ON THE DIRECT PRODUCT OF W^*-ALGEBRAS

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Introduction. From the algebraical point of view, the direct product of operator algebras is useful for the study of operator algebras. This notion is originally due to F. J. Murray and J. von Neumann [9, 10]. They have investigated chiefly the direct product of a factor of type I and a factor in general type. Recently, T. Turumaru [17, 18] has investigated the direct product of C^*-algebras and obtained many interesting results.

The present paper is devoted to a natural step in the direct product of W^*-algebras. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be W^*-algebras on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Then the direct product $\mathbf{M}_1 \otimes \mathbf{M}_2$ of $\mathbf{M}_1$ and $\mathbf{M}_2$ is defined as the weak closure of the algebraical direct product of $\mathbf{M}_1$ and $\mathbf{M}_2$ on the Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. Therefore the concept of the direct product of W^*-algebras depends on the underlying Hilbert spaces. One of purposes of this paper is to show that $\mathbf{M}_1 \otimes \mathbf{M}_2$ does not depend on underlying Hilbert spaces in algebraical sense. That is, if $\mathbf{M}_1$, $\mathbf{M}_2$ are represented as W^*-algebras on another Hilbert spaces $\mathfrak{K}_1$, $\mathfrak{K}_2$ respectively, then $\mathbf{M}_1 \otimes \mathbf{M}_2$ on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is algebraically $*$-isomorphic (in the following we shall state isomorphic) to the one on $\mathfrak{K}_1 \otimes \mathfrak{K}_2$ (Theorem 1).

The study of the relation between the semi-finite W^*-algebras and the Hilbert algebras is fostered by F. J. Murray and J. von Neumann and developed by J. Dixmier, H. A. Dye, R. Godement, I. E. Segal and many authors. R. Pallu de la Barrière has investigated the direct product of Hilbert algebras. Our second purpose is to study of the relations between the direct product of Hilbert algebras and the one of W^*-algebras which is stated in Theorem 2. This result is the central role for the study of the direct product of semi-finite W^*-algebras. For example, as an application of it, we shall prove the commutation theorem for the direct product in semi-finite case (Theorem 3).

The final section is devoted to the direct product of finite W^*-algebras. We shall show that the direct product of finite W^*-algebras is finite too (Theorem 4) and further we shall consider the type of direct product in semi-finite case. There is a $\natural$-operation in a finite W^*-algebra in the sense of J. Dixmier [2], and we shall consider the relation of $\natural$-operations between the direct product of finite W^*-algebras and them. An approximately finite factor is a factor of type II_1, which has simple construction, and seems fundamental for the study of factors of type II_1. We shall consider the direct product of these factors and prove that this product is approximately finite (Theorem 6). The last theorem states that the fundamental group of the direct product of two finite factors contains the fundamental group of each factors as its subgroup.

1. The direct product of W^*-algebras. In this section, we shall give the definition of the direct product and general theory of it.

By a W^*-algebra we mean a self adjoint weakly closed algebra of bounded linear operators on a Hilbert space. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be W^*-algebras on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. Then the algebraical direct product of $\mathbf{M}_1$ and $\mathbf{M}_2$ can be considered as an operator algebra on the Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ in the sense of F. J. Murray and J. von Neumann [9]. We shall define the *direct product* of $\mathbf{M}_1$ and $\mathbf{M}_2$ as the weak closure of it on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and denote this as $\mathbf{M}_1 \otimes \mathbf{M}_2$. We shall use the notations in [17, 18], for example, $\mathbf{M}_1 \odot \mathbf{M}_2$ means the algebraical direct product of $\mathbf{M}_1$, $\mathbf{M}_2$ and for $\Phi \in \mathbf{M}_1 \odot \mathbf{M}_2$

$$\Phi \simeq \sum_i A_i \times B_i$$

means that Φ contains $\Sigma A_i \times B_i$ as its expression.

In this definition the direct product of W^*-algebras depends on their underlying Hilbert spaces. Our first aim is to show that the direct product of W^*-algebras does not depend on their underlying Hilbert spaces in algebraical sense.

THEOREM 1. *Let* $\mathbf{M}_1$ *be a* W^**-algebra on Hilbert spaces* $\mathfrak{H}_1$ *and* $\mathfrak{K}_1$. *Let* $\mathbf{M}_2$ *be a* W^**-algebra on Hilbert spaces* $\mathfrak{H}_2$ *and* $\mathfrak{K}_2$. *Then the direct product of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *on* $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ *is isomorphic to the one of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *on* $\mathfrak{K}_1 \otimes \mathfrak{K}_2$.

To prove the above, we need some lemmas. The next lemma can be proved without difficulties and we shall omit its proof.

LEMMA 1. *Let* $\mathfrak{H}_1$, $\mathfrak{H}_2$ *be two Hilbert spaces and*

$$\mathfrak{H}_1 = \sum_\alpha \mathfrak{H}_{1,\alpha}, \quad \mathfrak{H}_2 = \sum_\beta \mathfrak{H}_{2,\beta}$$

be direct decompositions of $\mathfrak{H}_1$, $\mathfrak{H}_2$ *respectively, then*

$$\mathfrak{H}_1 \otimes \mathfrak{H}_2 = \sum_{\alpha,\beta} \mathfrak{H}_{1,\alpha} \otimes \mathfrak{H}_{2,\beta}.$$

LEMMA 2. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be two* W^**-algebras on Hilbert spaces* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$ *respectively and suppose that* $\mathbf{M}_1 = \sum_\alpha \mathbf{M}_{1,\alpha}$ *and* $\mathbf{M}_2 = \sum_\beta \mathbf{M}_{2,\beta}$ *be the direct decompositions. Then*

$$\mathbf{M}_1 \otimes \mathbf{M}_2 = \sum_{\alpha,\beta} \mathbf{M}_{1,\alpha} \otimes \mathbf{M}_{2,\beta}.$$

PROOF. By the assumptions, there exists families $\{E_\alpha\}$ in $\mathbf{M}_1$ and $\{F_\beta\}$ in $\mathbf{M}_2$ of mutually orthogonal non-zero central projections satisfying $\sum_\alpha E_\alpha = I_1$ and $\sum_\beta F_\beta = I_2$ where I_1 and I_2 are the units of $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively. Let $\mathfrak{H}_{1,\alpha}$ and $\mathfrak{H}_{2,\beta}$ be the ranges of E_α and F_β, then $\mathfrak{H}_1 = \sum_\alpha \mathfrak{H}_{1,\alpha}$ and $\mathfrak{H}_2 = \sum_\beta \mathfrak{H}_{2,\beta}$ and we can consider $\mathbf{M}_{1,\alpha}$, $\mathbf{M}_{2,\beta}$ as W^*-algebras on $\mathfrak{H}_{1,\alpha}$, $\mathfrak{H}_{2,\beta}$ respectively.

By Lemma 1 we have

$$\mathfrak{H}_1 \otimes \mathfrak{H}_2 = \sum_{\alpha,\beta} \mathfrak{H}_{1,\alpha} \otimes \mathfrak{H}_{2,\beta}.$$

Let Φ be an element of $\sum_{\alpha,\beta} \mathbf{M}_{1,\alpha} \otimes \mathbf{M}_{2,\beta}$, then we can decompose as following: $\Phi = \sum_{\alpha,\beta} \Phi_{\alpha,\beta}$ where $\Phi_{\alpha,\beta} \in \mathbf{M}_{1,\alpha} \otimes \mathbf{M}_{2,\beta}$. We can choose a directed set $\{\Phi_{\alpha,\beta,\lambda}\}$ in $\mathbf{M}_1 \odot \mathbf{M}_2$ which converges to $\Phi_{\alpha,\beta}$ strongly. Let

$$\Phi_{\alpha,\beta,\lambda} \simeq \sum_j A_j^{\alpha,\lambda} \times B_j^{\beta,\lambda} \quad \text{and} \quad \Phi_\lambda \simeq \sum_{\alpha,\beta,j} A_j^{\alpha,\lambda} \times B_j^{\beta,\lambda},$$

then it is obvious that $\Phi_\lambda \in \mathbf{M}_1 \otimes \mathbf{M}_2$ and Φ_λ converges to Φ in strongest topology. This proves that $\Phi \in \mathbf{M}_1 \otimes \mathbf{M}_2$.

