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ON THE FOURIER COEFFICIENTS OF JACOBI FORMS OF INDEX N OVER TOTALLY REAL NUMBER FIELDS

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Abstract. Skoruppa and Zagier established a bijective correspondence from the space of Jacobi forms ϕ of index m to that of elliptic modular forms f of level m. Gross, Kohnen and Zagier formulated this correspondence by means of kernel functions. Moreover, they proved that the squares of Fourier coefficients of ϕ are essentially equal to the critical values of the zeta functions $L(s, f, \chi)$ of f twisted by a quadratic character χ.

The purpose of this paper is to prove a generalization of such results concerning liftings and Fourier coefficients of Jacobi forms to the case of Jacobi forms of index N over totally real number fields F. Using kernel functions associated with the space of quadratic forms, we shall establish the existence of a lifting from the space of Jacobi forms ϕ of index N over F to that of Hilbert modular forms f of level N over F. Moreover, we determine explicitly the Fourier coefficients of f from those of ϕ. We prove that an analogue of Waldspurger's theorem in the case of Jacobi forms of index N over F holds.

Introduction. In [13], Skoruppa and Zagier succeeded in establishing a bijective correspondence from the space of Jacobi forms ϕ of index m to that of elliptic modular forms f of level m, which commutes with the action of Hecke operators. Gross, Kohnen and Zagier [4] formulated this correspondence by means of kernel functions. Moreover, they proved that the squares of Fourier coefficients of ϕ are essentially equal to the critical values of the zeta function $L(s, f, \chi)$ of f twisted by a quadratic character χ. Shimura [10] generalized Waldspurger's theorem [14] on elliptic modular forms of half integral weight to the case of Hilbert modular forms of half integral weight over totally real number fields.

The purpose of this paper is to prove a generalization of such results concerning liftings and Fourier coefficients of Jacobi forms in [4] and [13] to the case of Jacobi forms of index N over totally real number fields F. Using kernel functions associated with the space of quadratic forms, we shall establish the existence of a lifting Ψ_{D_0,r_0} from the space of Jacobi forms ϕ of index N over F to that of Hilbert modular forms f of level N over F. We prove that an analogue of Waldspurger's theorem in the case of Jacobi forms of index N over F holds. We refer to [15] for a generalization about an arithmetic of Heegner points in [4] to the case of Shimura curves over totally real number fields. We also refer to [12] for some property of Jacobi forms over totally real number fields.

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To prove our results, we need to generalize the methods in [4] to those of the case of totally real number fields. Our idea of the proof is to apply the reciprocity law for quadratic residue symbols due to Hecke [5] and Shimura's results about Gauss sums.

Section 0 is a preliminary section. In Section 1, we shall introduce Jacobi forms and quadratic forms over totally real number fields F. We discuss the property of genus characters of quadratic forms over F.

In Section 2, we study a relation between Gauss sums and genus characters of quadratic forms over F. We represent the genus character as a certain sum of Gauss sums in Proposition 2.1, which is a key lemma for our later arguments. Compared with the case of the rational number field, the proof of Proposition 2.1 is delicate and difficult because of the complexity of the computation of Gauss sums at even primes and its sign of Gauss sums. We overcome those by performing a precise calculation of Gauss sums. Furthermore, we need to determine the square of a certain Gauss sum $\varepsilon(a)$. It is difficult to compute $\varepsilon(a)$, but $\varepsilon(a)^2$ is determined by Shimura [9, p. 286]. By those fortunate circumstances, we may derive our results.

In Section 3, we discuss modular forms attached to the space of quadratic forms and Poincaré series for the Jacobi group. We shall determine explicitly Fourier coefficients of those. The constant terms of the former modular forms contain Gauss sums associated with quadratic character. Applying a results given in [11], we can calculate it explicitly.

In Section 4, we introduce a kernel function which is a sum of modular forms mentioned above. In Theorem 4.1, we shall deduce that this function can be represented as a sum of Poincaré series. Comparing Fourier coefficients of both sides, this formula may be reduced to Proposition 2.1. Employing this kernel function, we shall construct a lifting Ψ_{D_0,r_0} from the space of Jacobi forms ϕ of index N over F to that of Hilbert modular forms f of level N. By virtue of Theorem 4.1, we determine explicitly the Fourier coefficients of f from those of ϕ. Moreover, we show that Ψ_{D_0,r_0} is commutative with the action of Hecke operators.

In Section 5, we discuss a certain period integral of Hilbert modular forms attached to quadratic forms. As an application of a basic identity of kernel functions in Theorem 4.1, we shall deduce that the squares of Fourier coefficients of ϕ are essentially equal to the critical values of the zeta function $D(s, \chi, (\frac{*}{D_0}))$ of f twisted by a quadratic character $(\frac{*}{D_0})$ under the assumption of multiplicity one theorem concerning Hecke operators.

We mention that Shimura [10] formulated an analogue of Waldspurger's theorem in the case of Hilbert modular forms of level $2N$ and our result is a generalization of his result in the case of Hilbert modular forms of an odd level N. We note that Baruch and Mao [2] proved a Waldspurger-type formula in the totally real number field.

0. Notation and preliminaries. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For an associative ring R with identity element we denote by $R^\times$ the group of all its invertible element and by $M_{m,n}(R)$ the set of $m \times n$ matrices entries in R. We put $M_n(R) = M_{n,n}(R)$. Let $GL_n(R')$ (resp. $SL_n(R')$) denote the general linear group (resp. special linear group) of degree n over a commutative ring R'. Throughout this paper, we fix a totally real algebraic number field F of degree n with class number one and denote by a, h, $\mathfrak{o}$, d_F, and $\mathfrak{d}$, the set of all archimedean primes, the set of all non archimedean primes, the maximal order of F, the discriminant of F and the different of F relative to $\boldsymbol{Q}$, respectively. We denote by E the unit group of F. Let $\tau_1, \dots, \tau_n$ be the isomorphisms of F into $\boldsymbol{R}$. For each $\alpha \in F$, we put $\alpha^{(\nu)} = \tau_\nu(\alpha)$ $(1 \leq \nu \leq n)$. We consider an isomorphism $L : F \to \boldsymbol{R}^n$ defined by

$$L(\alpha) = (\alpha^{(1)}, \dots, \alpha^{(n)}) \quad \text{for every } \alpha \in F\,. \tag{0.1}$$

For each $\alpha \in F$, we put $N(\alpha) = \prod_{i=1}^n \alpha^{(i)}$.

We assume that

$$[E : E^+] = 2^n \quad \text{with} \quad E^+ = \{\varepsilon \in E; \varepsilon \gg 0\}\,,$$

where $\alpha \gg 0$ means $\alpha^{(i)} > 0$ $(1 \leq i \leq n)$ for $\alpha \in F$. We see that

$$E^+ = E^2 = \{\varepsilon^2; \varepsilon \in E\}\,.$$

We fix an element δ such that $\mathfrak{d} = (\delta)$ and $\delta \gg 0$. For an integer $c \in \mathfrak{o}$, the sum $\sum_{\alpha(c)}$ (resp. $\sum_{\alpha(c)^*}$) indicates the sum over representatives for all residue classes (primitive residue classes) modulo c.

1. Jacobi forms and quadratic forms over totally real number fields. Let N be an element of $\mathfrak{o}$ satisfying $N \gg 0$. We put

$$\begin{aligned} \Gamma_0(N) &= \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}) \,;\ N|c \right\}, \\ \tilde{\Gamma}_0(N) &= \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathfrak{o}) \,;\ N|c \text{ and } \det\gamma \gg 0 \right\}. \end{aligned} \tag{1.1}$$

We denote by $\mathfrak{H} = \{z \in \boldsymbol{C}\,;\ \Im(z) > 0\}$ the complex upper half plane. We define actions of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathfrak{o})$ on $\mathfrak{H}^n$ and $\mathfrak{H}^n \times \boldsymbol{C}^n$ by

(1.2)

$$\begin{aligned} z \to \gamma(z) &= \left(\frac{a^{(1)}z_1 + b^{(1)}}{c^{(1)}z_1 + d^{(1)}}, \dots, \frac{a^{(n)}z_n + b^{(n)}}{c^{(n)}z_n + d^{(n)}}\right) \\ (\tau, z) \to \gamma(\tau, z) &= \left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \dots, \frac{a^{(n)}\tau_n + b^{(n)}}{c^{(n)}\tau_n + d^{(n)}}; \frac{z_1}{c^{(1)}\tau_1 + d^{(1)}}, \dots, \frac{z_n}{c^{(n)}\tau_n + d^{(n)}}\right) \end{aligned}$$

for every $z = (z_1, \dots, z_n) \in \mathfrak{H}^n$ and for every $(\tau, z) = (\tau_1, \dots, \tau_n; z_1, \dots, z_n) \in \mathfrak{H}^n \times \boldsymbol{C}^n$ respectively. We also define an action of $(\lambda, \mu) \in \mathfrak{o}^2$ on $\mathfrak{H}^n \times \boldsymbol{C}^n$ by

$$(\tau, z) \to (\lambda, \mu)(\tau, z) = (\tau_1, \dots, \tau_n; z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}, \dots, z_n + \lambda^{(n)}\tau_n + \mu^{(n)})$$

for every $(\tau, z) \in \mathfrak{H}^n \times \boldsymbol{C}^n$. For $z = (z_1, \dots, z_n) \in \boldsymbol{C}^n$ and $\alpha \in F$, we put

$$\alpha \cdot z = (\alpha^{(1)}z_1, \dots, \alpha^{(n)}z_n) \quad \text{and} \quad \operatorname{tr} z = \sum_{i=1}^n z_i\,.$$

Given $z \in \boldsymbol{C}$, we put $e[z] = \exp(2\pi i z)$. Furthermore, for $z \in \boldsymbol{C}$ and $(k, p) \in \boldsymbol{Z}^2$ with $p > 0$, we define $z^{k/p} = (\sqrt[p]{z})^k$ with $-\pi/p < \arg \sqrt[p]{z} \leq \pi/p$. Given $z = (z_1, \dots, z_n) \in \boldsymbol{C}^n$, $k = (k_1, \dots, k_n) \in \boldsymbol{Q}^n$, $l \in \boldsymbol{Q}$ and $\alpha \in F$, we put $z^k = \prod_{i=1}^n z_i^{k_i}$ and $\alpha^{k-l} = \prod_{i=1}^n (\alpha^{(i)})^{k_i - l}$. Furthermore, for $d \in \boldsymbol{C}$, $I = (1, \dots, 1) \in \boldsymbol{C}^n$ and $k = (k_1, \dots, k_n) \in \boldsymbol{Q}^n$, we write as $d = dI$ and $d^k = (d \cdot I)^k$ if there is no fear of confusion. Let N and $k = (k_1, \dots, k_n)$ be elements such that $N \in \mathfrak{o}$, $k \in \boldsymbol{Z}^n$ and $k_i (1 \leq i \leq n)$. We consider a holomorphic function $\phi(\tau, z)$ on $\mathfrak{H}^n \times \boldsymbol{C}^n$ satisfying the conditions:

$$\text{(1.3)} \qquad \begin{aligned} &\text{(i)} \quad \phi(\gamma(\tau, z)) = (c\tau + z)^k e\left[\operatorname{tr}\left(\frac{N}{\delta}\left(\frac{cz^2}{c\tau + d}\right)\right)\right]\phi(\tau, z), \\ &\text{(ii)} \quad \phi((\lambda, \mu)(\tau, z)) = e\left[-\operatorname{tr}\left(\frac{N}{\delta}(\lambda^2\tau + 2\lambda z)\right)\right]\phi(\tau, z), \quad \text{and} \\ &\text{(iii)} \quad \phi((\tau, z)) = \sum_{(n,r) \in \mathfrak{o}^2} c(n, r) e\left[\operatorname{tr}\left(\frac{n}{\delta}\tau + \frac{r}{\delta}z\right)\right] \end{aligned}$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathfrak{o})$ and for every $(\lambda, \mu) \in \mathfrak{o}^2$, where

$$(c\tau + z)^k = \prod_{i=1}^n (c^{(i)}\tau_i + d^{(i)})^{k_i}, \quad e\left[\operatorname{tr}\left(\frac{N}{\delta}\left(\frac{cz^2}{c\tau + d}\right)\right)\right] = e\left[\sum_{i=1}^N \frac{N^{(i)}}{\delta^{(i)}}\left(\frac{c^{(i)} z_i^2}{c^{(i)}\tau_i + d^{(i)}}\right)\right],$$

$$e\left[-\operatorname{tr}\left(\frac{N}{\delta}(\lambda^2\tau + 2\lambda z)\right)\right] = e\left[-\sum_{i=1}^n \frac{N^{(i)}}{\delta^{(i)}}\left((\lambda^{(i)})^2\tau_i + 2\lambda^{(i)}(z_i)\right)\right],$$

$$e\left[\operatorname{tr}\left(\frac{n}{\delta}\tau + \frac{r}{\delta}z\right)\right] = e\left[\sum_{i=1}^n \left(\frac{n^{(i)}}{\delta^{(i)}}\tau_i + \frac{r^{(i)}}{\delta^{(i)}}z_i\right)\right], \quad \text{and} \quad 4Nn - r^2 = 0 \text{ or } 4Nn - r^2 \gg 0.$$

