINITION 1.1. A triplet (M, ∇, g) is called a *Hessian manifold* if (1) $\nabla g \in \Gamma(TM^{(0,3)})$ is symmetric, and (2) $R^{\nabla} = 0$.

The name comes from the following facts. For an arbitrary point p of a Hessian manifold (M, ∇, g), there exist a neighborhood U of p and a function $\varphi \in C^{\infty}(U)$ such that $g|_U = \nabla d\varphi$, that is, the Riemannian metric is locally expressed as the Hessian matrix of a function with respect to an affine coordinate system. Besides, we should recall that a triplet (M, ∇, g) is called a *statistical manifold* if (1) in Definition 1.1 holds. This notion comes from information geometry (See [1] and [3]).

DEFINITION 1.2. For a Hessian manifold (M, ∇, g), we denote the Levi-Civita connection of g by ∇^g or $\widehat{\nabla}$. We use the latter when the metric is clearly specified. We then define

the *difference tensor field* $K := K^{(\nabla, g)} \in \Gamma(TM^{(1,2)})$ as

$$K(X, Y) := \nabla_X Y - \widehat{\nabla}_X Y$$

for all $X, Y \in \Gamma(TM)$.

DEFINITION 1.3.   Let (M, ∇, g) be a Hessian manifold and $c \in \mathbf{R}$. (∇, g) is said to be *of constant Hessian sectional curvature c* if

$$(\nabla_X K)(Y, Z) := (\nabla K)(Y, Z; X) = -\frac{c}{2}\{g(X, Y)Z + g(X, Z)Y\}$$

for any $X, Y, Z \in \Gamma(TM)$. We abbreviate it as "of CHC c" in this paper.

We can construct a Kählerian metric g^T on the tangent bundle of a Hessian manifold M by the diagonal lifting (or the Sasaki lifting) and remark that M is of CHC c if and only if (TM, g^T) is of constant holomorphic sectional curvature $-c$.

PROPOSITION 1.4 ([4]).   *Let (M, ∇, g) be a Hessian manifold of CHC c. Then (M, g) is a Riemannian manifold of constant curvature $-c/4$. In fact, the following holds*:

$$R^{\widehat{\nabla}}(X, Y)Z = -\frac{1}{2}\{(\nabla K)(Y, Z; X) - (\nabla K)(Z, X; Y)\}$$

$$= -\frac{c}{4}\{g(Y, Z)X - g(X, Z)Y\}.$$

We denote the Euclidean space, the sphere of curvature $\kappa > 0$ (of radius $\kappa^{-1/2}$ with center $0 \in \mathbf{R}^{n+1}$), the hyperbolic space of curvature $\kappa < 0$ by $(\mathbf{R}^n, g_E)$, $(S^n(\kappa), g_S)$, $(H^n(\kappa), g_H)$, respectively, and by $(\widetilde{M}^n(\kappa), \widetilde{g})$ in general.

COROLLARY 1.5.   *Let (M, ∇, g) be an n-dimensional simply-connected Hessian manifold of CHC -4κ. Then there exists an isometric immersion ι of M to $\widetilde{M}^n(\kappa)$. In particular, g is not complete if $\iota(M) \neq \widetilde{M}^n(\kappa)$.*

2.   Examples of Hessian manifolds of CHC.

EXAMPLE 2.1.   The Euclidean space $(\mathbf{R}^n, \nabla^{g_E}, g_E)$ with the standard connection is a Hessian manifold of CHC 0.

EXAMPLE 2.2.   The triplet $((\mathbf{R}^+)^n, D, g)$ defined by

$$(\mathbf{R}^+)^n := \{y = {}^t(y^1, \dots, y^n) \in \mathbf{R}^n \; ; \; y^1 > 0, \dots, y^n > 0\},$$

$$g := g_E|_{(\mathbf{R}^+)^n} = \sum_{j=1}^n (dy^j)^2,$$

$$D : \qquad D_{\partial/\partial y^i}\frac{\partial}{\partial y^j} = -\delta_{ij}(y^j)^{-1}\frac{\partial}{\partial y^j},$$

is a Hessian manifold of CHC 0.

EXAMPLE 2.3. For $k \in \{0, 1, \ldots, n\}$, the triplet (N_k, ∇_k, g_k) defined by

$$N_k := \{y = {}^t(y^1, \ldots, y^n) \in \mathbf{R}^n \; ; \; y^{n-k+1} > 0, \ldots, y^n > 0\},$$

$$g_k := g_E|_{N_k} = \sum_{j=1}^n (dy^j)^2,$$

$$\nabla_k := \nabla^{g_k} + K_k, \quad \text{where } K_k := \sum_{j=n-k+1}^n (-y^j)^{-1} dy^j \otimes dy^j \otimes \frac{\partial}{\partial y^j},$$

is also a Hessian manifold of CHC 0.