By an analogous way, we can show that if $\Phi \in \mathbf{M}_1 \otimes \mathbf{M}_2$ then $\Phi \in \sum_{\alpha,\beta} \mathbf{M}_{1,\alpha} \otimes \mathbf{M}_{2,\beta}$. This proves the lemma.

Now we shall introduce the following notion according to I. E. Segal [14]. An operator algebra $\mathbf{M}$ on a Hilbert space $\mathfrak{H}$ is called an *α-fold copy* of an operator algebra $\mathbf{N}$ on a Hilbert space $\mathfrak{K}$, α being a cardinal number greater than 0, if

(1) there is a set S of power α such that $\mathfrak{H}$ consists of all functions f on S to $\mathfrak{K}$ for which the series $\sum_{x\in S} \|f(x)\|^2$ is convergent, with (f, g) defined as $\sum_{x\in S} (f(x), g(x))$, and

(2) $\mathbf{M}$ consists of all operators A of the form $(Af)(x) = Bf(x)$ for some B in $\mathbf{N}$.

Then we have the following lemma.

LEMMA 3. *Let $\mathbf{M}_1, \mathbf{M}_2$ be W^*-algebras on Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$ and $\mathbf{M}_1$ be an α-fold copy of W^*-algebra $\mathbf{N}_1$ on a Hilbert space $\mathfrak{K}_1$. Then $\mathbf{M}_1 \otimes \mathbf{M}_2$ on the Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is an α-fold copy of $\mathbf{N}_1 \otimes \mathbf{M}_2$ on the Hilbert space $\mathfrak{K}_1 \otimes \mathfrak{H}_2$.*

PROOF. By Lemma 1, there exists a set S of power α such that $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ consists of all functions on S to $\mathfrak{K}_1 \otimes \mathfrak{H}_2$ for which the series $\sum_{x\in S} \|\phi(x)\|^2$ is convergent, with

$$(\phi, \psi) = \sum_{x\in S} (\phi(x), \psi(x)) \text{ for all } \phi,\ \psi \in \mathfrak{H}_1 \otimes \mathfrak{H}_2.$$

Let Φ be any element of $\mathbf{M}_1 \odot \mathbf{M}_2$ and $\Phi \simeq \sum_i A_i \times B_i$. Let ϕ be any element of $\mathfrak{H}_1 \odot \mathfrak{H}_2$ and $\phi \simeq \sum_j f_j \times g_j$. Then there exists A_i' in $\mathbf{N}_1$ with

$$(A_i f_j)(x) = A_i' f_j(x) \quad \text{for all } j.$$

Therefore

$$\Big(\big(\sum_i A_i \times B_i\big)\big(\sum_j f_j \times g_j\big)\Big)(x) = \sum_{i,j} (A_i f_j)(x) \times B_i g_j$$
$$= \sum_{i,j} A_i' f_j(x) \times B_i g_j = \big(\sum_i A_i' \times B_i\big)\big(\sum_j f_j \times g_j\big)(x)$$

and this show that $(\Phi\phi)(x) = \Phi'\phi(x)$ where $\Phi' \simeq \sum_i A_i' \times B_i$. Since $\mathfrak{H}_1 \odot \mathfrak{H}_2$ is dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, we can easily prove that for any vector ϕ in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$

$$(\Phi\phi)(x) = \Phi'\phi(x).$$

It is known that the operator norm of $\Phi \in \mathbf{M}_1 \odot \mathbf{M}_2$ depends on $\mathbf{M}_1$ and $\mathbf{M}_2$, but not on the underlying spaces. Hence Φ is an element in the unit sphere of $\mathbf{M}_1 \odot \mathbf{M}_2$ if and only if Φ' is so. By the above consideration, we have

$$\|\Phi\phi\|^2 = \sum_{x\in S} \|\Phi'\phi(x)\|^2.$$

Now we shall show that the mapping $\Phi \to \Phi'$ is strongly bicontinuous. Let $\{\Phi_\lambda\}$ be a directed set in $\mathbf{M}_1 \odot \mathbf{M}_2$ which converges to 0 strongly, then it is obvious that Φ_λ' converges to 0 strongly. Conversely suppose that Φ_λ' converges to 0 strongly. Let ϕ be any element in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, then for any $\varepsilon > 0$ there exists a finite subset S' of S such that $\sum_{x\in S'} \|\phi(x)^2\| < \varepsilon^2/2$. Since S' is finite we can choose μ such that

$$\|\Phi_\lambda'\phi(x)\|^2 < \frac{\varepsilon^2}{2n} \qquad \text{for } \lambda > \mu$$

where n is the number of S'. Then we have

$$\|\Phi_\lambda\phi\|^2 = \sum_{x\in S} \|\Phi_\lambda'\phi(x)\|^2 < \sum_{x\in S} \|\Phi_\lambda'\phi(x)\|^2 + \sum_{x\in S'} \|\Phi_\lambda'\|^2 \|\phi(x)\|^2 < \varepsilon^2 \text{ for } \lambda > \mu.$$

That is $\|\Phi_\lambda\phi\| < \varepsilon$ for $\lambda > \mu$. This proves that $\Phi \to \Phi'$ is strongly bicontinuous on the unit spheres. Therefore, by a theorem due to I. Kaplansky [7], this mapping can be extended topologically to the mapping from the unit sphere of $\mathbf{M}_1 \otimes \mathbf{M}_2$ to the one of $\mathbf{N}_1 \otimes \mathbf{M}_2$. It is clear that this extended mapping holds the above equation. This proves the lemma.

By the above lemma, we can easily prove the following:

LEMMA 4. *Let $\mathbf{M}_1, \mathbf{M}_2$ be as in the above lemma and suppose that $\mathbf{M}_2$ be a β-fold copy of W^*-algebra $\mathbf{N}_2$ on a Hilbert space $\mathfrak{K}_2$. Then $\mathbf{M}_1 \otimes \mathbf{M}_2$ on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is an $\alpha\beta$-fold copy of $\mathbf{N}_1 \otimes \mathbf{N}_2$ on $\mathfrak{K}_1 \otimes \mathfrak{K}_2$.*

Two projections P, Q in a W^*-algebra is called *equivalent* if there exists a partially isometric operator V in the algebra with $P = VV^*$ and $Q = V^*V$.

LEMMA 5. *Let $\mathbf{M}_1, \mathbf{M}_2$ be W^*-algebras on Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$ respectively. Let E be a projection in $\mathbf{M}_1'$ which is not annihilated by any non-zero central projection of $\mathbf{M}_1$ and moreover suppose that there exist infinitely many projections in $\mathbf{M}_1'$ which are orthogonal to eah other and equivalent to E. Then*

$\mathbf{M}_1 \otimes \mathbf{M}_2$ is isomorphic to $\mathbf{M}_{1E} \otimes \mathbf{M}_2$ where $\mathbf{M}_{1E}$ is the contraction of $\mathbf{M}_1$ on the range of E.

PROOF. Since E is not annihilated by any non-zero central projection of $\mathbf{M}_1'$, $\mathbf{M}_{1E}$ is isomorphic to $\mathbf{M}_1$.