We denote by $J_{k,N}$ the set of all such functions ϕ. We call such ϕ a Jacobi form of index N and of weight k. Moreover, we say that ϕ is a cusp form if the following condition is satisfied.

$$\text{(iv)} \qquad c(n, r) = 0 \quad \text{if } 4Nn - r^2 \text{ is not totally positive.}$$

We denoted by $J_{k,N}^{\text{cusp}}$ the set of all cusp forms $\phi \in J_{k,N}$.

We introduce the Jacobi group $\Gamma(1)^J = \{(\gamma, (\lambda, \mu))\,;\ \gamma \in SL_2(\mathfrak{o}), \lambda, \mu \in \mathfrak{o}\}$ determined by the group law

$$\text{(1.4)} \qquad (\gamma, (\lambda, \mu)) \cdot (\gamma', (\lambda', \mu')) = (\gamma\gamma', (\lambda, \mu)\gamma' + (\lambda', \mu'))$$

for every $\gamma, \gamma' \in SL_2(\mathfrak{o})$, (λ, μ) and $(\lambda', \mu') \in \mathfrak{o}^2$. We define an action of $(\gamma, (\lambda, \mu)) \in \Gamma(1)^J$ on $\mathfrak{H}^n \times \boldsymbol{C}^n$ by

$$\text{(1.5)} \qquad (\tau, z) \to (\gamma, (\lambda, \mu))(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right)$$

for every $(\tau, z) \in \mathfrak{H}^n \times \boldsymbol{C}^n$. For a function ϕ on $\mathfrak{H}^n \times \boldsymbol{C}^n$ and $(\gamma, (\lambda, \mu)) \in \Gamma(1)^J$, we define a function $\phi|_{k,N}(\gamma, (\lambda, \mu))$ on $\mathfrak{H}^n \times \boldsymbol{C}^n$ by

$$\begin{aligned}\phi|_{k,N}(\gamma, (\lambda, \mu))(\tau, z) &= (c\tau + d)^{-k} \\ &\times e\left[\operatorname{tr}\left(\frac{N}{\delta}\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)\right)\right]\phi(\gamma, (\lambda, \mu))(\tau, z)\end{aligned} \tag{1.6}$$

for every $(\tau, z) \in \mathfrak{H}^n \times \boldsymbol{C}^n$. Given two Jacobi forms ϕ, ϕ' in $J_{k,N}^{\mathrm{cusp}}$, we introduce their inner product $\langle \phi, \phi' \rangle$ defined by

$$\langle \phi, \phi' \rangle = \int_{\Gamma(1)^J \backslash \mathfrak{H}^n \times \boldsymbol{C}^n} \phi(\tau, z)\overline{\phi'(\tau, z)} v^k e^{-4\pi N y^2/\delta v} v^{-3} dx dy du dv \tag{1.7}$$

with $z = x + iy \in \boldsymbol{C}^n$ and $\tau = u + iv \in \mathfrak{H}^n$. We refer to [3] for basic facts on Jacobi forms.

Here we recall the notion of the quadratic residue symbol given by Hecke [5]. For α and β in $\mathfrak{o}$ satisfying $(2, \beta) = 1$, we define a symbol $(\frac{\alpha}{\beta})$ by

$$\begin{aligned}&\left(\frac{\alpha}{\beta}\right) = \prod_{i=1}^{s} \left(\frac{\alpha}{\mathfrak{p}_i}\right)^{e_i} \qquad \text{and} \\ &\left(\frac{\alpha}{\mathfrak{p}_i}\right) = \#\{x \in \mathfrak{o}/\mathfrak{p}_i\ ;\ x^2 \equiv \alpha \pmod{\mathfrak{p}_i}\} - 1,\end{aligned} \tag{1.8}$$

where $(\beta) = \prod_{i=1}^{s} \mathfrak{p}_i^{e_i}$ with an odd prime ideal $\mathfrak{p}_i$ $(1 \le i \le s)$. Suppose that $\rho \in \mathfrak{o}$ and $\Delta \in \mathfrak{o}$ satisfy the following conditions:

$$\Delta \gg 0 \quad \text{and} \quad \Delta \equiv \rho^2 \pmod{4N}. \tag{1.9}$$

We consider a set of quadratic forms $L_{N,\Delta,\rho}$ defined by

$$\begin{aligned}L_{N,\Delta,\rho} = \Bigg\{ Q = [Na, b, c] = \begin{pmatrix} Na & b/2 \\ b/2 & c \end{pmatrix}\ ; \\ a, b, c \in \mathfrak{o}, b^2 - 4Nac = \Delta \text{ and } b \equiv \rho \pmod{2N} \Bigg\}.\end{aligned} \tag{1.10}$$

The group $\Gamma_0(N)$ acts on $L_{N,\Delta,\rho}$ by

$$Q \circ \gamma = {}^t\gamma Q \gamma \text{ for every } \gamma \in \Gamma_0(N) \text{ and } Q \in L_{N,\Delta,\rho}.$$

Let us assume that D_0 is an element of $\mathfrak{o}$ such that $D_0 | \Delta$ and Δ / D_0 is square modulo $4N$. Moreover, we impose the following condition:

ASSUMPTION 1.11. $D_0 \ll 0$, $(D_0, 4N) = 1$, the finite part of the conductor of the abelian extension $F(\sqrt{D_0})$ over F equals (D_0) and $D_0 = \pi_1^* \cdots \pi_l^*$ with distinct primary odd prime elements π_i^* of F $(1 \le i \le l)$.

Here an integer π_i^* is said to be primary if it is odd and congruent to the square of an integer in F modulo 4.

We define a genus character $\chi_{D_0}(Q)$ by

$$\chi_{D_0}(Q)=\begin{cases}\left(\dfrac{m}{D_0}\right) & \text{if } (a,b,c,D_0)=1\,,\\ 0 & \text{otherwise}\end{cases} \tag{1.12}$$

for every $Q=[Na,b,c]\in L_{N,\Delta,\rho}$, where m is an element of $\mathfrak{o}$ such that $(m,D_0)=1$ and $m=aN_1x^2+bxy+cN_2y^2$ for some N_1,N_2,x and $y\in\mathfrak{o}$ with $N=N_1N_2$ and $N_1\gg 0$, $N_2\gg 0$. By Hecke's reciprocity law for quadratic residue symbols, we see that

$$\left(\frac{m}{D_0}\right)=\left(\frac{F(\sqrt{D_0})/F}{(m)}\right)(\operatorname{sgn} m) \tag{1.13}$$

for every odd $m\in\mathfrak{o}-\{0\}$, where $\left(\frac{F(\sqrt{D_0})/F}{(m)}\right)$ means the Artin symbol of the abelian extension $F(\sqrt{D_0})/F$ and sgn $m=\prod_{i=1}^{n}m^{(i)}/|m^{(i)}|$. By some modification of the arguments in [4, p. 510], we may verify the following lemma.

LEMMA 1.1. *The notation being as above, suppose that D_0 satisfies Assumption* (1.11). *Then the function χ_{D_0} is $\Gamma_0(N)$ invariant and has the following properties.*

$$\chi_{D_0}([Na,b,c])=\chi_{D_0}([Na_1,b,ca_2])\chi_{D_0}([Na_2,b,ca_1]) \quad \text{if } a=a_1a_2,\ (a_1,a_2)=1, \tag{1.14}$$

$$\chi_{D_0}([Na,b,c])=\chi_{D_0}([Nc,-b,a])\,, \tag{1.15}$$

$$\chi_{D_0}([Na,b,c])=\left(\frac{N_1a}{D_1}\right)\left(\frac{N_2c}{D_2}\right) \tag{1.16}$$

for any splitting $D_0=D_1D_2$ of D_0 and $N=N_1N_2$ such that $N_1\gg 0$, $N_2\gg 0$, $(D_1,N_1a)=(D_2,N_2c)=1$ and $\chi_{D_0}([Na,b,c])=0$ if such splitting does not exist.

PROOF. Straightforward analysis using Assumption 1.11 and the quadratic reciprocity law proves the lemma and so we only prove (1.16). For simplicity, we may assume that $D_1=\prod_{i=1}^{k}\pi_i^*$ and $D_2=\prod_{i=k+1}^{l}\pi_i^*$. We have

$$\chi_{D_0}([Na,b,c])=\left(\frac{m}{D_0}\right)=\prod_{i=1}^{l}\left(\frac{m}{\pi_i^*}\right). \tag{1.17}$$

If $1\le i\le k$, then $\left(\frac{m}{\pi_i^*}\right)=\left(\frac{aN_1}{\pi_i^*}\right)$ since $m=aN_1x^2+bxy+cN_2y^2$ implies that $4aN_1m=(2aN_1x+by)^2-\Delta y^2$ and that π_i^* divides Δ. Similarly $\left(\frac{m}{\pi_i^*}\right)=\left(\frac{cN_2}{\pi_i^*}\right)$ if $k+1\le i\le l$. This yields that

$$\chi_{D_0}([Na,b,c])=\left(\frac{N_1a}{D_1}\right)\left(\frac{N_2c}{D_2}\right). \tag{1.18}$$

We may omit the details of the remainders of the proof. □

2. A relation between Gauss sums and genus characters and a key proposition. Suppose that $r_0, n_0, r, n, b \in \mathfrak{o}$ satisfy the condition:

$$D_0 = r_0^2 - 4Nn_0\,, \quad D = \Delta/D_0 = r^2 - 4Nn \quad \text{and} \quad b \equiv r_0 r \pmod{2N}. \tag{2.1}$$

We consider a polyominal $F(x, y)$ determined by

$$F(x, y) = Nx^2 + r_0xy + n_0y^2 + rx + sy + n \quad \text{with } s = (r_0r - b)/2N\,.$$

Given an integral ideal (a) in F, we define a sum F_a by

$$F_a = F_a(N, r_0, n_0, r, s, n) = |N(a)|^{-1} \sum_{\lambda(a)^*} \sum_{x,y \in (a)} e\left[\operatorname{tr}\left(\frac{\lambda F(x, y)}{a\delta}\right)\right]. \tag{2.2}$$

PROPOSITION 2.1. *Let r_0, n_0, r, n and $b \in \mathfrak{o}$ be elements of $\mathfrak{o}$ satisfying* (2.1). *Suppose that D_0 satisfies Assumption* 1.11.*Then*

$$\begin{aligned} &|N(a)|^{-1} \sum_{(d)|a, d\gg 0} \left(\frac{d}{D_0}\right) F_{a/d} \\ &= \begin{cases} \chi_{D_0}([Na, b, (b^2 - \Delta)/4Na]) & \text{if } a|(b^2 - \Delta)/4N\,, \\ 0 & \text{otherwise}\,. \end{cases} \end{aligned} \tag{2.3}$$

PROOF. By the property (1.14) in Lemma 1.1, we may reduce (2.3) to the case where $a = \pi^l$ with a positive prime element π. First we consider the case where $(\pi, 2) = 1$ and $\pi \not| D_0$. We can assume that $a|r_0$. Therefore

$$F_a = N(a)^{-1} \sum_{\lambda(a)^*} e[\operatorname{tr}(\lambda n/\delta a)] G_a(\lambda N, \lambda r) G_a(\lambda n_0, \lambda s)\,, \tag{2.4}$$

where

$$G_a(A, B) = \sum_{x(a)} e[\operatorname{tr}((Ax^2 + Bx)/\delta a)] = \sqrt{N(a)}\varepsilon(a)\left(\frac{A}{a}\right) e[-\operatorname{tr}(B^2(4A)^{-1}/\delta a)]$$

and $\varepsilon(a) = \sum_{x(a)} e[\operatorname{tr}(x^2/\delta a)] N(a)^{-1/2}$ with $(4l)^{-1}$ an element of $\mathfrak{o}$ such that $(4l)^{-1}.\ 4l \equiv 1 \pmod{a}$ for $(l, a) = 1$.