We remark that (N_k, ∇_k, g_k) coincides with $(\mathbf{R}^n, \nabla^{g_E}, g_E)$ when $k = 0$, and $((\mathbf{R}^+)^n, D, g)$ when $k = n$.

Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a Hessian manifold, and $f : M \to \widetilde{M}$ an immersion. We define g and ∇ on M by

$$g = f^*\widetilde{g}, \quad g(\nabla_X Y, Z) = \widetilde{g}(\widetilde{\nabla}_X f_* Y, f_* Z)$$

for any $X, Y, Z \in \Gamma(TM)$, where the connection induced from $\widetilde{\nabla}$ by f on the induced bundle $f^* T\widetilde{M} \to M$ is denoted by the same symbol $\widetilde{\nabla}$. Then the pair (∇, g) is a statistical structure on M, which is called the one *induced by f from* $(\widetilde{\nabla}, \widetilde{g})$ (See [2]).

EXAMPLE 2.4. Let $f : S_+^n(\kappa) := S^n(\kappa) \cap (\mathbf{R}^+)^{n+1} \to (\mathbf{R}^+)^{n+1}$ be the inclusion, and (∇^{S_+}, g_{S_+}) the statistical structure induced by f from (D, g) in Example 2.2. Then the triplet $(S_+^n(\kappa), \nabla^{S_+}, g_{S_+})$ is a Hessian manifold of CHC $-4\kappa \; (< 0)$.

We set

$$\widetilde{f} : \mathbf{R}^n \ni x = {}^t(x^1, \ldots, x^n) \mapsto \kappa^{-1/2} \Lambda^{-1/2}(x){}^t(e^{x^1/2}, \ldots, e^{x^n/2}, 1) \in (\mathbf{R}^+)^{n+1},$$

where $\Lambda(x) := 1 + \sum_{j=1}^n e^{x^j}$. Then it is easy to show that $\widetilde{f}(\mathbf{R}^n) = S_+^n(\kappa)$, and $(x^1, \ldots, x^n)$ are affine coordinates of $S_+^n(\kappa)$ with respect to ∇^{S_+}, that is, $\nabla^{S_+}(\partial/\partial x^j) = 0$. The metric is given in these coordinates by

$$\widetilde{f}^* g_E = g_{S_+} = (4\kappa)^{-1} \Lambda^{-1}(x) \sum \{\delta_{ij} e^{x^i} - \Lambda^{-1}(x) e^{x^i} e^{x^j}\} dx^i \otimes dx^j.$$

EXAMPLE 2.5. For $x \in \{1, 2, \ldots, n+1\}$ and $\Delta := \{\eta = {}^t(\eta^1, \ldots, \eta^n) \in \mathbf{R}^n \; ; \; \eta^j > 0, \sum_{j=1}^n \eta^j < 1\}$, the probability density function of the $(n+1)$-nomial distribution is given by

$$p(x; \eta) := \begin{cases} \eta^i & x = i \in \{1, 2, \ldots, n\}, \\ 1 - \sum_{j=1}^n \eta^j & x = n+1 \end{cases}$$

$$= \exp\left(\sum_{i=1}^n \log \frac{\eta^i}{1 - \sum_{j=1}^n \eta^j} \delta_{ix} - \log \frac{1}{1 - \sum_{j=1}^n \eta^j} \right).$$

We denote the Fisher information metric on Δ by g_F, and the exponential connection on Δ by $\nabla^{(e)}(= \nabla^{(1)})$ (See [1]). Then $(\Delta, \nabla^{(e)}, g_F)$ is a Hessian manifold of CHC -1.

It is known by [4] that there exists a diffeomorphism $\Phi : \Delta \to S^n_+(1/4)$ such that

$$\Phi^* \nabla^{S_+} = \nabla^{(e)} \quad \text{and} \quad \Phi^* g_{S_+} = g_F \,.$$