Let $\{E_\alpha\}$ be a maximal family of mutually orthogonal projections in $\mathbf{M}_1'$ each of which is equivalent to E. Then $\{E_\alpha\}$ is an infinite family by the assumption. By the comparison theorem, there exists a central projection F_1 such that

$$EF_1 \succsim (I - \sum E_\alpha)F_1, \; EF_1 \precsim (I - \sum E_\alpha)F_1.$$

It is obvious that each $E_\alpha F_1$ is equivalent to EF_1. Since $\{E_\alpha\}$ is infinite set

$$F_1 = (I - \sum E_\alpha)F_1 + \sum E_\alpha F_1 \precsim \sum E_\alpha F_1 \leqq F_1.$$

This shows that $\sum E_\alpha F_1 \sim F_1$ and hence there exists a family of mutually orthogonal projections $\{P_\mu^1\}$ such that $P_\mu^1 \leqq F_1, P_\mu^1 F_1 \sim P_\mu^1 E$ and $\sum P_\mu' = F_1$. By an analogous way to the above, we get a central projection F_2 which is orthogonal to F_1 and there exists a family of mutually orthogonal projections which are contained in F_2 and each of which is equivalent to $F_2 E$ and the upper bound of them equals to F_2. By the induction, we get a family of mutually orthogonal central projections in $\mathbf{M}_1$ such that for each F_β there exists a family of mutually orthogonal projections $\{P_\mu^\alpha\}$ such that

$$P_\mu^\alpha \sim EF, \; \sum P_\mu^\alpha = F_\alpha.$$

By Lemma 2

$$\mathbf{M}_1 \otimes \mathbf{M}_2 = \sum \mathbf{M}_{1F\alpha} \otimes \mathbf{M}_2 \text{ and } \mathbf{M}_{1E} \otimes \mathbf{M}_2 = \sum (\mathbf{M}_{1E})_{F\alpha} \otimes \mathbf{M}_2.$$

Hence it is sufficient to show that $\mathbf{M}_{1E\alpha} \otimes \mathbf{M}_2$ is isomorphic to $(\mathbf{M}_{1E})_{F\alpha} \otimes \mathbf{M}_2$.

By the above considerations, without loss of generality, we can assume that there exists a family $\{P_\beta\}_{\beta\in\Gamma}$ of mutually orthogonal projections in $\mathbf{M}_1'$ such that $\sum_\beta P_\beta = I$ and each P_β is equivalent to E. Then $\mathbf{M}_1$ is an α-fold copy of $\mathbf{M}_{1E}$ where α is the cardinal of Γ (cf. [8]). Therefore by Lemma 3, $\mathbf{M}_1 \otimes \mathbf{M}_2$ is an α-fold copy of $\mathbf{M}_{1E} \otimes \mathbf{M}_2$. This proves the lemma.

By the above lemma, we have the following:

LEMMA 6. *Let $\mathbf{M}_1$, $\mathbf{M}_2$ be as in the above lemma. Let F be a projection in $\mathbf{M}_2'$ which is not annihilated by any non-zero central projection in $\mathbf{M}_2$ and moreover suppose that there exist infinitely many projections which are orthogonal to each other and equivalent to F. Then $\mathbf{M}_1 \otimes \mathbf{M}_2$ is isomorphic to $\mathbf{M}_{1E} \otimes \mathbf{M}_{2F}$.*

PROOF OF THEOREM 1. We shall consider the identical mapping from $\mathbf{M}_1$ on $\mathfrak{H}_1$ to $\mathbf{M}_1$ on $\mathfrak{K}_1$ and denote this mapping as θ. Then by a theorem due to J. Dixmier [3, Proposition 2], θ can be expressed as product of the following three isomorphisms $\theta_1, \theta_2, \theta_3$, that is, $\theta = \theta_3\theta_2\theta_1$:

$\theta_1(\mathbf{M}_1)$ is an α-fold copy of $\mathbf{M}_1$ for suitable cardinal α,

$\theta_2(\theta_1(\mathbf{M}_1))$ is the contraction of the range of some projection E in $(\theta_1(\mathbf{M}_1))'$,

θ_3 is a spatial isomorphism.
We can choose α such that there exist infinitely many projections in $(\theta_1(\mathbf{M}_1))'$ which are orthogonal to each other and equivalent to E.

Analogously, the identical mapping θ' from $\mathbf{M}_2$ on $\mathfrak{H}_2$ to $\mathbf{M}_2$ on $\mathfrak{K}_2$ can be discribed as $\theta' = \theta'_3\theta'_2\theta'_1$ where θ'_1, θ'_2 and θ'_3 are analogous to θ_1, θ_2 and θ_3.

By Lemma 4 $\mathbf{M}_1 \otimes \mathbf{M}_2$ on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is isomorphic to $\theta_1(\mathbf{M}_1) \otimes \theta_2(\mathbf{M}_2)$ and by Lemma 6 this is isomorphic to $\theta_2(\theta_1(\mathbf{M}_1)) \otimes \theta'_2(\theta'_1(\mathbf{M}_2))$ and the later is isomorphic to $\mathbf{M}_1 \otimes \mathbf{M}_2$, on $\mathfrak{K}_1 \otimes \mathfrak{K}_2$, since θ_3, θ'_3 are spatial isomorphisms. This proves the theorem.

REMARK 1. Recently, T. Turumaru and Z. Takeda obtained another proofs of Theorem 1 independently of the author. These proofs will be appear in this journal.

REMARK 2. By Theorem 1, for the study of purely algebraical properties of direct products, we can consider it free from the underlying Hilbert spaces.

2. **The direct product of Hilbert algebras.** In this section we shall consider the direct product of Hilbert algebras and as an application of it we shall prove the commutation theorem for the direct product in semi-finite case in the sense of E. L. Griffin [5].

According to H. Nakano [11], we shall give the following definition:

A linear manifold $\mathfrak{A}$ in a Hilbert space $\mathfrak{H}$ is called a *Hilbert algebra* if it satisfies the following conditions (1)–(5):

(1) $\mathfrak{A}$ is dense in $\mathfrak{H}$.

(2) $\mathfrak{A}$ is a ring over the complex field.

(3) To each $a \in \mathfrak{A}$, there exists an element $a^* \in \mathfrak{A}$ such that $(ab, c) = (b, a^*), (ba, c) = (b, ca^*)$ for all $b, c \in \mathfrak{A}$.

(4) To each $a \in \mathfrak{A}$, there exists $\alpha_a \geqq 0$ such that $\|au\| \leqq \alpha_a \|u\|$ for all $u \in \mathfrak{A}$.

From the conditions (1)–(4), for each $a \in \mathfrak{A}$, we can define unique bounded linear operators L_a and R_a on $\mathfrak{H}$ with

$$L_a x = ax, \quad R_a x = xa \quad \text{for all } x \in \mathfrak{A}.$$

(5) $L_a x = 0$ ($R_a x = 0$) for all $a \in \mathfrak{A}$ implies $x = 0$.

A Hilbert algebra $\mathfrak{A}$ in a Hilbert space $\mathfrak{H}$ is called to be *maximal* if there is no Hilbert algebra which contains $\mathfrak{A}$ properly. It is known that any Hilbert algebra is uniquely extended to a maximal Hilbert algebra (cf. [6]) and $\mathfrak{H}$ is an *H-system* in the sense of W. Ambrose [1] and moreover its maximal Hilbert algebra is the *bounded algebra* of the *H*-system. Let $\mathfrak{A}$ is a Hilbert algebra in $\mathfrak{H}$ and $\mathfrak{B}$ be its maximal extension, then we can consider a bounded operator L_a on $\mathfrak{H}$ for each $a \in \mathfrak{B}$ which is defined as in (5). The weak closure of $\{L_a | a \in \mathfrak{A}\}$ coincide with that of $\{L_a | a \in \mathfrak{B}\}$ and this closure will be called the *right W*-algebra* of the given Hilbert algebra. Analogously we can define the *left W*-algebra* of the given Hilbert algebra $\mathfrak{A}$ as the weak closure of $\{R_a | a \in \mathfrak{A}\}$. By $\mathbf{L}(\mathfrak{A})$ and $\mathbf{R}(\mathfrak{A})$, we shall mean the left and right

W^*-algebras of $\mathfrak{A}$.