Therefore we have

$$F_a = N(a)^{-1} \sum_{\lambda(a)^*} N(a)\varepsilon(a)^2 \left(\frac{Nn_0}{a}\right) e\left[-\operatorname{tr}\left(\frac{\lambda(r^2(4N)^{-1} + s^2(4n_0)^{-1}) - \lambda n}{\delta a}\right)\right]. \tag{2.5}$$

Put $C = Ns^2 + r_0(-sr) + n_0r^2 + nD_0$. Then

$$D_0^{-1}C \equiv -(4n_0)^{-1}s^2 - (4N)^{-1}r^2 + n \pmod{a}\,,$$

which yields that

$$F_a = \sum_{\lambda(a)^*} \varepsilon(a)^2 \left(\frac{D_0}{a}\right)\left(\frac{-1}{a}\right) e[\operatorname{tr}(\lambda(D_0)^{-1}C/\delta a)]\,.$$

By virtue of Shimura [9, Prop. 1.2], we have

$$\varepsilon(a)^2 = \left(\frac{F(\sqrt{-1})/F}{(a)}\right) = \left(\frac{-1}{a}\right). \tag{2.6}$$

It implies that

$$F_a = \sum_{\lambda(a)^*} \left(\frac{D_0}{a}\right) e[\mathrm{tr}(\lambda(D_0)^{-1}C/\delta a)] = \left(\frac{D_0}{a}\right) \sum_{\lambda(a)^*} e[\mathrm{tr}(\lambda C/\delta a)]\,.$$

Therefore we obtain that

$$\sum_{(d)|a,d\gg 0} \left(\frac{d}{D_0}\right) F_{a/d} = \sum_{(d)|a,d\gg 0} \left(\frac{d}{D_0}\right)\left(\frac{D_0}{a/d}\right) \sum_{\lambda(a)^*,(\lambda,a)=d} e[\mathrm{tr}(\lambda C/\delta a)]. \tag{2.7}$$

Applying the quadratic reciprocity law, we find that

$$\sum_{(d)|a,d\gg 0} \left(\frac{d}{D_0}\right) F_{a/d} = \left(\frac{a}{D_0}\right) \sum_{\lambda(a)} e[\mathrm{tr}(\lambda C/\delta a)] = \begin{cases} \left(\frac{a}{D_0}\right) N(a) & \text{if } a|C\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

When $\pi | D_0$, it is easy to prove the required result using a method similar to that of [4] and the quadratic reciprocity law. So we may omit the details.

Next we treat the case where $\pi | 2$ and $\left(\frac{F(\sqrt{D_0})/F}{(\pi)}\right) = -1$. We put

$$\tilde{N}(\pi^l) = \#\{(x, y) \in (\mathfrak{o}/\pi^l)^2\ ;\ F(x, y) \equiv 0 \pmod{\pi^l}\}\,. \tag{2.9}$$

By the argument in [4, p. 510], the equality (2.3) for $a = \pi^l$, $(l = 1, \dots)$ is equivalent to

$$\sum_{l=0}^{\infty} \tilde{N}(\pi^l) N(\pi^l)^{-s-1} = \frac{1 + N(\pi)^{-s-1}}{1 - N(\pi)^{-s}} \sum_k (-1)^k N(\pi^k)^{-s}\,, \tag{2.10}$$

where k runs through $\mathbf{Z}$ under the condition that $k \geq 0$ and $\pi^k | (b^2 - \Delta)/4N$. We need to calculate $\tilde{N}(\pi^l)$ explicitly. Since we can reduce the problem to that of π-adic integers, we can assume that $N = 1$, $r_0 = 1$ and $r = 0$. Put $K = F(\sqrt{D_0})$, $\mathfrak{o}_K = \{\alpha \in K\ ;\ \alpha \text{ is an integer in } K\}$, $\omega_0 = (-1 + \sqrt{D_0})/2$ and $D_0 = 1 - 4n_0$. Let F_π, K_π, $\mathfrak{o}_\pi$ and $\mathfrak{o}_{K_\pi}$ denote the π-adic completions of F, K, $\mathfrak{o}$ and $\mathfrak{o}_K$, respectively. For x and y in $\mathfrak{o}_\pi$, we can check that

$$\begin{aligned} x^2 + xy + n_0 y^2 + sy + n &= \alpha\overline{\alpha} + sy + n \\ &= -D_0\left(\left(\frac{\alpha}{\sqrt{D_0}} + \frac{s}{D_0}\right)\overline{\left(\frac{\alpha}{\sqrt{D_0}} + \frac{s}{D_0}\right)}\right) + \frac{s^2 + nD_0}{D_0}, \end{aligned} \tag{2.11}$$

where $\alpha = x - \omega_0 y$, $l \in K_\pi$ and $\bar{l}$ is the conjugate of l in K_π. Observe that any integer l in K_π is written as $l = \alpha/\sqrt{D_0} + s/D_0$ for some $\alpha = x - \omega_0 y$ with $x, y \in \mathfrak{o}_\pi$. Hence it is sufficient to consider only the case where $s = 0$. Let us assume that $s = 0$. Note that

$$(b^2 - \Delta)/4N = n(1 - 4n_0)\,. \tag{2.12}$$

Let $e = \mathrm{ord}_\pi n$ be the largest number of l such that $\pi^l | n$. First we discuss the case where $e = 0$. By the property of the norm, we may derive that

$$\tilde{N}(\pi^{(l+1)}) = N(\pi)\tilde{N}(\pi^l)\,. \tag{2.13}$$

Therefore we need to calculate $\tilde{N}(\pi)$. We denote by U_{K_π} and U_{F_π} the unit groups of K_π and F_π, respectively. By the local class field theory, there is a surjective mapping $N_{K_\pi/F_\pi} : U_{K_\pi} \longrightarrow U_{F_\pi}$, where $N_{K_\pi/F_\pi}(\alpha) = \alpha\overline{\alpha}$ with $\alpha \in K_\pi$, which yields a surjective mapping $N_{K_\pi/F_\pi} : (\mathfrak{o}_{K_\pi}/\pi)^\times \longrightarrow (\mathfrak{o}_{F_\pi}/\pi)^\times$. Therefore we find that

$$\tilde{N}(\pi) = \#(\mathrm{Ker}\, N_{K_\pi/F_\pi}) = N(\pi) + 1 \text{ and } \tilde{N}(\pi^l) = (N(\pi)+1)N(\pi)^{l-1}\,. \tag{2.14}$$

Next we assume that $e = 2e'$ $(e' \geq 1)$. Put $\xi_0 = (\pi^e)^{-1} n/D_0$. Then

$$\tilde{N}(\pi^l) = \#\{\alpha \in \mathfrak{o}_{K_\pi}/\pi^l \; ; \; N_{K_\pi/F_\pi}(\alpha) \equiv \pi^{2e'}\xi_0 \pmod{\pi^l}\}\,. \tag{2.15}$$

Take an element α of $\mathfrak{o}_{K_\pi}/\pi^l$ and let $k = \mathrm{ord}_\pi \alpha$. Then we can represent α as $\alpha = \pi^k\alpha'$ for some $\alpha' \in U_{K_\pi}$. When $2e' < l$, an easy computation implies that

$$\begin{aligned} \tilde{N}(\pi^l) &= \#\{\alpha' \in (\mathfrak{o}_{K_\pi}/\pi^{l-e'})^\times \; ; \; N_{K_\pi/F_\pi}(\alpha') \equiv \xi_0 \pmod{\pi^{l-2e'}}\} \\ &= \#\{(x,y) \in (\mathfrak{o}_{F_\pi}/\pi^{l-e'})^2 \; ; \; x^2 + xy + n_0 y^2 + \pi^{-e} n \equiv 0 \pmod{\pi^{l-2e'}}\}\,. \end{aligned} \tag{2.16}$$

Put

$$x = c_0 + c_1\pi + \cdots + c_{l-2e'-1}\pi^{l-2e'-1} + \cdots + c_{l-e'-1}\pi^{l-e'-1}$$

and

$$y = c'_0 + c'_1\pi + \cdots + c'_{l-2e'-1}\pi^{l-2e'-1} + \cdots + c'_{l-e'-1}\pi^{l-e'-1}\,.$$

From (2.14), we find

$$\begin{aligned} \tilde{N}(\pi^l) &= (N(\pi)+1)N(\pi)^{l-2e'-1}(N(\pi)^{l-e'-1-(l-2e')+1})^2 \\ &= (N(\pi)+1)N(\pi)^{l-1}\,. \end{aligned} \tag{2.17}$$

When $2e' = l$, we have $\pi^{2k-l}N_{K_\pi/F_\pi}(\alpha') \equiv \xi_0$ (1). Therefore, $e' \leq k \leq l$. If $\pi^k\alpha' \equiv \pi^k\alpha'' \pmod{\pi^l}$ for some $\alpha', \alpha'' \in U_{K_\pi}$, then

$$\alpha' \equiv \alpha'' \pmod{\pi^{l-k}}\,. \tag{2.18}$$

For $\alpha \in U_{K_\pi}$, we can represent α as

$$\alpha = c_0 + c_1\pi + \cdots + c_{l-k-1}\pi^{l-k-1} + \cdots \text{ with } (c_0) \in (\mathfrak{o}_{K_\pi}/\pi)^\times$$

and $(c_i) \in (\mathfrak{o}_{K_\pi}/\pi)$ $(1 \leq i)$, where $c_i \in \mathfrak{o}_{K_\pi}$ $(i \geq 0)$ and (c_i) is the residue class containing c_i. Therefore we have

$$\#\{\alpha \in (\mathfrak{o}_{K_\pi}/\pi^{l-k})^\times\} = \begin{cases} (N(\pi^2)-1)N(\pi^2)^{l-k-1} & \text{if } e' \leq k < l\,, \\ 1 & \text{if } k = l\,, \end{cases} \tag{2.19}$$

which yields that $\tilde{N}(\pi^l) = N(\pi^2)^{l-l'}$. When $2e' < l$, we also obtain

$$\tilde{N}(\pi^l) = (N(\pi)+1)N(\pi)^{l-1}\,.$$

In the same fashion, we can determine $\tilde{N}(\pi^l)$ explicitly in the case of odd e. If $\pi|2$ and $\left(\frac{F(\sqrt{D_0})/F}{(\pi)}\right) = 1$, the calculation may be reduced to the case where $F(x, y) = xy$. Long and tedious calculation prove our assertions but we omit the details. □