3. Preliminaries.

PROPOSITION 3.1. *Let* (M, g) *be a Riemannian manifold of constant curvature* κ. *For* $K \in \Gamma(TM^{(1,2)})$, *we set* $\nabla := \widehat{\nabla} + K$. *The triplet* (M, ∇, g) *is a Hessian manifold of CHC* -4κ *if and only if* K *satisfies the following four conditions*:

$$(3.1) \qquad\qquad\qquad K(X, Y) = K(Y, X) \,,$$

$$(3.2) \qquad\qquad\qquad g(K(X, Y), Z) = g(Y, K(X, Z)) \,,$$

$$(3.3) \qquad \kappa\{g(Y, Z)X - g(X, Z)Y\} = K(Y, K(X, Z)) - K(X, K(Y, Z)) \,,$$

$$(3.4) \qquad \begin{aligned} (\widehat{\nabla}K)(X, Y; Z) &= 2\kappa\{g(Z, X)Y + g(Z, Y)X\} \\ &\quad - K(Z, K(X, Y)) + K(K(Z, X), Y) + K(X, K(Z, Y)) \,. \end{aligned}$$

PROOF. We use the following formulas (3.5) through (3.9). Equation (3.6) holds under (3.1). Equation (3.9) holds under (3.1) and (3.4).

$$(3.5) \qquad\qquad\qquad K(X, Y) - K(Y, X) = T^{\nabla}(X, Y) \,,$$

$$(3.6) \qquad \begin{aligned} &g(X, K(Y, Z)) - g(K(Y, X), Z) \\ &\quad = (\nabla g)(Y, Z; X) - (\nabla g)(X, Z; Y) \,, \end{aligned}$$

$$(3.7) \qquad \begin{aligned} &(\widehat{\nabla}K)(X, Y; Z) \\ &\quad + K(Z, K(X, Y)) - K(K(Z, X), Y) - K(X, K(Z, Y)) \\ &= (\nabla K)(X, Y; Z) \,, \end{aligned}$$

$$(3.8) \qquad \begin{aligned} &R^{\nabla}(X, Y)Z \\ &\quad = R^{\widehat{\nabla}}(X, Y)Z + (\widehat{\nabla}K)(Y, Z; X) - (\widehat{\nabla}K)(X, Z; Y) \\ &\qquad - K(Y, K(X, Z)) + K(X, K(Y, Z)) \,, \end{aligned}$$

$$(3.9) \qquad \begin{aligned} &= -\kappa\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + K(Y, K(X, Z)) - K(X, K(Y, Z)) \,. \end{aligned}$$

Assuming (3.1) through (3.4), we derive that (M, ∇, g) is a Hessian manifold of CHC -4κ. By (3.5), (3.1) implies that ∇ is of torsion free. By (3.6), (3.2) implies that ∇g is symmetric. By (3.7), (3.4) implies that (∇, g) is of CHC -4κ. By (3.9), (3.3) implies that ∇ is flat. The converse is proved in the similar way.

The formulas above are derived by direct calculations. Here we give the proof of only (3.8) and (3.9) as follows. Since $\nabla_Y Z = \widehat{\nabla}_Y Z + K(Y, Z)$, we have

$$\nabla_X \nabla_Y Z = \widehat{\nabla}_X \widehat{\nabla}_Y Z + \widehat{\nabla}_X K(Y, Z) + K(X, \widehat{\nabla}_Y Z) + K(X, K(Y, Z)) \,,$$

and

$$-\nabla_{[X,Y]}Z = -\widehat{\nabla}_{[X,Y]}Z - K(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X, Z).$$

Then

$$R^\nabla(X,Y)Z$$
$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$
$$= R^{\widehat{\nabla}}(X,Y)Z$$
$$+ \widehat{\nabla}_X K(Y,Z) + K(X,\widehat{\nabla}_Y Z) + K(X, K(Y,Z))$$
$$- \widehat{\nabla}_Y K(X,Z) - K(Y,\widehat{\nabla}_X Z) - K(Y, K(X,Z))$$
$$- K(\widehat{\nabla}_X Y, Z) + K(\widehat{\nabla}_Y X, Z),$$

which is the right-hand side of (3.8). Since g is of constant curvature κ, we have, using (3.4) and (3.1),

$$\text{RHS of (3.8)}$$
$$= \kappa\{g(Y,Z)X - g(X,Z)Y\}$$
$$+ [2\kappa\{g(X,Y)Z + g(X,Z)Y\}$$
$$- K(X, K(Y,Z)) + K(K(X,Y), Z) + K(Y, K(X,Z))]$$
$$- [2\kappa\{g(Y,X)Z + g(Y,Z)X\}$$
$$- K(Y, K(X,Z)) + K(K(Y,X), Z) + K(X, K(Y,Z))]$$
$$- K(Y, K(X,Z)) + K(X, K(Y,Z)),$$

which is the right-hand side of (3.9). □

PROPOSITION 3.2. *Let (M, g) be a Riemannian manifold of constant curvature κ. Suppose that $K \in \Gamma(TM^{(1,2)})$ satisfies the differential equation (3.4) on M. If the algebraic conditions (3.1) through (3.3) hold at some point of M, then they hold on M, that is, the triplet $(M, \nabla := \widehat{\nabla} + K, g)$ is a Hessian manifold of CHC -4κ.*