THEOREM 2. *Let* $\mathfrak{A}_1$, $\mathfrak{A}_2$ *be two Hilbert algebras in Hilbert spaces* $\mathfrak{H}_1$, $\mathfrak{H}_2$. *Then* $\mathfrak{A} = \mathfrak{A}_1 \odot \mathfrak{A}_2$ *is a Hilbert algebra in* $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ *and* $\mathbf{L}(\mathfrak{A}) = \mathbf{L}(\mathfrak{A}_1) \otimes \mathbf{L}(\mathfrak{A}_2)$, $\mathbf{R}(\mathfrak{A}) = \mathbf{R}(\mathfrak{A}_1) \otimes \mathbf{R}(\mathfrak{A}_2)$.

PROOF. At first, we shall prove that $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is a Hilbert algebra in $\mathfrak{H}$, that is, $\mathfrak{A}_1 \odot \mathfrak{A}_2$ satisfies the conditions (1)–(5). It is obvious that $\mathfrak{A}_1 \odot \mathfrak{A}_2$ satisfies (1). For any $\phi, \psi \in \mathfrak{A}_1 \odot \mathfrak{A}_2$, we define

$$\phi\,\psi \simeq \sum_{i,j} a_i c_j \times b_i d_j, \quad \phi^* \simeq \sum_i a_i^* \times b_i^*$$

where $\phi \simeq \sum_i a_i \times b_i$ and $\psi \simeq \sum_j c_j \times d_j$. Then it is not difficult to show that $\mathfrak{A}_1 \odot \mathfrak{A}_2$ satisfies (2) and (3).

Let ϕ be as above and ψ_1, ψ_2 be arbitrary elements in $\mathfrak{A}_1 \odot \mathfrak{A}_2$ and $\psi_1 \simeq \sum_j u_j \times v_j, \psi_2 \simeq \sum_k s_k \times t_k$. Then

$$\begin{aligned}(\phi\psi_1, \psi_2) &= \Big(\big(\sum_i a_i \times b_i\big)\big(\sum_j u_j \times v_j\big), \sum_k s_k \times t_k\Big)\\ &= \Big(\sum_{i,j} a_i u_j \times b_i v_j, \sum_k s_k \times t_k\Big) = \sum_{i,j,k} (a_i u_j, s_k)(b_i a_j, t_k)\\ &= \sum_{i,j,k} (u_j, a_i^* s_k)(v_j, b_i^* t_k) = \Big(\sum_j u_j \times v_j, \sum_{i,k} a_i^* s_k \times b_i^* t_k\Big)\\ &= (\psi, \phi^* \psi_2).\end{aligned}$$

Analogously we have $(\psi_1\phi, \psi_2) = (\psi_1, \psi_2\phi^*)$. This shows that $\mathfrak{A}_1 \odot \mathfrak{A}_2$ satisfies the condition (3).

Let ϕ, ψ be as above and put

$$\eta = \max_{1 \leqq i \leqq n} (\|L_{a_i}\|, \|L_{b_i}\|).$$

Then

$$\begin{aligned}\|\phi\psi\|^2 &= \sum_{i,j,k,l} (a_i c_k, a_j c_l)(b_i d_k, b_j d_l)\\ &= \sum_{i,j,k,l} (L_{a_j}^* L_{a_i} c_k, c_l)(L_{b_j}^* L_{b_i} d_k, d_l)\\ &\leqq \sum_{i,j,k,l} \|L_{a_j}^*\| \|L_{a_i}\| \|L_{b_j}^*\| \|L_{b_i}\| \|c_k\| \|c_l\| \|d_k\| \|d_l\|\\ &\leqq m^2\eta^2 \sum_k \|c_k\| \|d_k\| \leqq 2m^2\eta^2 \|\phi\|^2.\end{aligned}$$

This shows that $\mathfrak{A}_1 \odot \mathfrak{A}_2$ satisfies the condition (4).

Let I_1 and I_2 be the unites of $\mathbf{L}(\mathfrak{A}_1)$ and $\mathbf{L}(\mathfrak{A}_2)$ respectively. Then we can choose a directed set $\{L_{a\alpha}\}$ and $\{L_{b\beta}\}$ in the unit spheres of $\{L_a | a \in \mathfrak{A}_1\}$ and $\{L_b | b \in \mathfrak{A}_2\}$ such that $L_{a\alpha}$ and $L_{b\beta}$ converge to I_1 and I_2 strongly. Obviously $\Phi_{\alpha,\beta} \simeq L_{a\alpha} \times L_{b\beta}$ converges to the identity operator on $\mathfrak{H}$ strongly. Let ψ be an element in $\mathfrak{H}$ with $L_\phi\psi = 0$ for all $\phi \in \mathfrak{A}_1 \odot \mathfrak{A}_2$. Then

$$\Phi_{\alpha,\beta}\phi = 0 \text{ for all } \alpha, \beta.$$

This shows that $\phi = 0$, since $\Phi_{\alpha,\beta}$ converges to the identity on $\mathfrak{H}$. That is, $\mathfrak{A}_1 \odot \mathfrak{A}_2$ satisfies the condition (5). Thus $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is a Hilbert algebra in $\mathfrak{H}$.

Now we shall consider the second part of the theorem. It is obvious that $\{L_\phi | \phi \in \mathfrak{A}\}$ is contained in $\mathbf{L}(\mathfrak{A}_1) \odot \mathbf{L}(\mathfrak{A}_2)$. Hence it is sufficient to show that $\{L_\phi | \phi \in \mathfrak{A}\}$ is strongly dense in $\mathbf{L}(\mathfrak{A}_1) \odot \mathbf{L}(\mathfrak{A}_2)$. Let $\Phi \simeq \sum_{i=1}^{n} A_i \times B_i$ be an arbitrary element in $\mathbf{L}(\mathfrak{A}_1) \odot \mathbf{L}(\mathfrak{A}_2)$ and put

$$\lambda = \max_{1 \leqq i \leqq n} (\| A_i \|, \| B_i \|).$$

For any $\psi \in \mathfrak{H}$ and $\varepsilon > 0$, we can choose $\psi_1 \in \mathfrak{H}_1 \odot \mathfrak{H}_2$ such that

$$\| \psi - \psi_1 \| < \varepsilon / 4n\lambda^2.$$

Let $\psi_1 \simeq \sum_{j=1}^{m} x_j \times y_j$ and put

$$\mu = \max_{1 \leqq j \leqq m} (\| x_j \|, \| y_j \|).$$

Then for $A_i, x_1, \ldots, x_m$ there exists A_i' in $\{L_a | a \in \mathfrak{A}_1\}$ with

$$\| (A_i - A_i') x_j \| < \varepsilon / 4nm\lambda\mu \qquad \text{for } j = 1, \ldots, m,$$

and for $B_i, y_1, \ldots, y_m$, there exists B_i' in $\{L_b | b \in \mathfrak{A}_2\}$ with

$$\| (B_i - B_i') y_j \| < \varepsilon / 4nm\lambda\mu \qquad \text{for } j = 1, \ldots, m.$$