3. Modular forms associated with quadratic forms and Poincaré series for the Jacobi groups. Let $k = (k_1, \dots, k_n)$ be an element of $\mathbf{Z}^n$ with $k_i > 1$ $(1 \le i \le n)$. We denote by $S_{2k}(\Gamma_0(N))$ (resp. $S_{2k}(\tilde{\Gamma}_0(N))$) the space of cusp forms of weight $2k$ with respect to $\Gamma_0(N)$ (resp. $\tilde{\Gamma}_0(N)$). Since $E^+ = E^2$, we easily see that $S_{2k}(\Gamma_0(N)) = S_{2k}(\tilde{\Gamma}_0(N))$. Given Δ and D_0 satisfying (1.9) and Assumption 1.11, we define a function $f_{k,N,\Delta,\rho,D_0}(z)$ on $\mathfrak{H}^n$ by

$$f_{k,N,\Delta,\rho,D_0}(z) = \sum_{Q \in L_{N,\Delta,\rho}} \frac{\chi_{D_0}(Q)}{Q(z,1)^k} \quad \text{for every } z = (z_1, \dots, z_n) \in \mathfrak{H}^n , \tag{3.1}$$

where $Q(z,1)^k = \prod_{i=1}^n (N^{(i)}a^{(i)}z_i^2 + b^{(i)}z_i + c^{(i)})^{k_i}$ with $Q = [Na, b, c]$. This series converges absolutely and uniformly on compact sets. If $D_0 \neq 1$, by Lemma 1.1, we may verify that $f_{k,N,\Delta,\rho,D_0}(z)$ belongs to the space

$$M_{2k}(N)^{\operatorname{sgn} D_0} = \left\{ f \in S_{2k}(\tilde{\Gamma}_0(N)) \; ; \; f\left(-\frac{1}{Nz}\right) = (-Nz^2)^k (\operatorname{sgn} D_0) f(z) \right\} \tag{3.2}$$

with $\operatorname{sgn} D_0 = \prod_{i=1}^n \operatorname{sgn} D_0^{(i)}$. For $a \in \mathfrak{o}$ and $\rho \in \mathfrak{o}/2N\mathfrak{o}$, we put

$$S_{N,a,\rho,\Delta} = \{ b \in \mathfrak{o}/2Na\mathfrak{o} \; ; \; b - \rho \in 2N\mathfrak{o} \text{ and } b^2 - \Delta \in 4Na\mathfrak{o} \} .$$

We determine Fourier coefficients of $f_{k,N,\Delta,\rho,D_0}(z)$ explicitly as follows.

PROPOSITION 3.1. *The notation being as above, suppose that Δ and D_0 satisfy* (1.9) *and Assumption* 1.11. *Then the Fourier coefficients of $f_{k,N,\Delta,\rho,D_0}(z)$ is given by*

$$f_{k,N,\Delta,\rho,D_0}(z) = \sum_{m \gg 0, m \in \mathfrak{o}} c_{k,N}(m, \Delta, \rho, D_0) e[\operatorname{tr}(mz/\delta)] \tag{3.3}$$

$$\begin{aligned} c_{k,N}(m, \Delta, \rho, D_0) &= i^k (\operatorname{sgn} D_0)^{-1/2} \frac{(2\pi)^k}{(k-1)!} (m^2/\delta^2\Delta)^{(k-1)/2} \\ &\times \Bigg[\frac{1}{\sqrt{N(\delta)}} \prod_{i=1}^n |D_0^{(i)}|^{-1/2} \varepsilon_{N/\delta}^{\pm}(m, \Delta, \rho, D_0) + \frac{i^k (\operatorname{sgn} D_0)^{1/2}}{\sqrt{N(\delta)}} (\sqrt{2}\pi)^n (N(m^2/\delta^2\Delta))^{1/4} \\ &\times \sum_{a \in \mathfrak{o} - \{0\}} \prod_{i=1}^n (N^{(i)}|a^{(i)}|)^{-1/2} (\operatorname{sgn} a)^k S_{Na}(m, \Delta, \rho, D_0) J_{k-1/2}\left(\frac{\pi m \sqrt{\Delta}}{N|a|\delta} \right) \Bigg] , \end{aligned}$$

where

$$(\operatorname{sgn} D_0)^{-1/2} = \prod_{i=1}^n (\operatorname{sgn} D_0^{(i)})^{-1/2}, \quad (\operatorname{sgn} D_0)^{1/2} = \prod_{i=1}^n (\operatorname{sgn} D_0^{(i)})^{1/2},$$

$$(\operatorname{sgn} a)^k = \prod_{i=1}^n (\operatorname{sgn} a^{(i)})^{k_i} ,$$

$$\varepsilon_{N/\delta}(m, \Delta, \rho, D_0) = \begin{cases} \left(\dfrac{m/f}{D_0}\right) & \textit{if } \Delta = D_0^2 f^2 \ (f \gg 0),\ f|m,\ D_0 f \equiv \rho \pmod{2N}, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\varepsilon^{\pm}_{N/\delta}(m, \Delta, \rho, D_0) = \begin{cases} \varepsilon_{N/\delta}(m, \Delta, \rho, D_0) + \varepsilon_{N/\delta}(m, \Delta, -\rho, D_0) & \textit{if } (-1)^k \operatorname{sgn} D_0 = 1, \\ \varepsilon_{N/\delta}(m, \Delta, \rho, D_0) - \varepsilon_{N/\delta}(m, \Delta, -\rho, D_0) & \textit{otherwise}, \end{cases}$$

$$S_{Na}(m, \Delta, \rho, D_0) = \sum_{b \in S_{N,a,\rho,\Delta}} \chi_{D_0}\left(\left[Na, b, \frac{b^2-\Delta}{4Na}\right]\right) e\left[\operatorname{tr}\left(\frac{mb}{2Na\delta}\right)\right],\ J_{k-1/2}\left(\frac{\pi m\sqrt{\Delta}}{N|a|\delta}\right)$$
$$= \prod_{i=1}^{n} J_{k_i-1/2}\left(\frac{\pi m^{(i)}\sqrt{\Delta^{(i)}}}{N^{(i)}|a^{(i)}|\delta^{(i)}}\right) \textit{ and } J_{k_i-1/2}(t) = \sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{(t/2)^{k_i+2\nu_i-1/2}}{\nu!\Gamma(k_i+\nu+1/2)}.$$

PROOF. For each $a \in \mathfrak{o} - \{0\}$, we put

$$\begin{aligned} f^a_{k,N}(z) &= \sum \chi_{D_0}\left(\left[Na, b, \frac{b^2-\Delta}{4Na}\right]\right)\left(Naz^2 + bz + \frac{b^2-\Delta}{4Na}\right)^{-k} \\ &= \sum_{b \in S_{N,a,\rho,\Delta}} \chi_{D_0}\left(\left[Na, b, \frac{b^2-\Delta}{4Na}\right]\right) \\ &\quad \times \sum_{n \in \mathfrak{o}}\left(Na(z+n)^2 + b(z+n) + \frac{b^2-\Delta}{4Na}\right)^{-k} \\ &= \sum_{m \in \mathfrak{o}} c'_{k,N}(m, \Delta, \rho, D_0) e[\operatorname{tr}(mz/\delta)], \end{aligned} \tag{3.4}$$

where the sum $\sum$ is taken over all $b \in \mathfrak{o}$ satisfying $b \equiv \rho \pmod{2N}$, $b^2 \equiv \Delta \pmod{4Na}$. Applying the Poisson summation formula, we have

$$\sum_{l \in \mathfrak{o}}\left(Na(z+l)^2 + b(z+l) + \frac{b^2-\Delta}{4Na}\right)^{-k} = \sum_{m \in \mathfrak{o}} c_{a,b}(m) e[\operatorname{tr}(mz/\delta)], \tag{3.5}$$

where

$$c_{a,b}(m) = \frac{1}{\sqrt{N(\delta)}} \int_{-\infty+ic_1}^{\infty+ic_1} \cdots \int_{-\infty+ic_n}^{\infty+ic_n} \left(Naz^2 + bz + \frac{b^2-\Delta}{4Na}\right)^{-k} e\left[-\operatorname{tr}\left(\frac{mz}{\delta}\right)\right] dz$$

with a constant c_i (> 0) $(1 \le i \le n)$. The substitution $t = -i(z + b/2Na)$ produces the relation

$$Naz^2 + bz + \frac{b^2-\Delta}{4Na} = -(\operatorname{sgn} a)\left(N|a|t^2 + \frac{\Delta}{4N|a|}\right). \tag{3.6}$$

Therefore we obtain

$$\begin{aligned} c_{a,b}(m) &= (-\operatorname{sgn} a)^k e^{\pi i \operatorname{tr}(bm/\delta Na)} \\ &\quad \times \frac{1}{\sqrt{N(\delta)}} \int_{-\infty+ic_1}^{\infty+ic_1} \cdots \int_{-\infty+ic_n}^{\infty+ic_n} e^{2\pi \operatorname{tr}((m/\delta)t)}\left(N|a|t^2 + \frac{\Delta}{4N|a|}\right)^{-k} (i^n) dt. \end{aligned} \tag{3.7}$$

Here we recall the following formula (cf. [1, 29.3.57]):

$$\left(\frac{1}{2\pi i}\right)^n \int_{-\infty+ic_1}^{\infty+ic_1} \cdots \int_{-\infty+ic_n}^{\infty+ic_n} e^{2\pi\,\mathrm{tr}\,lt}\left(N|a|t^2+\frac{\Delta}{4N|a|}\right)^{-k} dt$$
$$= \begin{cases} \dfrac{\pi^k (2l)^{k-1/2}}{\prod_{i=1}^n (N^{(i)}|a^{(i)}|)^{1/2}\Delta^{k/2-1/4}(k-1)!} J_{k-1/2}\left(\dfrac{\pi\sqrt{\Delta}l}{N|a|}\right) & \text{if } l \gg 0, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$c'_{k,N}(m,\Delta,\rho,D_0) = \begin{cases} \dfrac{(-1)^k(\operatorname{sgn} a)^k 2^{k+1/2}\pi^{k+1}(m/\delta)^{k-1/2}}{|\Delta^{k/2-1/4}|\sqrt{\prod_{i=1}^n N^{(i)}|a^{(i)}|}(k-1)!\sqrt{N(\delta)}} & \\ \times S_{Na}(m,\Delta,\rho,D_0) J_{k-1/2}\left(\dfrac{\pi m\sqrt{\Delta}}{N|a|\delta}\right) & \text{if } m \gg 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

We put $\Delta = D_0^2 f^2$ $(f \gg 0)$ and

$$f^0_{k,N}(z) = \sum_{b,c} \chi_{D_0}([0,b,c])(bz+c)^{-k}, \tag{3.9}$$

where b and c run through $\mathfrak{o}$ under the condition that $b^2 = \Delta$ and $b \equiv \rho \pmod{2N}$. Observe that $b = \pm D_0 f$. Let us consider the case where

$$(-1)^k(\operatorname{sgn} D_0) = -1,\ D_0 f \not\equiv \rho \pmod{2N} \ \text{ and } \ -D_0 f \equiv \rho \pmod{2N}.$$

We easily check that

$$f^0_{k,N}(z) = \sum_{c\in\mathfrak{o}} \left(\frac{c}{D_0}\right)(D_0 f z + c)^{-k}. \tag{3.10}$$

By the quadratic reciprocity law, we have

$$\left(\frac{-c}{D_0}\right) = \left(\frac{c}{D_0}\right)\left(\frac{-1}{D_0}\right) = \left(\frac{c}{D_0}\right)(\operatorname{sgn} D_0).$$

Therefore we obtain

$$f^0_{k,N}(z) = -|D_0^{-k}| \sum_{r(D_0)^*} \sum_{n\in\mathfrak{o}} \left(\frac{r}{D_0}\right)\left(fz + \frac{r}{|D_0|} + n\right)^{-k}. \tag{3.11}$$