PROOF. As the first step, we define $\alpha \in \Gamma(TM^{(1,2)})$ by

$$\alpha(X,Y) := K(X,Y) - K(Y,X),$$

and get, by using (3.4),

$$(3.10) \quad \begin{aligned} &(\widehat{\nabla}\alpha)(X,Y;Z) \\ &= -K(Z, \alpha(X,Y)) + \alpha(X, K(Z,Y)) - \alpha(Y, K(Z,X)) \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. If α vanishes at some point, $\alpha = 0$ is the unique solution of the differential equation (3.10) with the condition. It implies that (3.1) holds on M. The formula (3.10) is obtained by a direct calculation.

As the second step, we define $\beta \in \Gamma(TM^{(0,3)})$ and $\gamma \in \Gamma(TM^{(1,3)})$ by

$$\beta(X,Y,Z) := g(K(X,Y), Z) - g(Y, K(X,Z)),$$
$$\gamma(X,Y,Z) := \kappa g(Y,Z)X + K(X, K(Y,Z))$$

$$-\{\kappa\,g(X, Z)Y + K(Y, K(X, Z))\}$$

and show that β and γ vanish identically in the same way. We can calculate the following differential equations by the conditions (3.1) and (3.4):

(3.11)
$$
\begin{aligned}
(\widehat{\nabla}\beta)&(X, Y, Z; V) \\
&= 2\beta(Y, K(Z, V), X) - 2\beta(Z, K(Y, V), X) \\
&\quad - \beta(V, K(X, Y), Z) + \beta(X, K(V, Y), Z) \\
&\quad + \beta(V, K(X, Z), Y) - \beta(X, K(V, Z), Y) \\
&\quad + \beta(Y, K(V, X), Z) - \beta(Z, K(V, X), Y) \\
&\quad - 2g(\gamma(Y, Z, V), X)\,,
\end{aligned}
$$

(3.12)
$$
\begin{aligned}
(\widehat{\nabla}\gamma)&(X, Y, Z; V) \\
&= K(V, -\gamma(X, Y, Z)) + \gamma(X, Y, K(Z, V)) \\
&\quad + \gamma(X, K(Y, V), Z) + \gamma(K(X, V), Y, Z) \\
&\quad + \gamma(\widehat{\nabla}_V X, Y, Z) + \kappa\{\beta(Z, Y, V) - \beta(Y, V, Z)\}X \\
&\quad - \kappa\{\beta(Z, X, V) - \beta(X, V, Z)\}Y\,.
\end{aligned}
$$

The same argument to the first step works.

   Here we give the proof of (3.11). The formula (3.12) is obtained in the similar calculation. By definition, we have

(3.13)
$$
\begin{aligned}
- g(\gamma&(Y, Z, V), X) \\
&= \kappa\{g(Y, V)g(Z, X) - g(Z, V)g(Y, X)\} \\
&\quad + g(K(Z, K(Y, V)), X) - g(K(Y, K(Z, V)), X) \\
&= \kappa\{g(Y, V)g(Z, X) - g(Z, V)g(Y, X)\} \\
&\quad + g(K(Y, V), K(Z, X)) - g(K(Z, V), K(Y, X)) \\
&\quad + \beta(Z, K(Y, V), X) - \beta(Y, K(Z, V), X)\,.
\end{aligned}
$$

We calculate by using (3.4) and (3.1),

$$
\begin{aligned}
(\widehat{\nabla}\beta)&(X, Y, Z; V) \\
&= g((\widehat{\nabla}K)(X, Y; V), Z) - g(Y, (\widehat{\nabla}K)(X, Z; V)) \\
&= 2\kappa\{g(V, Y)g(X, Z) - g(V, Z)g(X, Y)\} \\
&\quad - g(K(V, K(X, Y)), Z) + g(K(X, K(V, Y)), Z) \\
&\quad + g(K(V, K(X, Z)), Y) - g(K(X, K(V, Z)), Y) \\
&\quad + g(K(Y, K(V, X)), Z) - g(K(Z, K(V, X)), Y)\,,
\end{aligned}
$$

and from (3.13) that

$$
\begin{aligned}
&= 2\{-g(K(Y, V), K(Z, X)) + g(K(Z, V), K(Y, X)) \\
&\quad - \beta(Z, K(Y, V), X) + \beta(Y, K(Z, V), X) \\
&\quad - g(\gamma(Y, Z, V), X)\} \\
&\quad - g(K(X, Y), K(V, Z)) - \beta(V, K(X, Y), Z) \\
&\quad + g(K(V, Y), K(X, Z)) + \beta(X, K(V, Y), Z) \\
&\quad + g(K(X, Z), K(V, Y)) + \beta(V, K(X, Z), Y) \\
&\quad - g(K(V, Z), K(X, Y)) - \beta(X, K(V, Z), Y) \\
&\quad + g(K(V, X), K(Y, Z)) + \beta(Y, K(V, X), Z) \\
&\quad - g(K(V, X), K(Z, Y)) - \beta(Z, K(V, X), Y) \\
&= \text{RHS of } (3.11).
\end{aligned}
$$