By a theorem due to I. Kaplansky [7], we may assume that $\| A_i' \| < \lambda$ and $\| B_i' \| < \lambda$. Let $\Phi \simeq \sum_i A_i' \times B_i'$, then

$$\| (\Phi - \Psi)(\psi - \psi_1) \| \leqq \| \Phi - \Psi \| \, \| \psi - \psi_1 \| \leqq \| \sum_i A_i \times B_i - \sum_i A_i' \times B_i' \| \varepsilon / 4n\lambda^2$$

$$\leqq \Big(\sum_i \| A_i \times B_i + \sum_i A_i' \times B_i' \| \Big) \varepsilon / 4n\lambda^2 \leqq 2n\lambda^2 \varepsilon / 4n\lambda^2 = \varepsilon / 2.$$

$$\| (\Phi - \Psi)\psi_1 \| = \| \Big(\sum_i A_i \times B_i - \sum_i A_i' \times B' \Big) \sum_j x_j \times y_j \|$$

$$\leqq \| \Big(\sum_i A_i \times B_i - \sum_i A' \times B_i \Big) \sum_j x_j \times y_j \|$$

$$+ \| \Big(\sum_i A_i' \times B_i - \sum_i A_i' \times B_i' \Big) \sum_j x_j \times y_j \|$$

$$\leqq \sum_{i,j} \| (A_i - A_i') x_j \| \, \| B_i y_j \| + \sum_{i,j} \| A' x_j \| \cdot \| (B_i - B_i') y_j \|$$

$$< 2mn\lambda\mu\varepsilon / 4mn\lambda\mu = \varepsilon / 2.$$

Hence we have

$$\| (\Phi - \Psi)\psi \| \leqq \| (\Phi - \Psi)(\psi - \psi_1) \| + \| (\Phi - \Psi)\psi_1 \| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that $\{L_\phi | \phi \in \mathfrak{A}\}$ is strongly dense in $\mathbf{L}(\mathfrak{A}_1) \odot \mathbf{L}(\mathfrak{A}_2)$, since Ψ belongs to $\{L_\phi | \phi \in \mathfrak{A}\}$. That is $\mathbf{L}(\mathfrak{A}) = \mathbf{L}(\mathfrak{A}_1) \otimes \mathbf{L}(\mathfrak{A}_2)$. By an analogous way, we can prove that $\mathbf{R}(\mathfrak{A}) = \mathbf{R}(\mathfrak{A}_1) \otimes \mathbf{R}(\mathfrak{A}_2)$.

Our next step is to show the following[1]:

THEOREM 3. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be two semi-finite* W^**-algebras on Hilbert spaces* $\mathfrak{H}_1$ *and* $\mathfrak{H}_2$ *respectively, then we have*

$$(*) \qquad \mathbf{M}_1' \otimes \mathbf{M}_2' = (\mathbf{M}_1 \otimes \mathbf{M}_2)'.$$

In order to prove the above we shall show some lemmas. We shall use the following terminology which was introduced by I. E. Segal [15]. A W^*-algebra is called *standard* if it is unitarily equivalent to the right (or left) W^*-algebra of some Hilbert algebra.

LEMMA 7. *Let* $\mathbf{M}_1, \mathbf{M}_2$ *be two standard* W^**-algebras, then the equation* (*) *is valid.*

PROOF. By the assumptions, $\mathbf{M}_1$ and $\mathbf{M}_2$ can be considered as the right W^*-algebra of Hilbert algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in their underlying Hilbert spaces respectively. By Theorem 2, $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is the right W^*-algebra of the Hilbert algebra $\mathfrak{A}_1 \odot \mathfrak{A}_2$. There $(\mathbf{M}_1 \otimes \mathbf{M}_2)'$ is the left W^*-algebra of $\mathfrak{A}_1 \odot \mathfrak{A}_2$. On the other hand, $\mathbf{M}_1' \otimes \mathbf{M}_2'$ is the left algebra of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ since $\mathbf{M}_1'$ and $\mathbf{M}_2'$ are left W^*-algebras of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. This proves the lemma.

Let $\mathbf{M}$ be an arbitrary W^*-algebra and $\mathbf{N}$ be a full operator algebra on an α-dimensional Hilbert space. According to F. J. Murray and J. von Neumann [10], we shall denote $\mathbf{N} \otimes \mathbf{M}$ by $\mathbf{M}_\alpha$. Then there exist a family $\{P_\mu\}$ in $\mathbf{N}$, of power α, where each P_μ is a minimal projection. Let I_1 and I_2 be units of $\mathbf{N}$ and $\mathbf{M}$ and put

$$\Phi_\mu \simeq P \times I_2,$$

then each Φ_μ is a projection in $\mathbf{N} \otimes \mathbf{M}$ and further they are equivalent to each other. It is clear that the contraction of $(\mathbf{N} \otimes \mathbf{M})'$ on the range of Φ_μ is isomorphic to $I_1 \otimes \mathbf{M}_2'$. Therefore, by an analogous method to [8, Theorem 2], we have the following lemma.

LEMMA 8. *Let* $\mathbf{M}$ *be a* W^**-algebra, then* $(\mathbf{M}^\alpha)'$ *is unitarily equivalent to an* α*-fold copy of* $\mathbf{M}'$.

LEMMA 9. *Let* $\mathbf{M}_1, \mathbf{M}_2$ *be standard* W^**-algebras, then the equation* (*) *is valid for* $\mathbf{M}_1^\alpha, \mathbf{M}_2^\beta$.

PROOF. As we noticed above, $(\mathbf{M}_1^\alpha)'$, and $(\mathbf{M}_2^\beta)'$ are α and β-fold copies of $\mathbf{M}_1'$ and $\mathbf{M}_2'$ respectively. Hence, by Lemma 5, $(\mathbf{M}_1^\alpha)' \otimes (\mathbf{M}_2^\beta)'$ is an $\alpha\beta$-fold copy of $\mathbf{M}_1' \otimes \mathbf{M}_2'$.

By the definition,

$$\mathbf{M}_1^\alpha \otimes \mathbf{M}_2^\beta = \mathbf{N}_1 \otimes \mathbf{M}_1 \otimes \mathbf{N}_2 \otimes \mathbf{M}_2$$

where $\mathbf{N}_1$ and $\mathbf{N}_2$ are factors of type I_α and I_β. It is clear that $\mathbf{N}_1 \otimes \mathbf{M}_1 \otimes \mathbf{N}_2 \otimes \mathbf{M}_2$ is unitarily equivalent to $\mathbf{N}_1 \otimes \mathbf{M}_1 \otimes \mathbf{N}_2 \otimes \mathbf{M}_2$. Therefore

$$\mathbf{M}_1^\alpha \otimes \mathbf{M}_2^\beta = (\mathbf{M}_1 \otimes \mathbf{M}_2)^{\alpha\beta},$$

1) This theorem is proposed by Turumaru in a conversation.

since $\mathbf{N}_1 \otimes \mathbf{N}_2$ is of type $\mathrm{I}_{\alpha\beta}$ by the above lemma. This shows that $(\mathbf{M}_1^\alpha \otimes \mathbf{M}_2^\beta)'$ is an $\alpha\beta$-fold copy of $(\mathbf{M}_1 \otimes \mathbf{M}_2)'$. By the above lemma

$$(\mathbf{M}_1 \otimes \mathbf{M}_2)' = \mathbf{M}_1' \otimes \mathbf{M}_2'.$$

This proves the lemma.

LEMMA 10. *$\mathbf{M}_1$ and $\mathbf{M}_2$ be an α-fold copy of standard W^*-algebra and a β-one of $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively. Then the equation $(*)$ is valid for $\mathbf{M}_1$ and $\mathbf{M}_2$.*

PROOF. By the assumptions $\mathbf{M}_1' = \mathbf{N}_1'^{\alpha}$ and $\mathbf{M}_2' = \mathbf{N}_2'^{\beta}$. Hence

$$\mathbf{M}_1' \otimes \mathbf{M}_2' = \mathbf{N}_1'^{\alpha} \otimes \mathbf{N}_2'^{\beta} = (\mathbf{N}_1' \otimes \mathbf{N}_2')^{\alpha\beta} = (\mathbf{N}_1 \otimes \mathbf{N}_2)'^{\alpha\beta}$$

by Lemma 7 and the proof of the preceding lemma. On the other hand, by Lemma 5, $\mathbf{M}_1 \otimes \mathbf{M}_2$ is an $\alpha\beta$-fold copy of $\mathbf{N}_1 \otimes \mathbf{N}_2$, that is, $(\mathbf{M}_1 \otimes \mathbf{M}_2)' = (\mathbf{N}_1 \otimes \mathbf{N}_2)'^{\alpha\beta}$. This proves the lemma.