Applying the Poisson summation formula, we confirm that

$$\begin{aligned} f^0_{k,N}(z) &= -|D_0^{-k}| \frac{(2\pi)^k i^k}{\sqrt{N(\delta)}(k-1)!} \\ &\quad \times \sum_{\mu\in\mathfrak{o}^+} (\mu/\delta)^{k-1}\left(\sum_{r(D_0)^*}\left(\frac{r}{D_0}\right) e\left[\mathrm{tr}\left(\frac{\mu r}{\delta|D_0|}\right)\right]\right) e[\mathrm{tr}(\mu f z/\delta)]. \end{aligned} \tag{3.12}$$

For each r coprime to D_0, we may find an element $r' \in \mathfrak{o}$ such that

$$r \equiv r' \pmod{D_0} \text{ and } (2, r') = 1 .$$

Hence by (1.13), we have

$$\sum_{r(D_0)^*} \left(\frac{r}{D_0}\right) e\left[\operatorname{tr}\left(\frac{\mu r}{\delta|D_0|}\right)\right] = \sum_{r(D_0)^*} \left(\frac{F(\sqrt{D_0})/F}{(r)}\right)(\operatorname{sgn} r) e\left[\operatorname{tr}\left(\frac{\mu r}{\delta|D_0|}\right)\right].$$

By Shimura [11, A 6.3,4], we obtain that

$$\sum_{r(D_0)^*} \left(\frac{r}{D_0}\right) e\left[\operatorname{tr}\left(\frac{\mu r}{\delta|D_0|}\right)\right] = \left(\frac{\mu}{D_0}\right) i^n \prod_{i=1}^{n} |D_0^{(i)}|^{1/2} . \tag{3.13}$$

Therefore,

$$\begin{aligned} f_{k,N}^0(z) = &\sum_{m \gg 0, m \in \mathfrak{o}} (\operatorname{sgn} D_0)^{-1/2} i^k \frac{(2\pi)^k}{(k-1)!} \\ &\times (m^2/\delta^2 \Delta)^{(k-1)/2} \frac{1}{\sqrt{N(\delta)}} |D_0|^{-1/2} \varepsilon_{N/\delta}^{\pm}(m, \Delta, \rho, D_0) e[\operatorname{tr}(mz/\delta)] , \end{aligned} \tag{3.14}$$

where $|D_0|^{-1/2} = \prod_{i=1}^n |D_0^{(i)}|^{-1/2}$. We can prove the following formula similarly for the remaining cases.

$$\begin{aligned} f_{k,N}^0(z) = &\sum_{m \gg 0, m \in \mathfrak{o}} (\operatorname{sgn} D_0)^{-1/2} i^k \frac{(2\pi)^k}{(k-1)!} \\ &\times (m^2/\delta^2 \Delta)^{(k-1)/2} \frac{1}{\sqrt{N(\delta)}} |D_0|^{-1/2} \varepsilon_{N/\delta}^{\pm}(m, \Delta, \rho, D_0) e[\operatorname{tr}(mz/\delta)] . \end{aligned} \tag{3.15}$$

Therefore this proves the proposition. □

Next we introduce a Poincaré series for the Jacobi group. We define

$$\Gamma_\infty^J(1) = \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) \in \Gamma^J(1) \; ; \; n, \mu \in \mathfrak{o} \right\} \quad \text{and}$$
$$e^{n,r}|_{k,N}(\tau, z) = e[\operatorname{tr}((n/\delta)\tau + (n/\delta)z)] .$$

Given a $(n, r) \in \mathfrak{o}^2$ satisfying $r^2 - 4Nn \ll 0$, we define a function $P_{k,N,(n,r)}(\tau, z)$ on $\mathfrak{H}^n \times \boldsymbol{C}^n$ by

$$P_{k,N,(n,r)}(\tau, z) = \sum_{\gamma \in \Gamma_\infty^J(1) \backslash \Gamma^J(1)} e^{n,r}|_{k,N} \gamma(\tau, z) . \tag{3.16}$$

It is easy to verify that $P_{k,N,(n,r)}(\tau, z) \in J_{k,N}^{\mathrm{cusp}}$ for every $r^2 - 4Nn \ll 0$. The function $P_{k,N,(n,r)}(\tau, z)$ is characterized by the property:

$$\begin{aligned} &\langle \phi, P_{k,N,(n,r)} \rangle \\ &= p(n, r) N(\delta)(\delta N)^{k-3/2} 2^{-n} (N(N/\delta))^{-1/2} \frac{\Gamma(k-3/2)}{\pi^{k-3/2} |(4Nn - r^2)^{k-3/2}|} \end{aligned} \tag{3.17}$$

for every $\phi(\tau, z) = \sum_{(n',r')\in C(N)} p(n', r')e[\mathrm{tr}((n'/\delta)\tau + (r'/\delta)z)] \in J_{k,N}^{\mathrm{cusp}}$, where $C(N) = \{(n, r) \in \mathfrak{o}^2\ ;\ 4Nn - r^2 \gg 0\}$ and $\Gamma(k - 3/2) = \prod_{i=1}^{n} \Gamma(k_i - 3/2)$.

PROPOSITION 3.2. *Let $k = (k_1, \dots, k_n)$ be an element of $\mathbf{Z}^n$ such that $k_i > 1$ $(1 \leq i \leq n)$ and put*

$$P_{k,N,(n,r)}(\tau, z) = \sum_{(n',r')\in C(N)} c_{k,N,(n,r)}(n', r')e\left[\mathrm{tr}\left(\frac{n'}{\delta}\tau + \frac{r'}{\delta}z\right)\right].$$

Then the Fourier coefficient $c_{k,N,(n,r)}(n', r')$ is determined by

$$\begin{aligned} c_{k,N,(n,r)}(n', r') &= \delta^{\pm}_{N/\delta}(n, r, n', r') + i^k(\sqrt{2}\pi)^n(N(N/\delta))^{-1/2} \\ &\quad \times N(\delta)^{-1}(D'/D)^{k/2-3/4} \sum_{c\in\mathfrak{o}-\{0\}} (\operatorname{sgn} c)^k H_{N,c}(n, r, n', r') J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{N|c|\delta}\right), \end{aligned} \tag{3.18}$$

where $D' = (r')^2 - 4Nn'$, $D = r^2 - 4Nn$,

$$\delta_{N/\delta}(n, r, n', r') = \begin{cases} 1 & \text{if } D' = D, r' \equiv r \pmod{2N}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\delta^{\pm}_{N/\delta}(n, r, n', r') = \begin{cases} \delta_{N/\delta}(n, r, n', r') + \delta_{N/\delta}(n, r, n', -r') & \text{if } (-1)^k = 1, \\ \delta_{N/\delta}(n, r, n', r') - \delta_{N/\delta}(n, r, n', -r') & \text{otherwise}, \end{cases}$$

$$\begin{aligned} H_{N,c}(n, r, n', r') &= \prod_{i=1}^{n} |c^{(i)}|^{-3/2} \\ &\times \sum_{\rho(c)^*,\lambda(c)} e\left[\mathrm{tr}\left(\frac{1}{c}\left(\left(\frac{N}{\delta}\lambda^2 + \frac{r}{\delta}\lambda + \frac{n}{\delta}\right)\rho^{-1} + \frac{n'}{\delta}\rho + \frac{r'}{\delta}\lambda\right)\right)\right] e\left[\mathrm{tr}\left(\frac{rr'}{2N\delta c}\right)\right]. \end{aligned}$$

PROOF. We see that

$$\begin{aligned} P_{k,N,(n,r)}(\tau, z) &= \sum_{c,d,\lambda} (c\tau + d)^{-k} e\left[\mathrm{tr}\left(\frac{N}{\delta}\left(\frac{-cz^2}{c\tau + d} + \lambda^2\frac{a\tau + b}{c\tau + d} + \frac{2\lambda z}{c\tau + d}\right)\right)\right] \\ &\quad \times e\left[\mathrm{tr}\left(\frac{n}{\delta}\left(\frac{a\tau + b}{c\tau + d}\right)\right)\right] e\left[\mathrm{tr}\left(\frac{r}{\delta}\left(\frac{z}{c\tau + d} + \lambda\frac{a\tau + b}{c\tau + d}\right)\right)\right] \end{aligned} \tag{3.19}$$

with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathfrak{o})$, where c, d and λ run through elements of $\mathfrak{o}$ with $(c, d) = 1$ and $\lambda \in \mathfrak{o}$. Let us consider the contribution of the summation (3.18) from $c = 0$. Put

$$\begin{aligned} &\sum_{\varepsilon\in E,\lambda\in\mathfrak{o}} \varepsilon^{-k} e\left[\mathrm{tr}\left(\frac{N}{\delta}(\lambda^2\varepsilon^2\tau + 2\lambda\varepsilon z)\right)\right] e\left[\mathrm{tr}\left(\frac{n}{\delta}\varepsilon^2\tau\right)\right] e\left[\mathrm{tr}\left(\frac{r}{\delta}(\varepsilon z + \lambda\varepsilon^2\tau)\right)\right] \\ &\qquad = \sum_{n',r'} \tilde{c}(n', r')e\left[\mathrm{tr}\left(\frac{n'}{\delta}\tau + \frac{r}{\delta}z\right)\right] \end{aligned} \tag{3.20}$$

with $n' = \varepsilon^2(N\lambda^2 + n + r\lambda)$ and $r' = \varepsilon(2\lambda N + r)$. Then $(r')^2 - 4Nn' = \varepsilon^2(r^2 - 4Nn)$, which yields that

$$\varepsilon = \pm\sqrt{\frac{(r')^2 - 4Nn'}{r^2 - 4Nn}}\,.$$

Since the Fourier coefficient $c(n, r)$ of a Jacobi form equals $c(\varepsilon^2 n, \varepsilon r)$ for every $\varepsilon \in E$, it is sufficient to consider only the case where $(r')^2 - 4Nn' = r^2 - 4Nn$. We assume that $\varepsilon = \pm 1$. Therefore, we can calculate $\tilde{c}(n', r')$ explicitly.

Next we discuss the following summation.

$$\begin{aligned}(3.21)\quad &\sum_{c,d,\lambda\in\mathfrak{o},c\neq 0,(c,d)=1} (c\tau + d)^{-k} e\Big[\operatorname{tr}\Big(\frac{N}{\delta}\Big(\frac{-cz^2}{c\tau + d} + \lambda^2\frac{a\tau + b}{c\tau + d} + \frac{2\lambda z}{c\tau + d}\Big)\Big)\Big] \\ &\qquad \times e\Big[\operatorname{tr}\Big(\frac{n}{\delta}\Big(\frac{a\tau + b}{c\tau + d}\Big)\Big)\Big] e\Big[\operatorname{tr}\Big(\frac{N}{\delta}\Big(\frac{z}{c\tau + d} + \lambda\frac{a\tau + b}{c\tau + d}\Big)\Big)\Big].\end{aligned}$$

Note that

$$\frac{a\tau + b}{c\tau + d} = \frac{a}{c} - \frac{1}{c(c\tau + d)}, \quad \frac{z}{c\tau + d} + \lambda\frac{a\tau + b}{c\tau + d} = \frac{z - \lambda/c}{c\tau + d} + \lambda\frac{a}{c}\,,$$

$$\lambda^2\frac{a\tau + b}{c\tau + d} + 2\lambda\frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} = -\frac{c(z - \lambda/c)^2}{c\tau + d} + \lambda^2\frac{a}{c}\,.$$

Therefore (3.21) becomes

$$(3.22)\quad \sum_{c\in\mathfrak{o}-\{0\}} c^{-k} \sum_{d(c)^*,\lambda(c)} e\Big[\operatorname{tr}\Big(\frac{1}{c}\Big(\frac{N}{\delta}\lambda^2 + \frac{r\lambda}{\delta} + \frac{n}{\delta}\Big)d^{-1}\Big)\Big] F_{k,N,c,(n,r)}\Big(\tau + \frac{d}{c}, z - \frac{\lambda}{c}\Big)$$

with

$$\begin{aligned}F_{k,N,c,(n,r)}(\tau, z) &= \sum_{\alpha,\beta\in\mathfrak{o}} (\tau + \alpha)^{-k} e\Big[\operatorname{tr}\Big(\frac{N}{\delta}\Big(- \frac{(z - \beta)^2}{\tau + \alpha}\Big)\Big)\Big] \\ &\qquad \times e\Big[\operatorname{tr}\Big(\frac{n}{\delta}\Big(\frac{-1}{c^2(c\tau + d)}\Big)\Big)\Big] e\Big[\operatorname{tr}\Big(\frac{r}{\delta}\Big(\frac{z - \beta}{c(\tau + \alpha)}\Big)\Big)\Big] \\ &= \sum_{n',r'\in\mathfrak{o}} r(n', r') e\Big[\operatorname{tr}\Big(\frac{n'}{\delta}\tau + \frac{r'}{\delta}z\Big)\Big].\end{aligned}$$