$\square$

LEMMA 3.3. *The differential equation* (3.4) *for K satisfies the integrability condition provided* (3.1) *through* (3.3).

PROOF. We have only to derive the following Ricci's identity from Equation (3.4) :

$$
\begin{aligned}
&(\widehat{\nabla}^2 K)(X, Y; V, W) - (\widehat{\nabla}^2 K)(X, Y; W, V) \\
(3.14) \quad &= R^{\widehat{\nabla}}(W, V)K(X, Y) \\
&\quad - K(R^{\widehat{\nabla}}(W, V)X, Y) - K(X, R^{\widehat{\nabla}}(W, V)Y).
\end{aligned}
$$

The left hand side of (3.14) is calculated as follows:

$$
\begin{aligned}
&(\widehat{\nabla}^2 K)(X, Y; V, W) - (\widehat{\nabla}^2 K)(X, Y; W, V) \\
&= \kappa\{g(W, Y)K(V, X) + g(K(V, X), Y)W \\
(3.15) \quad &\quad + g(W, X)K(V, Y)\} \\
&\quad - \kappa\{g(V, Y)K(W, X) + g(K(W, X), Y)V \\
&\quad + g(V, X)K(W, Y)\}.
\end{aligned}
$$

In fact, by (3.3) we can get

$$
\begin{aligned}
&(\widehat{\nabla}^2 K)(X, Y; V, W) \\
&= (\widehat{\nabla}K)(K(V, X), Y; W) + K((\widehat{\nabla}K)(V, X; W), Y) \\
&= \kappa\{g(K(V, X), Y)W + g(W, Y)K(V, X) \\
&\quad + 2g(K(V, X), W)Y\} + K(K(W, K(V, X)), Y) \\
(3.16) \quad &\quad + K(\kappa\{g(V, X)W + g(W, X)V + 2g(W, V)X\} \\
&\quad + K(K(W, V), X), Y) \\
&= \kappa\{g(W, Y)K(V, X) + g(K(V, X), Y)W\} \\
&\quad + K(K(W, K(V, X)), Y) + S(X, Y, V, W),
\end{aligned}
$$

where $S \in \Gamma(TM^{(1,4)})$ is a term symmetric with respect to V and W, which is given by

$$
\begin{aligned}
S(X&, Y, V, W) \\
&= \kappa\{2g(K(V, X), W)Y + 2g(V, W)K(X, Y) \\
&\quad + g(W, X)K(V, Y) + g(V, X)K(W, Y)\} \\
&\quad + K(K(K(W, V), X), Y).
\end{aligned}
$$

It is easy to calculate (3.15) from (3.16) and (3.3). On the other hand, the right side of (3.14) is calculated as follows:

$$
\begin{aligned}
R^{\widehat{\nabla}}&(W, V)K(X, Y) - K(R^{\widehat{\nabla}}(W, V)X, Y) - K(X, R^{\widehat{\nabla}}(W, V)Y) \\
&= \kappa\{g(V, K(X, Y))W - g(W, K(X, Y))V\} \\
&\quad - \kappa\{g(V, X)K(W, Y) - g(W, X)K(V, Y)\} \\
&\quad - \kappa\{g(V, Y)K(X, W) - h(W, Y)K(X, V)\},
\end{aligned}
$$

which coincides with (3.15), and thus, we obtain the equation (3.14). □

The uniqueness derived from this integrability condition implies

COROLLARY 3.4. *Let (M, ∇, g), $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be n-dimensional simply-connected Hessian manifolds of CHC -4κ with difference tensor fields K, $\widetilde{K}$, respectively. Let $\iota : M \to \widetilde{M}$ be an isometric immersion. If $\iota^*\widetilde{K} = K$ at some point, then $\iota^*\widetilde{\nabla} = \nabla$ and $\iota^*\widetilde{g} = g$ on M.*

4. Hessian manifolds of CHC zero. In this section, we show that the n-dimensional Hessian manifolds (N_k, ∇_k, g_k) in Example 2.3 are the only maximal ones of CHC zero. Namely, we prove the following theorem.