LEMMA 11. *Let $\mathbf{M}$ be the direct product of W^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$. Let $\Phi \simeq P \times Q$ where P and Q are projections in $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively, then Φ is a projection in $\mathbf{M}$ and $\mathbf{M}_\Phi$ is isomorphic to $\mathbf{M}_{1P} \otimes \mathbf{M}_{2Q}$.*

PROOF. It is obvious that Φ is a projection. $\mathbf{M}_\Phi$, $\mathbf{M}_{1P} \otimes \mathbf{M}_{2Q}$ are isomorphic to $\Phi\mathbf{M}\Phi$, $P\mathbf{M}_1P \otimes Q\mathbf{M}_2Q$ respectively, hence it is sufficient to show that $(\Phi\mathbf{M}\Phi)$ and $P\mathbf{M}_1P \otimes Q\mathbf{M}_2Q$ are isomorphic to each other. It is clear that $\Phi(\mathbf{M}_1 \odot \mathbf{M}_2)\Phi$ is isomorphic to $P\mathbf{M}_1P \odot Q\mathbf{M}_2Q$ and moreover this isomorphism is spatial one. Since $\Phi(\mathbf{M}_1 \odot \mathbf{M}_2)\Phi$ and $P\mathbf{M}_1P \odot Q\mathbf{M}_2Q$ are weakly closed in $\Phi\mathbf{M}\Phi$ and $P\mathbf{M}_1P \otimes Q\mathbf{M}_2Q$, the above isomorphism can be extended to the isomorphism from $\Phi\mathbf{M}\Phi$ to $P\mathbf{M}_1P \otimes Q\mathbf{M}_2Q$. This proves the lemma.

REMARKS. In the above lemma, we assumed that P and Q lie in $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively. This assumptions is not necessary. The above lemma is true for $P \in \mathbf{M}_1'$ and $Q \in \mathbf{M}_2'$.

LEMMA 12. *Suppose that $(*)$ is valid for W^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$. Let P, Q be projections in $\mathbf{M}_1, \mathbf{M}_2$ respectively and $\Phi \simeq P \times Q$, then we have*

$$(\mathbf{M}_{1P})' \otimes (\mathbf{M}_{2Q})' = (\mathbf{M}_1 \otimes \mathbf{M}_2)'_\Phi.$$

PROOF. It is known that $(\mathbf{M}_{1P})' = \mathbf{M}_{1P}'$ and $(\mathbf{M}_{2Q})' = \mathbf{M}_{2Q}'$ (cf. [9, Lemma 11. 3. 2]). Hence, by the above remark, we have

$$(\mathbf{M}_{1P})' \otimes (\mathbf{M}_{2Q})' = \mathbf{M}_{1P}' \otimes \mathbf{M}_{2Q}' = (\mathbf{M}_1' \otimes \mathbf{M}_2')_\Phi = (\mathbf{M}_1 \otimes \mathbf{M}_2)'_\Phi.$$

PROOF OF THEOREM 3. As we have noticed, $\mathbf{M}_1$ and $\mathbf{M}_2$ isomorphic to the left W^*-algebras of some Hilbert algebras in Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, that is, $\mathbf{M}_1$ and $\mathbf{M}_2$ can be represented as standard W^*-algebras on $\mathfrak{K}_1$ and $\mathfrak{K}_2$ respectively. By an analogous way to the proof of Theorem 1, the identical mapping θ from $\mathbf{M}_1$ on $\mathfrak{K}_1$ to $\mathbf{M}_1$ on $\mathfrak{H}_1$ can be described as $\theta = \theta_3\theta_2\theta_1$ where θ_1, θ_2 are θ_3 are analogous to those in the proof of Theorem 1. We shall write the identical mapping from $\mathbf{M}_2$ on $\mathfrak{K}_2$ to $\mathbf{M}_2$ on $\mathfrak{H}_1$ as $\theta' = \theta_3'\theta_2'\theta_1'$ where θ_1', θ_2' and θ_3' are analogous to those of Theorem 1. Then, by Lemma

7, (*) is true on $\mathfrak{K}_1 \otimes \mathfrak{K}_2$. By Lemmas 10, 11, (*) is valid for $\theta_2(\theta_1(\mathbf{M}_1))$ and $\theta_2'(\theta_1(\mathbf{M}_2))$. Therefore

$$(\mathbf{M}_1 \otimes \mathbf{M}_2)' = \mathbf{M}_1' \otimes \mathbf{M}_2',$$

since $\mathbf{M}_1$ on $\mathfrak{H}_1$ and $\mathbf{M}_2$ on $\mathfrak{H}_2$ are unitarily equivalent to $\theta_2(\theta_1(\mathbf{M}_1))$ and $\theta_2'(\theta_1'(\mathbf{M}))$ respectively.

REMARK. In Theorem 3, we assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are semi-finite. In the general case, Theorem 3 is still open.

3. **The direct product of finite W-*algebras.** In this section, we shall consider the direct product of finite W^*-algebras. A projection P in a W^*-algebra is called *finite* if $P \geq Q$ and $P \sim Q$ imply $P = Q$. A W^*-algebra is called *finite* if the unite is finite. Then a W^*-algebra is *semi-finite* if any non zero central projection contains a non zero finite projection. Let $\mathbf{M}$ be a semi-finite W^*-algebra on a Hilbert space $\mathfrak{H}$, then there exists a Hilbert algebra $\mathfrak{A}$ in a suitable Hilbert space $\mathfrak{K}$ such that $\mathbf{M}$ is isomorphic to the left W^*-algebra of $\mathfrak{A}$ (cf. [12, 15]). Moreover if $\mathbf{M}$ is finite and *σ-finite*, that is, any orthogonal family of projections is at most countable, then $\mathbf{M}$ can be taken as a maximal Hilbert algebra in a suitable Hilbert space [6]. Conversely, it is clear that such maximal Hilbert algebra can be considered as a finite W^*-algebra. The next theorem is stated in [15, §5].

THEOREM 4. *Let* $\mathbf{M}_1, \mathbf{M}_2$ *be finite W^*-algebras, then* $\mathbf{M}_1 \otimes \mathbf{M}_2$ *is a finite W^*-algebra.*

PROOF. Let I_1 and I_2 be the units of $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively. There exists a family $\{E_\alpha\}$ of mutually orthogonal central projections in $\mathbf{M}_1$ such that $\sum E_\alpha = I_1$ and the contraction of $\mathbf{M}_1$ on the range of E_α is σ-finite. Then $\mathbf{M}_1 = \sum \mathbf{M}_{1,\alpha}$ where $\mathbf{M}_{1,\alpha}$ is the contraction of $\mathbf{M}_1$ on the range of E_α. Analogously, we can find a family $\{F_\beta\}$ of mutually orthogonal central projections in $\mathbf{M}_2$ and let $\mathbf{M}_{2,\beta}$ be the contraction of $\mathbf{M}_2$ on the range of F_β. Then, by Lemma 2, we have

$$\mathbf{M}_1 \otimes \mathbf{M}_2 = \sum_{\alpha,\beta} \mathbf{M}_{1,\alpha} \otimes \mathbf{M}_{2,\beta}.$$

If each $\mathbf{M}_{1,\alpha} \otimes \mathbf{M}_{2,\beta}$ is finite, then $\mathbf{M}_1 \otimes \mathbf{M}_2$ is finite. Therefore we can assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are σ-finite without loss of generality.

Now, let $\mathbf{M}_1$, $\mathbf{M}_2$ be finite σ-finite W^*-algebras. Then $\mathbf{M}_1$, $\mathbf{M}_2$ are isomorphic to left W^*-algebras of suitable Hilbert algebra $\mathfrak{A}$, $\mathfrak{A}_2$ in Hilbert spaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ respectively. By Theorem 1, it is sufficient to consider $\mathbf{M}_1 \otimes \mathbf{M}_2$ on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. By Theorem 2,

$$\mathbf{M}_1 \otimes \mathbf{M}_2 = \mathbf{L}(\mathfrak{A}_1) \otimes \mathbf{L}(\mathfrak{A}_2) = \mathbf{L}(\mathfrak{A}_1 \odot \mathfrak{A}_2).$$

It is clear that $\mathbf{L}(\mathfrak{A}_1 \odot \mathfrak{A}_2)$ is isomorphic to the maximal extension of the Hilbert algebra $\mathfrak{A}_1 \odot \mathfrak{A}_2$, since the unit is contained in $\mathfrak{A}_1 \odot \mathfrak{A}_2$. Therefore $\mathbf{M}_1 \otimes \mathbf{M}_2$ is finite.