Applying the Poisson summation formula, we obtain

$$\begin{aligned}(3.23)\quad r(n', r') &= (N(\delta)^{-1/2})^2 \int_{c_1^{(1)}-i\infty}^{c_1^{(1)}+i\infty} \cdots \int_{c_1^{(n)}-i\infty}^{c_1^{(n)}+i\infty} \tau^{-k} e\Big[\operatorname{tr}\Big(- \frac{n'}{\delta}\tau\Big)\Big] \\ &\quad \times \int_{c_2^{(1)}-i\infty}^{c_2^{(1)}+i\infty} \cdots \int_{c_2^{(n)}-i\infty}^{c_2^{(n)}+i\infty} e\Big[\operatorname{tr}\Big(- \frac{Nz^2}{\delta\tau} + \frac{rz}{\delta c\tau} - \frac{n}{\delta c^2\tau} - \frac{r'z}{\delta}\Big)\Big] dz d\tau\end{aligned}$$

with constants $c_1^{(i)} > 0$ and $c_2^{(i)} > 0$ $(1 \le i \le n)$. We can check the following equality.

$$
\begin{aligned}
&-\frac{N}{\delta\tau}z^2 + \frac{rz}{\delta c\tau} - \frac{n}{\delta c^2\tau} - \frac{r'}{\delta}z \\
&\quad = -\frac{N}{\delta\tau}\left(z - \frac{1}{2N}\left(\frac{r}{c} - \tau r'\right)\right)^2 + \left(\frac{D}{4Nc^2\delta\tau} + \frac{D'\tau}{4N\delta}\right) + \frac{n'\tau}{\delta} - \frac{rr'}{2Nc\delta}\,.
\end{aligned} \tag{3.24}
$$

Therefore we find that

$$
\begin{aligned}
r(n', r') &= (N(2N/\delta))^{-1/2} e\left[\operatorname{tr}\left(\frac{-rr'}{2Nc}\right)\right] \\
&\quad \times \int_{c_1^{(1)}-i\infty}^{c_1^{(1)}+i\infty} \cdots \int_{c_1^{(n)}-i\infty}^{c_1^{(n)}+i\infty} \left(\frac{\tau}{i}\right)^{1/2} \tau^{-k} e\left[\operatorname{tr}\left(\frac{D'\tau}{4N\delta} + \frac{D\tau^{-1}}{4N\delta c^2}\right)\right] d\tau\,.
\end{aligned} \tag{3.25}
$$

If D' is not totally negative, we see that the integral (3.25) vanishes. If $D' \ll 0$, we confirm that

$$
\begin{aligned}
r(n', r') &= (2\pi)^n (N(2N/\delta))^{-1/2} N(\delta)^{-1} \\
&\quad \times e\left[\operatorname{tr}\left(\frac{-rr'}{2N\delta c}\right)\right] i^{-k} |c^{k-3/2}| (D'/D)^{k/2-3/4} \\
&\quad \times \frac{1}{(2\pi i)^n} \int_{c_1^{(1)}-i\infty}^{c_1^{(1)}+i\infty} \cdots \int_{c_1^{(n)}-i\infty}^{c_1^{(n)}+i\infty} s^{-k+1/2} e^{(2\pi/4N\delta|c|)(D'D)^{(s-s^{-1})/2}} ds\,.
\end{aligned} \tag{3.26}
$$

Observe that the integral in (3.26) is equal to

$$
(2\pi i)^n J_{k-1/2}\left(\frac{\pi (D'D)^{1/2}}{N|c|\delta}\right) \text{ (cf. [1, 29.3.80])}\,.
$$

This completes our proof. □

4. A basic identity between Gauss sums and Kloosterman sums. In this section, using a kernel function, we construct a lifting from the space of Jacobi forms of index N to that of modular forms of level N. Define a function $\Omega_{k,N,D_0,r_0}(w; \tau, z)$ on $\mathfrak{H}^n \times (\mathfrak{H}^n \times \boldsymbol{C}^n)$ by

$$
\begin{aligned}
\Omega_{k,N,D_0,r_0}(w; \tau, z) &= c_{k,N,D_0} \\
&\quad \times \sum_{(n,r)} (4Nn - r^2)^{k-1/2} f_{k,N,D_0(r^2-4Nn),r_0 r,D_0}(w) e\left[\operatorname{tr}\left(\frac{n}{\delta}\tau + \frac{r}{\delta}z\right)\right]
\end{aligned} \tag{4.1}
$$

with $c_{k,N,D_0} = (-2i)^{k-1} 2^{n-1} N(\delta)^{-3/2} N^{1-k} \pi^{-k} |D_0^{k-1/2}|$, where (n, r) runs over $c(N)$. We first prove the following theorem concerning the property of the above function $\Omega_{k,N,D_0,r_0}(w; \tau, z)$.

THEOREM 4.1 (A basic identity). *Suppose that $D_0 = r_0^2 - 4Nn_0$ satisfies Assumption 1.11. Then*

$$
\begin{aligned}
(4.2)\quad & \Omega_{k,N,D_0,r_0}(w;\tau,z) = c_{k,N,D_0}(2\pi)^k i^{k-1}(k-1)!^{-1}\delta^{-k+1/2} \\
& \times \sum_m m^{k-1}\Bigg(\sum_{dd'=m,\, d\in\mathfrak{o}^+/E^+}\left(\frac{d}{D_0}\right)(d')^k P_{k+1,N,(n_0(d')^2, r_0 d')}(\tau,z)\Bigg) e\left[\operatorname{tr}\left(\frac{mw}{\delta}\right)\right],
\end{aligned}
$$

where m runs through all positive integers in $\mathfrak{o}$.

PROOF. Comparing Fourier coefficients of the both sides of (4.2), it is enough to prove the relations

$$
\begin{aligned}
& (D/D_0)^{k/2}\varepsilon_{N/\delta}(m, DD_0, rr_0, D_0) \\
(4.3)\quad & = \sum_{(d)\mid m,\, d\gg 0}\left(\frac{d}{D_0}\right)(m/d)^k \delta_{N/\delta}\left(\frac{m^2}{d^2}n_0, \frac{m}{d}r_0, n, r\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& S_{Na}(m, DD_0, rr_0, D_0) \\
(4.4)\quad & = \sum_{(d)\mid (m,a),\, d\gg 0}\left(\frac{d}{D_0}\right)|N(a/d)|^{1/2} H_{N,a/d}\left(\frac{m^2}{d^2}n_0, \frac{m}{d}r_0, n, r\right)
\end{aligned}
$$

with $D = r^2 - 4Nn \ll 0$. The proof of (4.3) is easy. So we may omit the details. The proof of (4.4) may be reduced to the following equality.

$$
\begin{aligned}
(4.5)\quad & \sum_{b\in S_{N,a,rr_0,DD_0}} \chi_{D_0}\left(\left[Na, b, \frac{b^2 - D_0 D}{4Na}\right]\right) e\left[\operatorname{tr}\left(\frac{(b - r_0 r)m}{2Na\delta}\right)\right] \\
& = |N(a)|^{-1}\sum_{(d)\mid (a,m),\, d\gg 0}\left(\frac{d}{D_0}\right)|N(d)| \\
& \quad \times \sum_{\rho\in(a/d)^*,\, \lambda\in(a/d)} e\left[\operatorname{tr}\left(\frac{1}{(a/d)\delta}\left(\left(N\lambda^2 + \frac{m}{d}r_0\lambda + \left(\frac{m}{d}\right)^2 n_0\right)\rho^{-1} + n\rho + r\lambda\right)\right)\right]
\end{aligned}
$$

for every $a \in \mathfrak{o} - \{0\}$. Since both sides are periodic with period a as functions of $m \in \mathfrak{o}$, we apply the Fourier transform to the above functions on $\mathfrak{o}/a\mathfrak{o}$. Consequently we need to verify that

$$
(4.6)\quad
\begin{aligned}
&|N(a)|^{-1}\sum_{b\in S_{N,a,rr_0,DD_0}}\sum_{m(a)}\chi_{D_0}\left(\left[Na,b,\frac{b^2-DD_0}{4Na}\right]\right)\\
&\qquad\times e\left[\operatorname{tr}\left(\frac{((b-rr_0)/2N-h')m}{a\delta}\right)\right]\\
&\quad=N(a)^{-2}\sum_{m(a)}\sum_{(d)|(a,m),d\gg 0}\left(\frac{d}{D_0}\right)N(d)^{-1}\sum_{\rho(a/d)^*,\lambda(a/d)}\\
&\qquad\times e\left[\operatorname{tr}\left(\frac{(N\lambda^2+(m/d)r_0\lambda+(m/d)^2n_0)\rho^{-1}+n\rho+r\lambda-h'(m/d)}{(a/d)\delta}\right)\right].
\end{aligned}
$$

Put $h=2Nh'+r_0r$. Then the left-hand side of (4.6) is equal to

$$
(4.7)\quad
\begin{cases}\chi_{D_0}([Na,h,(h^2-D_0D)/4Na]) & \text{if } h^2\equiv D_0D \pmod{4Na}\,,\\ 0 & \text{otherwise}\end{cases}
$$

and the right-hand side of (4.6) becomes

$$
(4.8)\quad
\begin{aligned}
&|N(a)|^{-1}\sum_{(d)|a,d\gg 0}\left(\frac{d}{D_0}\right)|N(a/d)|^{-1}\\
&\times\sum_{\rho(a/d)^*,\lambda,m(a/d)}e\left[\operatorname{tr}\left(\frac{\rho(N\lambda^2+r_0m\lambda+n_0m^2+r\lambda-h'm+n)}{(a/d)\delta}\right)\right].
\end{aligned}
$$

By virtue of Proposition 2.1, (4.7) is equal to (4.8). This proves our assertion. □

Let $\phi\in J^{\mathrm{cusp}}_{k+1,N}$. Define a function $\Psi_{D_0,r_0}(\phi)$ on $\mathfrak{H}^n$ by

$$
(4.9)\quad \Psi_{D_0,r_0}(\phi)(w)=\langle\phi,\Omega_{k,N,D_0,r_0}(-\overline{w};*)\rangle\,.
$$

Then, by virtue of Theorem 4.1, we have

$$
(4.10)\quad
\begin{aligned}
&\Psi_{D_0,r_0}(\phi)(w)\\
&=\sum_{m\in\mathfrak{o},m\gg 0}\left(\sum_{(d)|m,d\gg 0}\left(\frac{d}{D_0}\right)d^{k-1}c((m/d)^2n_0,(m/d)r_0)\right)e\left[\operatorname{tr}\left(\frac{mw}{\delta}\right)\right]
\end{aligned}
$$

with $\phi(\tau,z)=\sum_{(n,r)\in C(N)}c(n,r)e[\operatorname{tr}((n/\delta)\tau+(r/\delta)z)]$. By our definition and the relation (3.2), we can see that $\Psi_{D_0,r_0}(\phi)$ belongs to $M_{2k}(N)^{\operatorname{sgn}D_0}$.