THEOREM 4.1. *Let (M, ∇, g) be an n-dimensional simply-connected Hessian manifold of CHC 0. Then there exist $k \in \{0, 1, \ldots, n\}$ and an immersion $\iota : M \to N_k$ such that $\iota^*\nabla_k = \nabla$ and $\iota^*g_k = g$.*

PROOF. Take a point $p \in M$ and fix it. We claim that there exist orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM with respect to g, and real numbers λ_j such that $0 \geq \lambda_1 \geq \cdots \geq \lambda_n$ and $K(e_i, e_j) = \lambda_i \delta_{ij} e_j$.

By (3.2), a linear transformation $K(X, \cdot) : T_pM \to T_pM$ is symmetric with respect to g for any $X \in T_pM$. Since linear transformations $K(X, \cdot)$ and $K(Y, \cdot)$ are mutually commutable for any $X, Y \in T_pM$ by (3.3), they are simultaneously diagonalizable. Accordingly, there exist orthonormal basis $\{\widetilde{e}_1, \ldots, \widetilde{e}_n\}$ of T_pM and $\widetilde{\lambda}_{ij} \in \mathbf{R}$ such that $K(\widetilde{e}_i, \widetilde{e}_j) = \widetilde{\lambda}_{ij}\widetilde{e}_j$. By (3.1), $\widetilde{\lambda}_{ij}$ is written as $\widetilde{\lambda}_{ij} = \widetilde{\lambda}_i \delta_{ij}$ for some $\widetilde{\lambda}_i \in \mathbf{R}$. In the case that $\widetilde{\lambda}_i > 0$, we set $e_i := -\widetilde{e}_i$ and $\lambda_i := -\widetilde{\lambda}_i(< 0)$. Otherwise, we set $e_i := \widetilde{e}_i$ and $\lambda_i := \widetilde{\lambda}_i(\leq 0)$. Renumbering them if necessary, we may assume that

$$0 \geq \lambda_1 \geq \cdots \geq \lambda_n \text{ and } K(e_i, e_j) = \lambda_i \delta_{ij} e_j.$$

Let $k \in \{0, 1, \ldots, n\}$ be the minimal number such that $0 = \lambda_1 = \cdots = \lambda_{n-k}$. From Corollary 1.5, we have an isometric immersion $\iota : (M, g) \to (\boldsymbol{R}^n, g_E)$, and may assume that

$$\iota(p) = \begin{bmatrix} 0_{n-k} \\ (-\lambda_{n-k+1})^{-1} \\ \vdots \\ (-\lambda_n)^{-1} \end{bmatrix} \in N_k \subset \boldsymbol{R}^n$$

$$\text{and } (d\iota)_p(e_i) = \left(\frac{\partial}{\partial y^i} \right)_{\iota(p)}, \quad i = 1, \ldots, n.$$

We next claim that $(d\iota)_p K(e_i, e_j) = K_k((d\iota)_p e_i, (d\iota)_p e_j)$. In fact, it is easy to see that both sides vanish when $i \neq j$ or $i = j \leq n - k$. By definition, we have that each side is $\lambda_i \left(\partial/\partial y^i \right)_{\iota(p)}$ when $i = j > n - k$.

Using Corollary 3.4, we complete the proof of the theorem. □

5. Hessian manifolds of CHC negative. In this section, we show that $(S_+^n(\kappa), \nabla^{S_+}, g_{S_+})$ in Example 2.4 is the only maximal Hessian manifolds of CHC $-4\kappa < 0$. Accordingly, the Hessian manifold consisting of multinomial distributions is characterized as the maximal one of CHC -1.

PROPOSITION 5.1. *Let (M, ∇, g) be an n-dimensional Hessian manifold of CHC $-4\kappa < 0$. We define the affine connection $\widetilde{\nabla}$ on the Riemannian cone $(\widetilde{M} := M \times \boldsymbol{R}^+, \widetilde{g} := t^2 \kappa g + dt^2)$ over (M, g) by*

$$\widetilde{\nabla}_X Y = \nabla_X Y - 2t\kappa g(X, Y)T,$$
$$\widetilde{\nabla}_Z T = \widetilde{\nabla}_T Z = 0,$$
$$\widetilde{\nabla}_T T = -t^{-1}T,$$

where $T = d/dt$ and we identify vector fields X, Y, Z on M with those on $\widetilde{M}$ in the natural way. Then $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is an $(n+1)$-dimensional Hessian manifold of CHC 0.