According to I. E. Segal [14], we shall introduce the following notion:

A W^*-algebra is called *hyper-reducible* if its commutor is commutative. Then the following lemma is obvious.

LEMMA 13. *If* $\mathbf{M}_1, \mathbf{M}_2$ *are hyper-reducible* W^**-algebras, then* $\mathbf{M}_1 \otimes \mathbf{M}_2$ *is so.*

Now we shall consider the type of the direct product.

THEOREM 5. *Let* $\mathbf{M}$ *be the direct product of semi-finite* W^**-algebras* $\mathbf{M}_1$ *and* $\mathbf{M}_2$. *If both* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are of type* I, *then* $\mathbf{M}$ *is of type* I. *If one of* $\mathbf{M}_1$, $\mathbf{M}_2$ *is of type* II *then* $\mathbf{M}$ *is of type* II.

PROOF. Suppose that $\mathbf{M}_1$ and $\mathbf{M}_2$ are of type I, then they can be decomposed as following:

$$\mathbf{M}_1 = \sum_\alpha \mathbf{M}_{1,\alpha} \quad \text{and} \quad \mathbf{M}_2 = \sum_\beta \mathbf{M}_{2,\beta},$$

where $\mathbf{M}_{1,\alpha}$ and $\mathbf{M}_{2,\beta}$ are of type I_α and I_β respectively. By Lemma 2,

$$\mathbf{M} = \sum_{\alpha,\beta} \mathbf{M}_{1,\alpha} \otimes \mathbf{M}_{2,\beta}.$$

If every $\mathbf{M}_{1,\alpha} \otimes \mathbf{M}_{2,\beta}$ is of type I, then $\mathbf{M}$ is so, hence we assume without loss of generality that $\mathbf{M}_1$ and $\mathbf{M}_2$ are of type I_α and I_β respectively. Then $\mathbf{M}_1'$ and $\mathbf{M}_2'$ are α and β-fold copies of hyper-reducible algebras where α and β are suitable cardinals (cf. [9, Theorem 2]). Therefore, by Lemma 4, and the preceding lemma, $\mathbf{M}_1' \otimes \mathbf{M}_2'$ is an $\alpha\beta$-fold copy of a hyper-reducible algebra. Hence, by Theorem 3, $(\mathbf{M}_1 \otimes \mathbf{M}_2)'$ is so, that is, $\mathbf{M}_1 \otimes \mathbf{M}_2$ is of type I.

Next, we shall show that $\mathbf{M}$ is semi-finite. As we noticed, $\mathbf{M}_1$ and $\mathbf{M}_2$ are isomorphic to left W^*-algebras of some Hilbert algebras, say $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. Then $\mathbf{M}_1 \otimes \mathbf{M}_2$ is isomorphic to $\mathbf{L}(\mathfrak{A}_1) \otimes \mathbf{L}(\mathfrak{A}_2) = \mathbf{L}(\mathfrak{A}_1 \odot \mathfrak{A}_2)$. This shows that $\mathbf{M}_1 \otimes \mathbf{M}_2$ is semi-finite.

Finally we suppose that $\mathbf{M}_1$ is of type II. It is known that $\mathbf{M}_1$ and $\mathbf{M}_2$ can be represented as W^*-algebras faithfully on some Hilbert spaces such that $\mathbf{M}_1'$ and are finite on these spaces. It is sufficient to show that $(\mathbf{M}_1 \otimes \mathbf{M}_2)' = \mathbf{M}_1' \otimes \mathbf{M}_2'$ is of type II. If $\mathbf{M}_1' \otimes \mathbf{M}_2'$ is of type I, then we can choose a central projection Φ in $\mathbf{M}_1' \otimes \mathbf{M}_2'$ and an integer p so large that the contraction of $\mathbf{M}_1' \otimes \mathbf{M}_2'$ is of type I_p, since $\mathbf{M}_1' \otimes \mathbf{M}_2'$ is finite. Therefore any family of mutually orthogonal equivalent projections contains at most p elements. On the other hand, it is clear that there exists in this contraction a family of mutually orthogonal equivalent projections which contains more than p elements. This is a contradiction, that is, $\mathbf{M}_1' \otimes \mathbf{M}_2'$ is of type II.

Analogously, we can prove the remainder part of the theorem.

REMARK. In the above theorem, we assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are semi-finite. We can not indicate the status of Theorem 5 when we pass from semi-finite case to general one.

By Theorem 4, $\mathbf{M} = \mathbf{M}_2 \otimes \mathbf{M}_2$ is finite if $\mathbf{M}_1$ and $\mathbf{M}_2$ is finite. Hence there exist $\natural$-operations in $\mathbf{M}, \mathbf{M}_1$ and $\mathbf{M}_2$ in the sense of J. Dixmier [2]. We shall show the following:

THEOREM 6. *Let $\mathbf{M}_1, \mathbf{M}_2$ be finite W^*-algebras and Φ be an element in $\mathbf{M}_1 \odot \mathbf{M}_2$ and let $\Phi \simeq \sum_i A_i \times B_i$. Then $\Phi^\natural \simeq \sum_i A_i^\natural \times B_i^\natural$.*

PROOF. It is sufficient to prove in the case $\Phi \simeq A \times B$. Let $\Phi_0 \simeq A^\natural \times B^\natural$ and $\mathbf{Z}$ be the center of $\mathbf{M} = \mathbf{M}_1 \otimes \mathbf{M}_2$ and $\mathbf{M}_u$ be the unitary group of $\mathbf{M}$. Let $\mathbf{K}_\Phi$ be the smallest uniformly closed convex set which contains $\{\Psi \Phi \Psi^{-1} | \Psi \in \mathbf{M}_u\}$. Then, J. Dixmier [2] proved that $\mathbf{K}_\Phi \cap \mathbf{Z}$ is not empty and contains only the element $\Phi^\natural$. On the other hand, for every $\varepsilon > 0$ there exist unitary elements $U_1, \ldots, U_n$ in $\mathbf{M}_1, V_1, \ldots, V_m$ in $\mathbf{M}_2$ and $\lambda_i > 0, \mu_j > 0$ $(\Sigma\lambda_i = \Sigma\mu_j = 1)$ such that

$$\| A^\natural - \sum_i \lambda_i U_i A U_i \| < \frac{\varepsilon}{2\nu}, \ \| B^\natural - \sum_j \mu_j V_j B V_j \| < \frac{\varepsilon}{2\nu}$$

where $\nu = \max (\| A \|, \| B^\natural \|)$. Let

$$\Phi_1 \simeq \sum_{i,j} \lambda_i \mu_j (U_i \times V_j)(A \times B)(U_i \times V_j)^{-1}.$$

Then

$$\|\Phi_0 - \Phi_1\| = \|A^\natural \times B^\natural - \sum_{ji,} \lambda_i \mu_j (U_i \times V_j)(A \times B)(U_i \times V_j)^{-1}\|$$

$$\leqq \| A^\natural \times B^\natural - \sum_i \lambda_i U_i A U_i \times B^\natural \| + \| \sum_i \lambda_i U_i A U_i \times B^\natural -$$

$$- \sum_{i,j} \lambda_i \mu_j (U_i \times V_j)(A \times B)(U_i \times V_j)^{-1} \|$$

$$\leqq \| A^\natural - \sum_i \lambda_i U_i A U_i^{-1} \| \cdot \| B \| + \| \sum_i \lambda_i U_i A U_i \| \cdot \| B^\natural - \sum_j \mu_j V_j B V_j^{-1} \|$$

$$< (\| A^\natural \| + \| A \|)\varepsilon/2\nu \leqq \varepsilon.$$

That is, Φ_0 can be approximated by the elements in $\mathbf{K}_\Phi$ uniformly. This proves $\Phi_0 = \Phi^\natural$ since Φ_0 is in $\mathbf{Z}$.