We recall the definition of Hecke operators on the space $J^{\mathrm{cusp}}_{k+1,N}$ (cf. [3]). Let $\mathfrak{A}=(l)$ denote an odd ideal satisfying $(\mathfrak{A},2D_0N)=1$ and $l\gg 0$. Given $\phi\in J^{\mathrm{cusp}}_{k,N}$, we define a function $T_{k,N}(\mathfrak{A})\phi$ by

$$
(4.11)\quad T_{k,N}(\mathfrak{A})\phi(\tau,z)=l^{k-4}\sum_{M}\sum_{x\in\mathfrak{o}^2/l\mathfrak{o}^2}\phi|_{k,N}(M,x)(\tau,z)\,,
$$

where $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ runs through $SL_2(\mathfrak{o})\backslash M_2(\mathfrak{o})$ under the conditions that $\det(M)=l^2$ and the ideal g.c.d(a,b,c,d) is square. We see that $T_{k,N}(\mathfrak{A})\phi$ belongs to $J^{\mathrm{cusp}}_{k,N}$ and its Fourier coefficients are determined by the relation

$$(T_{k,N}(\mathfrak{A})\phi)(\tau,z) = \sum_{(n,r)\in C(N)} c^*(n,r)e\left[\operatorname{tr}\left(\frac{n}{\delta}\tau + \frac{r}{\delta}z\right)\right] \tag{4.12}$$

with

$$c^*(n,r) = \sum_{(a),(n',r')\in C(N)} a^{k-2}\varepsilon_{r^2-4Nn}(a)c(n',r')\,,$$

where the summation is over all ideals (a) and $(n',r')\in C(N)$ such that

$$a|l^2,\quad a^2|l^2(r^2-4Nn)\,,\quad a^{-2}l^2(r^2-4Nn)\equiv \text{square} \pmod 4\,,$$
$$a^{-2}l^2(r^2-4Nn)=(r')^2-4Nn'\,,\quad ar'\equiv lr \pmod{2N}\,,\quad \text{and}$$

$$\varepsilon_{r^2-4Nn}(a) = \begin{cases} |N(f)|\left(\dfrac{(r^2-4Nn)/f^2}{a/f^2}\right) & \text{if there exists } f \text{ such that} \\ & (r^2-4Nn)/f^2 \equiv \text{square} \pmod 4,\ f^2|a \\ & \text{and } (a/f^2, (r^2-4Nn)/f^2)=1\,, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 4.2. *Suppose that* $k_i > 1$ $(1\le i\le n)$ *and* ϕ *belongs to* $J^{\text{cusp}}_{k+1,N}$. *Then* $\Psi_{D_0,r_0}(\phi)(w)$ *belongs to* $M_{2k}(N)^{\operatorname{sgn} D_0}$ *and the following diagram is commutative*:

$$\begin{array}{ccc} J^{\text{cusp}}_{k+1,N} & \xrightarrow{\Psi_{D_0,r_0}} & M_{2k}(N)^{\operatorname{sgn} D_0} \\ {\scriptstyle T_{k+1,N}(\mathfrak{p})}\big\downarrow & & \big\downarrow{\scriptstyle T_{2k,N}(\mathfrak{p})} \\ J^{\text{cusp}}_{k+1,N} & \xrightarrow[\Psi_{D_0,r_0}]{} & M_{2k}(N)^{\operatorname{sgn} D_0} \end{array}\,, \tag{4.13}$$

where $\mathfrak{p}$ *is an odd prime satisfying* $(\mathfrak{p}, 2D_0N)=1$ *and* $T_{2k,N}(\mathfrak{p})$ *is the Hecke operator on* $M_{2k}(N)^{\operatorname{sgn} D_0}$.

PROOF. We prove that the diagram is commutative. We put $\mathfrak{p}=(l)$. Let $\phi(\tau,z) = \sum_{(n,r)\in C(N)} c(n,r)e[\operatorname{tr}((n/\delta)\tau + (r/\delta)z)]$ be an elelemt of $J^{\text{cusp}}_{k+1,N}$. Then

$$T_{k+1,N}(\mathfrak{p})\phi(\tau,z) = \sum_{(n,r)\in C(N)} c^*(n,r)e\left[\operatorname{tr}\left(\frac{n}{\delta}\tau + \frac{r}{\delta}z\right)\right]$$

and

$$c^*(n,r) = \sum_{(a),(n',r')\in C(N)} a^{k-1}\varepsilon_{r^2-4Nn}(a)c(n',r')\,.$$

We have

$$\begin{aligned} &\Psi_{D_0,r_0}(T_{k+1,N}(\mathfrak{p})\phi)(w) \\ &= \sum_{m\in\mathfrak{o},m\gg 0}\left(\sum_{(d)|m,d\gg 0}\left(\frac{d}{D_0}\right)d^{k-1}c^*\left(\left(\frac{m}{d}\right)^2 n_0, \frac{m}{d}r_0\right)\right)e\left[\operatorname{tr}\left(\frac{mw}{\delta}\right)\right]. \end{aligned} \tag{4.14}$$

We see that

$$c^*\left(\left(\frac{m}{d}\right)^2 n_0, \frac{m}{d}r_0\right) = \sum_{(a),(n',r')\in C(N)} a^{k-1}\varepsilon_{(m/d)^2(r_0^2-4Nn_0)}(a)c(n',r')\,, \tag{4.15}$$

where $a|l^2$, $a^2|l^2(m/d)^2(r_0^2-4Nn_0)$, $(l/a)^2(m/d)^2(r_0^2-4Nn_0)=(r')^2-4Nn'$.
If $a=l^2$, then

$$l\left|\frac{m}{d},\ \left(\frac{m}{ld}\right)^2(r_0^2-4Nn_0)=(r')^2-4Nn' \text{ and } \varepsilon_{(m/d)^2(r_0^2-4Nn_0)}(a)=N(l)\,. \right. \tag{4.16}$$

If $a=l$, then

$$\left(\frac{m}{d}\right)^2(r_0^2-4Nn_0)=(r')^2-4Nn' \text{ and } \varepsilon_{(m/d)^2(r_0^2-4Nn_0)}(a)=\left(\frac{l}{D_0}\right). \tag{4.17}$$

(See [5, Th. 167].) If $a=1$, then

$$l^2\left(\frac{m}{d}\right)^2(r_0^2-4Nn_0)=(r')^2-4Nn' \text{ and } \varepsilon_{(m/d)^2(r_0^2-4Nn_0)}(a)=1\,. \tag{4.18}$$

We see that

$$\begin{aligned}&\Psi_{D_0,r_0}(T_{k+1,N}(\mathfrak{p})\phi)(w)\\&=\sum_{m\in\mathfrak{o},m\gg 0}\left(\sum_{(d)|m,d\gg 0}\left(\frac{d}{D_0}\right)d^{k-1}c^*\left(\left(\frac{m}{d}\right)^2 n_0, \frac{m}{d}r_0\right)e\left[\operatorname{tr}\left(\frac{mw}{\delta}\right)\right].\right.\end{aligned}$$

We put

$$\overline{c}(m)=\sum_{(d)|m,d\gg 0}\left(\frac{d}{D_0}\right)d^{k-1}c\left(\left(\frac{m}{d}\right)^2 n_0, \frac{m}{d}r_0\right)$$

and

$$T_{2k,N}(\mathfrak{p})\Psi_{D_0,r_0}(\phi)(w)=\sum_{m'\in\mathfrak{o},m'\gg 0}\tilde{c}(m')e\left[\operatorname{tr}\left(\frac{m'w}{\delta}\right)\right].$$

By definition, we have

$$\tilde{c}(m')=\begin{cases}\overline{c}(lm') & \text{if } l\nmid m'\,,\\ \overline{c}(lm')+l^{2k-1}\overline{c}(m'/l) & \text{if } l\mid m'\,.\end{cases}$$

When, $l\nmid m'$, we have

$$\begin{aligned}\overline{c}(lm')&=\sum_{(d)|lm'}\left(\frac{d}{D_0}\right)d^{k-1}c\left(\left(\frac{lm'}{d}\right)^2 n_0, \frac{lm'}{d}r_0\right)\\&=\left(\frac{l}{D_0}\right)\sum_{(d')|m'}l^{k-1}\left(\frac{d'}{D_0}\right)(d')^{k-1}c\left(\left(\frac{m'}{d'}\right)^2 n_0, \frac{m'}{d'}r_0\right)\end{aligned}$$

$$+\sum_{(d)|m'}\left(\frac{d}{D_0}\right)d^{k-1}c\left(l^2\left(\frac{m'}{d}\right)^2 n_0, \frac{lm'}{d}r_0\right).$$

By (4.14), (4.15), (4.17) and (4.18), we have

$$\tilde{c}(m') = \sum_{(d)|m'}\left(\frac{d}{D_0}\right)d^{k-1}c^*\left(\left(\frac{m'}{d}\right)^2 n_0, \frac{m'}{d}r_0\right).$$

When $l|m'$, we have

$$\begin{aligned}
&\bar{c}(lm') + l^{2k-1}\bar{c}\left(\frac{m'}{l}\right)\\
&= \sum_{(d)|m'}\left(\frac{d}{D_0}\right)d^{k-1}c\left(\left(\frac{lm'}{d}\right)^2 n_0, \frac{lm'}{d}r_0\right) + l^{2k-1}\sum_{(d)|(m'/l)}\left(\frac{d}{D_0}\right)d^{k-1}c\left(\left(\frac{m'}{dl}\right)^2 n_0, \frac{m'}{dl}r_0\right)\\
&= \sum_{(d')|m'}\left(\frac{d'}{D_0}\right)\left(\frac{l}{D_0}\right)l^{k-1}(d')^{k-1}c\left(\left(\frac{m'}{d'}\right)^2 n_0, \frac{m'}{d'}r_0\right)\\
&\quad + \sum_{(d)|m'}\left(\frac{d}{D_0}\right)d^{k-1}c\left(\left(\frac{lm'}{d}\right)^2 n_0, \frac{lm'}{d}r_0\right)\\
&\quad + l^{2k-1}\sum_{(d)|(m'/l)}\left(\frac{d}{D_0}\right)d^{k-1}c\left(\left(\frac{m'}{dl}\right)^2 n_0, \frac{m'}{dl}r_0\right).
\end{aligned}$$

By (4.14), (4.15), (4.16), (4.17) and (4.18), we have

$$\begin{aligned}
\tilde{c}(m') &= \bar{c}(lm') + l^{2k-1}\bar{c}\left(\frac{m'}{l}\right)\\
&= \sum_{(d)|m'}\left(\frac{d}{D_0}\right)d^{k-1}c^*\left(\left(\frac{m'}{d}\right)^2 n_0, \frac{m'}{d}r_0\right).
\end{aligned}$$

This completes our proof. □

5. Fourier coefficients of Jacobi forms and the critical values of the zeta function associated with Hilbert modular forms. In this section, using a basic identity of kernel functions, we shall derive a relation between Fourier coefficients of Jacobi forms and the critical values of zeta functions attached to Hilbert modular forms. Define a function $\Psi^*_{D_0,r_0}(f)$ on $\mathfrak{H}^n \times \boldsymbol{C}^n$ by

$$\Psi^*_{D_0,r_0}(f)(\tau, z) = \langle f, \Omega_{k,N,D_0,r_0}(*; -\bar{\tau}, -\bar{z})\rangle \tag{5.1}$$

for every $f \in M_{2k}(N)^{\operatorname{sgn} D_0}$. Then

$$\begin{aligned}
&\Psi^*_{D_0,r_0}(f)(\tau, z) = \overline{c_{k,N,D_0}}\\
&\quad\times \sum_{(n,r)\in C(N)}(4Nn - r^2)^{k-1/2}\langle f, f_{k,N,D_0(r^2-4Nn),r_0r,D_0}\rangle e\left[\operatorname{tr}\left(\frac{n}{\delta}\tau + \frac{r}{\delta}z\right)\right].
\end{aligned}\tag{5.2}$$