PROOF. The following formulas (5.1) through (5.6) hold for vector fields T, X, Y, Z on $\widetilde{M}$ as above:

(5.1)
$$(\widetilde{\nabla}\widetilde{g})(Y, Z; X) = t^2 \kappa (\nabla g)(Y, Z; X),$$
$$(\widetilde{\nabla}\widetilde{g})(X, Y; T) = (\widetilde{\nabla}\widetilde{g})(X, T; Y) = (\widetilde{\nabla}\widetilde{g})(T, X; Y)$$
$$= 2t\kappa g(X, Y),$$
$$(\widetilde{\nabla}\widetilde{g})(X, T; T) = (\widetilde{\nabla}\widetilde{g})(T, X; T) = (\widetilde{\nabla}\widetilde{g})(T, T; X) = 0,$$
$$(\widetilde{\nabla}\widetilde{g})(T, T; T) = 2t^{-1},$$

(5.2)
$$T^{\widetilde{\nabla}}(X, Y) = T^{\nabla}(X, Y),$$
$$T^{\widetilde{\nabla}}(T, \cdot) = 0,$$

$$R^{\widetilde{\nabla}}(X, Y)Z = R^{\nabla}(X, Y)Z\,,$$

(5.3)

$$R^{\widetilde{\nabla}}(\cdot, \cdot)T = R^{\widetilde{\nabla}}(T, \cdot)\cdot = 0\,,$$

$$\nabla^{\widetilde{g}}_X Y = \nabla^g_X Y - t\kappa\, g(X, Y)T\,,$$

(5.4)

$$\nabla^{\widetilde{g}}_X T = \nabla^{\widetilde{g}}_T X = t^{-1}X\,,$$

$$\nabla^{\widetilde{g}}_T T = 0\,,$$

$$\widetilde{K}(X, Y) = K(X, Y) - t\kappa\, g(X, Y)T\,,$$

(5.5)

$$\widetilde{K}(X, T) = -t^{-1}X\,, \quad \widetilde{K}(T, T) = -t^{-1}T\,,$$

$$(\widetilde{\nabla}\widetilde{K})(Y, Z; X)$$

(5.6)

$$= (\nabla K)(Y, Z; X) - 2\kappa\{g(X, Y)Z + g(X, Z)Y\}\,,$$

$$(\widetilde{\nabla}\widetilde{K})(T, \cdot; \cdot) = (\widetilde{\nabla}\widetilde{K})(\cdot, T; \cdot) = (\widetilde{\nabla}\widetilde{K})(\cdot, \cdot; T) = 0\,.$$

Equations (5.1), (5.2) and (5.3) show that $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is a Hessian manifold. Equations (5.6) shows that it is of CHC 0. □

THEOREM 5.2. *Let (M, ∇, g) be an n-dimensional simply-connected Hessian manifold of CHC $-4\kappa < 0$. Then there exists an immersion $\iota : M \to S^n_+(\kappa)$ such that $\iota^*\nabla^{S_+} = \nabla$ and $\iota^* g_{S_+} = g$.*

PROOF. Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be the $(n+1)$-dimensional Hessian manifold as in Proposition 5.1. If we set $\eta(x) := (x, \kappa^{-1/2})$ for $x \in M$, then (∇, g) is induced by $\eta : M \to \widetilde{M}$ from $(\widetilde{\nabla}, \widetilde{g})$. Let $((\mathbf{R}^+)^{n+1} = N_{n+1},\ D = \nabla_{n+1},\ g_{n+1} = \langle\cdot, \cdot\rangle|_{(\mathbf{R}^+)^{n+1}})$ be the $(n+1)$-dimensional Hessian manifold as in Examples 2.2 and 2.3 $(k = n+1)$, where $\langle\cdot, \cdot\rangle := g_E$ is the Euclidean metric. We denote by f the natural inclusion of the hypersphere $S^n(\kappa)$ of radius $\kappa^{-1/2}$ with center 0 into $\mathbf{R}^{n+1}$ and also its restriction on $S^n_+(\kappa)$. We recall that (∇^{S_+}, g_{S_+}) is induced by $f : S^n_+(\kappa) \to (\mathbf{R}^+)^{n+1}$ from (D, g_{n+1}). To prove the theorem, we construct immersions $\varphi : \widetilde{M} \to (\mathbf{R}^+)^{n+1}$ and $\iota : M \to S^n_+(\kappa)$ such that $\varphi^*D = \widetilde{\nabla}$, $\varphi^*g_{n+1} = \widetilde{g}$ and the following diagram is commutative.

$$
\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\ \varphi\ } & (\mathbf{R}^+)^{n+1} \\[4pt]
{\scriptstyle \eta}\big\uparrow & & \big\uparrow{\scriptstyle f} \\[4pt]
M & \xrightarrow[\ \iota\ ]{} & S^n_+(\kappa)
\end{array}
$$

By Proposition 5.1, $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is of CHC 0. By Theorem 4.1, there exist $k \in \{0, 1, \ldots, n+1\}$ and $\varphi : \widetilde{M} \to N_k \subset \mathbf{R}^{n+1}$ such that $\varphi^*\nabla_k = \widetilde{\nabla}$ and $\varphi^*g_k = \widetilde{g}$.