In the following, we shall study the direct product of factors.

LEMMA 14. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ are factors on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively, then $\mathbf{M}_1 \otimes \mathbf{M}_2$ is a factor.*

PROOF. It is clear that $\mathbf{M}_1 \otimes \mathbf{M}_2$ is generated by $\mathbf{M}_1 \otimes I_2$ and $I_1 \otimes \mathbf{M}_2$ where I_1 and I_2 are units of $\mathbf{M}_1$ and $\mathbf{M}_2$. Let $\mathbf{B}_1$, $\mathbf{B}_2$ be the full operator algebras on $\mathfrak{H}_1$, $\mathfrak{H}_2$ respectively and put $\mathbf{P} = \mathbf{B}_1 \otimes I_2$. Then

$$\mathbf{M}_1 \otimes I_2 \subset \mathbf{P} \text{ and } I_1 \otimes \mathbf{M}_2 \subset \mathbf{P}'.$$

By Theorem 5, $\mathbf{P}$ and $\mathbf{P}'$ are of type I and so they are normal. Hence, by a lemma due to F. J. Murray and J. von Neumann [9, Lemma 11.1.3], $\mathbf{M}_1 \otimes \mathbf{M}_2$ is a factor.

By Theorem 6 and Lemma 14, we have the following:

COROLLARY. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be finite factors, then* $\mathbf{M} = \mathbf{M}_1 \otimes \mathbf{M}_2$ *is a finite factor. Let* τ, τ_1, τ_2 *be the traces of* $\mathbf{M}, \mathbf{M}_1, \mathbf{M}_2$ *respectively. Then*

$$\tau(\Phi) = \sum_i \tau_1(A_i)\tau_2(B_i)$$

where $\Phi \in \mathbf{M}_1 \odot \mathbf{M}_2$ *and* $\Phi \simeq \sum_i A_i \times B_i$.

In the following we shall consider only direct product of finite factors. The really interesting is the direct product of factors of type II_1. We shall use terminologies *algebraical type*, *approximate finiteness* and *fundamental group* which are introduced by F. J. Murray and J. von Neumann[10].

THEOREM 7. *Let* $\mathbf{M}_1$, $\mathbf{M}_2$ *be two approximately finite factors, then* $\mathbf{M}_1 \otimes \mathbf{M}_2$ *is an approximately finite factor.*

PROOF. Let τ, τ_1, τ_2 be traces of $\mathbf{M} = \mathbf{M}_1 \otimes \mathbf{M}_2, \mathbf{M}_1, \mathbf{M}_2$ and $[[\cdot,\cdot]], [[\cdot,\cdot]]_1, [[\cdot,\cdot]]_2$ be norms of the pre-hilbert spaces which are generated by $\mathbf{M}, \tau$; $\mathbf{M}_1, \tau_1$; $\mathbf{M}_2, \tau_2$ respectively. Then for any $\Phi_1, \ldots, \Phi_m \in \mathbf{M}$ and $\varepsilon > 0$, there exist $\Psi_1, \ldots, \Psi_m \in \mathbf{M}_1 \odot \mathbf{M}_2$ such that

$$[[\Phi_n - \Psi_n]] < \frac{\varepsilon}{2} \text{ for } n = 1, \ldots, m.$$

Let $\Phi_n \simeq \sum_{i=1}^{p(n)} A_{n,i} \times B_{n,i}$ and put

$$\mu = \max(\|A_{n,i}\|, \|B_{n,i}\|) \text{ and } p = \max_n (p(n)).$$

Since $\mathbf{M}_1$ is approximately finite, for $A_{n,i}$ $(n = 1, \ldots, m; i = 1, \ldots (pn))$ and $\varepsilon/4\mu p$, there exists a subring $\mathbf{N}_1$ of $\mathbf{M}_1$ with following properties:

(i) $\mathbf{N}_1$ is of finite order,

(ii) there exist $A'_{n,i}$ such that

$$[[A_{n,i} - A'_{n,i}]]_1 < \varepsilon/4\mu p \text{ for } n = 1, \ldots, m;\ i = 1, \ldots, p(n).$$

We choose an analogous subring $\mathbf{N}_2$ of $\mathbf{M}_2$ and its elements $B'_{n,i}$.

Let

$$\Psi'_n \simeq \sum_{i=1}^{p(n)} A'_{n,i} \times B'_{n,i}$$

then

$$[[\Psi_n - \Psi'_n]] = [[\sum_i A_{n,i} \times B_{n,i} - \sum_i A'_{n,i} \times B'_{n,i}]]$$

$$\leqq [[\sum_i A_{n,i} \times B_{n,i} - \sum_i A'_{n,i} \times B_{n,i}]] + [[\sum_i A'_{n,i} \times B_{n,i} - \sum_i A'_{n,i} \times B'_{n,i}]]$$

$$\leqq \sum_i \mu[[A_{n,i} - A'_{n,i}]]_1 + \sum_i \mu[[B_{n,i} - B'_{n,i}]]_2 < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

Accordingly

$$[[\Phi_n - \Psi'_n]] \leqq [[\Phi_n - \Psi_n]] + [[\Psi_n - \Psi'_n]] < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

It is obvious that $\mathbf{N} = \mathbf{N}_1 \otimes \mathbf{N}_2$ is of finite order and $\Psi'_n \in \mathbf{N}$. This proves the theorem.

Since all approximately finite factors are isomorphic [10], we have the following:

COROLLARY. *Let* $\mathbf{M}_1, \mathbf{M}_2$ *be two approximately finite factors, then* $\mathbf{M}_1 \otimes \mathbf{M}_2$ *and* $\mathbf{M}_1$, $\mathbf{M}_2$ *have the same algebraical type.*

Finally we shall consider the fundamental groups of direct products.

THEOREM 8. *Let* $\mathbf{M}$ *be the direct product of finite factors* $\mathbf{M}_1$, $\mathbf{M}_2$. *Then the fundamental group of* $\mathbf{M}$ *contains the fundamental groups of* $\mathbf{M}_1, \mathbf{M}_2$ *as its subgroups.*

PROOF. Let $\mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2$ be the fundamental groups of $\mathbf{M}, \mathbf{N}_1, \mathbf{M}_2$ respectively. It is sufficient to prove that if $\theta \in \mathfrak{G}_1$ then θ is contained in $\mathfrak{G}$.

Let τ, τ_1, τ_2 be traces of $\mathbf{M}, \mathbf{M}_1, \mathbf{M}_2$ respectively. By the assumption, there exists a projection P in $\mathbf{M}_1$ such that $\tau_1(P) = \theta$ and $\mathbf{M}_{1P}$ is isomorphic to $\mathbf{M}_1$. Let $\Phi \simeq P \times I_2$ where I_2 is the unit of $\mathbf{M}_2$. Then

$$\tau(\Phi) = \tau_1(P) \cdot \tau_2(I_2) = \theta.$$

Hence it is sufficient to show that $\mathbf{M}_\Phi$ is isomorphic to $\mathbf{M}$. By an analogous way to the proof of Theorem 1, we can show that $\mathbf{M}_{1P} \otimes \mathbf{M}_2$ and $\mathbf{M}_1 \otimes \mathbf{M}_2$ have same algebraical type. By Lemma 11, $\mathbf{M}_{1P} \otimes \mathbf{M}_2$ is isomorphic to $\mathbf{M}_\Phi$. That is, $\mathbf{M}$ is isomorphic to $\mathbf{M}_\Phi$. This proves the theorem.

It is known that the fundamental group of an approximately finite factor contains all θ in $0 < \theta < \infty$ [10]. Hence we have the following:

COROLLARY. *Let* $\mathbf{M}_1, \mathbf{M}_2$ *be finite factors and suppose that one of them is approximately finite, then the fundamental group of* $\mathbf{M}_1 \otimes \mathbf{M}_2$ *contains all* θ *in* $0 < \theta < \infty$.

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