We assume that $\Delta = D_0^2$ and $\rho = r_0^2$. We easily see that

$$L_{N,D_0^2,r_0^2}/\Gamma_0(N) = \{Q = [0, D_0, \mu]\ ;\ \mu \pmod{D_0}\}$$

and

$$\Gamma_0(N)_Q = \{\gamma \in \Gamma_0(N)\ ;\ Q \circ \gamma = Q\} = \left\{\begin{pmatrix} \varepsilon & \frac{\mu(\varepsilon^2-1)}{D_0\varepsilon} \\ 0 & \varepsilon^{-1} \end{pmatrix}\ ;\ \varepsilon \in E, \varepsilon^2 \equiv 1 \pmod{D_0}\right\}$$

for every $Q = [0, D_0, \mu]$ and μ such that $(\mu, D_0) = 1$. Putting $\theta = \arg(z-\beta)$ and $r = |z-\beta|$ with $\beta = -(\mu/D_0)$, we obtain

$$Q(\overline{z}, 1)^{-k} y^{2k-2} dxdy = D_0^{-k}(z-\beta)^{k-1} \sin^{2k-2}\theta dz d\theta\,.$$

Employing the arguments in Kohnen [6, p.265–p.266] and Shimura [8], we have

$$\begin{aligned}(5.3)\qquad &\int_{\Gamma_0(N)_Q\backslash\mathfrak{H}^n} f(z)Q(\overline{z}, 1)^{-k} y^{2k-2} dxdy \\ &= D_0^{-k}\prod_{i=1}^{n}\left(\int_0^{\pi}\sin^{2k_i-2}\theta_i d\theta_i\right)\int_{(\boldsymbol{R}^+)^n/E_0} f(ri+\beta)i^k r^{k-1} dr\end{aligned}$$

with $E_0 = \{\varepsilon^2\ ;\ \varepsilon \in E \text{ and } \varepsilon^2 \equiv 1 \pmod{D_0}\}$. We consider the integral.

$$\begin{aligned}\langle f, f_{k,N,D_0^2,r_0^2,D_0}\rangle(s) = &\sum_{\mu(D_0)^*}\left(\frac{\mu}{D_0}\right)D_0^{-k}\prod_{i=1}^{n}\left(\int_0^{\pi}\sin^{2k_i-2}\theta_i d\theta_i\right) \\ &\times\int_{(\boldsymbol{R}^+)^n/E_0} f(ri+\beta)(ir)^{k-1} i^n r^s dr \quad (s \in \boldsymbol{C})\,.\end{aligned}$$

Then

$$\begin{aligned}(5.4)\qquad \langle f, f_{k,N,D_0^2,r_0^2,D_0}\rangle(s) &= i^k\left(\prod_{i=1}^{n}\int_0^{\pi}\sin^{2k_i-2}\theta_i d\theta_i\right)D_0^{-k} \\ &\times\sum_{\alpha\in\mathfrak{o}^+}\sum_{\mu(D_0)^*}\left(\frac{\mu}{D_0}\right)e\left[\operatorname{tr}\left(\frac{\alpha\mu}{\delta|D_0|}\right)\right]a(\alpha)\int_{(\boldsymbol{R}^+)^n/E_0} e^{-2\pi\operatorname{tr}(\alpha t/\delta)} t^{s+k-1} dt\,,\end{aligned}$$

where $f(z) = \sum_{\alpha\in\mathfrak{o}^+} a(\alpha)e[\operatorname{tr}(\alpha/\delta)z]$. Observe that

$$\int_0^{\pi}\sin^{2k_i-2}\theta_i d\theta_i = \pi\frac{(2k_i-3)!!}{(2k_i-2)!!},\quad \sum_{\mu(D_0)^*}\left(\frac{\mu}{D_0}\right)e\left[\operatorname{tr}\left(\frac{\alpha\mu}{\delta|D_0|}\right)\right] = \left(\frac{\mu}{D_0}\right)i^n\prod_{i=1}^{n}|D_0^{(i)}|^{1/2}$$

and $a(\mu\varepsilon) = \varepsilon^k a(\mu)$ for every $\varepsilon \in E^+$, where $(2k_i-3)!! = 1\times 3\times\cdots\times(2k_i-3)$ and $(2k_i-2)!! = 2\times 4\times\cdots\times(2k_i-2)$. Decomposing

$$(\boldsymbol{R}^+)^n/E_0 = \bigcup_{\varepsilon'\in E^+/E_0}\varepsilon'((\boldsymbol{R}^+)^n/E^+)\,,$$

we get

$$\langle f, f_{k,N,D_0^2,r_0^2,D_0}\rangle(s) = i^{k+1}\frac{\pi^n}{D_0^k}\frac{(2k-3)!!}{(2k-2)!!}(E^+ : E_0)(2\pi)^{-k}\delta^k\Gamma(k) \times \prod_{i=1}^{n}|D_0^{(i)}|^{1/2}\Bigg(\sum_{\mu\in\mathfrak{o}^+/E^+} a(\mu)\left(\frac{\mu}{D_0}\right)\mu^{-k-s}\Bigg). \tag{5.5}$$

We take the modular form $\mathfrak{f}$ attached to f given in [7, Section 2]. For an integral ideal $\mathfrak{A} = (\mu)$ ($\mu \gg 0$), put $c(\mathfrak{A}, \mathfrak{f}) = a(\mu)\mu^{-k}$. Then $c(\mathfrak{A}, \mathfrak{f})$ is independent of the choice of μ. Putting $C(\mathfrak{A}, \mathfrak{f}) = N(\mathfrak{A})^{k_0}c(\mathfrak{A}, \mathfrak{f})$ with $k_0 = \max\{k_1, \ldots, k_n\}$, we consider the Dirichlet series determined by

$$D\Bigg(s, \mathfrak{f}, \left(\frac{*}{D_0}\right)\Bigg) = \sum_{\mathfrak{A}=(\mu),\mu\gg 0} C(\mathfrak{A}, \mathfrak{f})\left(\frac{\mu}{D_0}\right)N(\mathfrak{A})^{-s}. \tag{5.6}$$

Then

$$\langle f, f_{k,N,D_0^2,r_0^2,D_0}\rangle = i^{k+1}\frac{\pi^k}{D_0^k}\frac{(2k-3)!!}{(2k-2)!!} \times \prod_{i=1}^{n}|D_0^{(i)}|^{1/2}(E^+ : E_0)(2\pi)^{-k}\delta^k\Gamma(k)D\Bigg(k_0, \mathfrak{f}, \left(\frac{*}{D_0}\right)\Bigg). \tag{5.7}$$

Here we assume that $\mathfrak{f}$ is a normalized Hecke eigenform satisfying

$$\mathfrak{f}|\tilde{T}_{2k,N}(\mathfrak{A}) = \chi(\mathfrak{A})\mathfrak{f}, \quad C(\mathfrak{A}, \mathfrak{f}) = \chi(\mathfrak{A})C(\mathfrak{o}, \mathfrak{f}) \quad \text{and} \quad C(\mathfrak{o}, \mathfrak{f}) = 1, \tag{5.8}$$

where $\mathfrak{f}|\tilde{T}_{2k,N}(\mathfrak{A})$ is the action of the Hecke operators given in [7, Section 2]. Consider the Dirichlet series $D(s, \chi, (\frac{*}{D_0}))$ defined by

$$D\Bigg(s, \chi, \left(\frac{*}{D_0}\right)\Bigg) = \prod_{\mathfrak{p}}\Bigg(1 - \chi(\mathfrak{p})\left(\frac{\pi}{D_0}\right)N(\mathfrak{p})^{-s} + 1_N(\mathfrak{p})\left(\frac{\pi}{D_0}\right)^2 N(\mathfrak{p})^{2k_0-1-2s}\Bigg)^{-1}, \tag{5.9}$$

where $1_N(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \nmid N \\ 0 & \text{otherwise.}\end{cases}$ Then

$$\begin{aligned} D\Big(s, \chi, \Big(\frac{*}{D_0}\Big)\Big) &= \prod_{\mathfrak{p}|2,\mathfrak{p}=(\pi),\pi\gg 0}\Big(1 - \chi(\mathfrak{p})\Big(\frac{\pi}{D_0}\Big)N(\mathfrak{p})^{-s} + 1_N(\mathfrak{p})\Big(\frac{\pi}{D_0}\Big)^2 N(\mathfrak{p})^{2k_0-1-2s}\Big)^{-1} \\ &\times \prod_{\mathfrak{p}\nmid 2}\Big(1 - \chi(\mathfrak{p})\Big(\frac{F(\sqrt{D_0})/F}{\mathfrak{p}}\Big)N(\mathfrak{p})^{-s} \\ &+ 1_N(\mathfrak{p})\Big(\frac{F(\sqrt{D_0})/F}{\mathfrak{p}}\Big)^2 N(\mathfrak{p})^{2k_0-1-2s}\Big)^{-1}. \end{aligned} \tag{5.10}$$

Let $\phi(\tau, z) = \sum_{(n,r)\in C(N)} c(n,r)e[\mathrm{tr}((n/\delta)\tau + (r/\delta)z)]$ be an element of $J^{\mathrm{cusp}}_{k+1,N}$. We impose the following assumptions:

ASSUMPTION 5.11.
1. $\phi|T_{k+1,N}(l) = \chi(l)\phi$ for every odd prime l satisfying $(l, 2D_0N) = 1$.
2. $\Psi_{D_0,r_0}(\phi)$ is a new form of $M_{2k}(N)^{\mathrm{sgn}\, D_0}$ and f is the primitive form associated with it.
3. If ϕ' is a nonzero element of $J^{\mathrm{cusp}}_{k+1,N}$ such that $\phi'|T_{k+1,N}(l) = \chi(l)\phi'$ for every prime l $((l, 2D_0N) = 1)$, then there is a constant c such that $\phi' = c\phi$.

From the above assumption, we have

$$\Psi_{D_0,r_0}(\phi) = c(n_0, r_0)f\,. \tag{5.12}$$

Since $T_{k+1,N}(l)$ $((l, 2D_0N) = 1)$ is Hermitian and Ψ_{D_0,r_0} commutes with the action of $T_{k+1,N}(l)$ $((l, 2D_0N) = 1)$, we conclude that

$$T_{k+1,N}(l)(\Psi^*_{D_0,r_0}(f)) = \chi(l)\Psi^*_{D_0,r_0}(f) \tag{5.13}$$

for every odd prime l $((l, 2D_0N) = 1)$. Assumption 5.11 implies that $\Psi^*_{D_0,r_0}(f) = \alpha\phi$ for some constant α. Therefore we obtain

$$\alpha c(n,r) = \overline{c_{k,N,D_0}}(4Nn - r^2)^{k-1/2}\langle f, f_{k,N,D_0(r^2-4Nn),r_0r,D_0}\rangle\,, \tag{5.14}$$

which yields that

$$\begin{aligned}\alpha c(n,r)\langle\phi,\phi\rangle &= c(n,r)\langle\Psi^*_{D_0,r_0}(f),\phi\rangle \\ &= c(n,r)\langle f, \Psi_{D_0,r_0}(\phi)\rangle = c(n,r)\overline{c(n_0,r_0)}\langle f, f\rangle\,.\end{aligned} \tag{5.15}$$

This implies that

$$\overline{c_{k,N,D_0}}(4Nn - r^2)^{k-1/2}\langle f, f_{k,N,D_0(r^2-4Nn),r_0r,D_0}\rangle\langle\phi,\phi\rangle = \overline{c(n_0,r_0)}c(n,r)\langle f, f\rangle\,. \tag{5.16}$$

Therefore, we deduce our main theorem.

THEOREM 5.1. *Let D_0 be an element satisfying Assumption* 1.11. *Suppose that ϕ satisfies Assumption* 5.11. *Then*

$$\langle f, f\rangle\langle\phi,\phi\rangle^{-1}|c(n_0,r_0)|^2 = (k-1)!\frac{\delta^{k-3/2}|D_0^{k-1/2}|}{2^{2k-1}N^{k-1}\pi^k}(E^+ : E_0)D\left(k_0, \chi, \left(\frac{*}{D_0}\right)\right). \tag{5.17}$$

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