As the first step, we claim that φ is given in the form $\varphi(x, t) = t\sqrt{\kappa}\, f \circ \iota(x) + c$ for some $\iota : M \to S^n(\kappa)$ and $c \in \mathbf{R}^{n+1}$.

From the definition of $\widetilde{g}$, we have that

$$(5.7) \qquad 1 = \left\langle \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial t} \right\rangle, \quad 0 = \left\langle \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x^i} \right\rangle, \quad t^2 \kappa\, g_{ij} = \left\langle \frac{\partial \varphi}{\partial x^i}, \frac{\partial \varphi}{\partial x^j} \right\rangle$$

and, by differentiating them, that

$$0 = \left\langle \frac{\partial^2 \varphi}{\partial t\, \partial t}, \frac{\partial \varphi}{\partial t} \right\rangle, \quad 0 = \left\langle \frac{\partial^2 \varphi}{\partial x^i\, \partial t}, \frac{\partial \varphi}{\partial t} \right\rangle, \quad 0 = \left\langle \frac{\partial^2 \varphi}{\partial t\, \partial t}, \frac{\partial \varphi}{\partial x^i} \right\rangle,$$

which imply that $\partial^2 \varphi / \partial t\, \partial t = 0$. Accordingly, there exist maps $\psi, \xi : M \to \boldsymbol{R}^{n+1}$ such that $\varphi(x, t) = t\psi(x) + \xi(x)$. We then find that ξ is a constant map because

$$\left(\left| \frac{\partial \psi}{\partial x^i} \right|^2 - \kappa\, g_{ii} \right) t^2 + 2 \left\langle \frac{\partial \psi}{\partial x^i}, \frac{\partial \xi}{\partial x^i} \right\rangle t + \left| \frac{\partial \xi}{\partial x^i} \right|^2 = 0$$

holds for any t by the third equality of (5.7) and so $\left| \partial \xi / \partial x^i \right| = 0$. Moreover, it follows from the first equality of (5.7) that $\psi(M) \subset S^n(1) \subset \boldsymbol{R}^{n+1}$.

In the second step, we claim that $c = 0$ and $k = n + 1$, so that the above diagram is commutative.

Since $\varphi_*(tT) = \sum_{j=1}^{n+1}(y^j - c^j)\partial / \partial y^j$ for $T = d/dt \in \Gamma(T\widetilde{M})$, the third equality of (5.5) implies that

$$0 = t^2 \varphi_* \left\{ \widetilde{K}(T, T) + t^{-1} T \right\} = K_k(\varphi_*(tT), \varphi_*(tT)) + \varphi_*(tT)$$

$$= \sum_{i,j=1}^{n+1}(y^i - c^i)(y^j - c^j) K_k\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) + \sum_{l=1}^{n+1}(y^l - c^l)\frac{\partial}{\partial y^l}$$

$$= \sum_{l=n+1-k+1}^{n+1} \frac{(y^l - c^l)^2}{-y^l} \frac{\partial}{\partial y^l} + \sum_{l=1}^{n+1}(y^l - c^l)\frac{\partial}{\partial y^l}.$$

Since the right-hand side can not vanish if $k < n + 1$, we have $k = n + 1$. Then the right-hand side is given by

$$\sum_{l=1}^{n+1} \left\{ \frac{(y^l - c^l)^2}{-y^l} + (y^l - c^l) \right\} \frac{\partial}{\partial y^l} = \sum_{l=1}^{n+1} c^l \frac{y^l - c^l}{y^l} \frac{\partial}{\partial y^l},$$

which implies that $c = 0$.

To sum up, we have $\varphi : \widetilde{M} \to (\boldsymbol{R}^+)^{n+1}$ and $\iota : M \to S^n_+(\kappa)$ such that $\varphi^* D = \widetilde{\nabla}$, $\varphi^* g_{n+1} = \widetilde{g}$ and $\varphi(x, t) = t\sqrt{\kappa} f \circ \iota(x)$, that is, $\varphi \circ \eta = f \circ \iota$. It is easy to show that $\iota^* \nabla^{S_+} = \nabla$ and $\iota^* g_{S_+} = g$. $\qquad \square$

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