ON GENERAL ERGODIC THEOREMS

SHIGERU TSURUMI

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1. The purpose of this paper is to state the general ergodic theorems which are the extensions of those of J. L. Doob [1][1], E. Hopf [3] and A. Khintchine [4]. The idea of the proofs, which appeal to the representation into an infinite product space, is due to Doob [1].

2. Let $(X, \mathfrak{B}, \mu)$ be the measure space such that X is an abstract set, $\mathfrak{B}$ is a Borel field of subsets of X, and μ is a σ-finite measure on $\mathfrak{B}$.

Throughout this paper, unless otherwise stated, m, n will be any positive integers, h any non-negative integer, $(n_1, n_2, \ldots, n_m)$ any finite sequence of positive integers with $n_1 < n_2 < \ldots < n_m$, and $(n_1, n_2, \ldots)$ any infinite sequence of positive integers with $n_1 < n_2 < \ldots$.

DEFINITION 1. A sequence of measurable functions $\{f_j(x)\}_{j=1,2,\ldots}$ is said to have the ***homogeneity property*** (which we denote by h. p.) provided that, for any Borel set A in m-dimensional space, $\mu(\{x;\ (f_{n_1+h}(x), f_{n_2+h}(x), \ldots, f_{n_m+h}(x)) \in A\})$ is independent of h.

DEFINITION 2. Two sequences of measurable functions $\{f_j(x)\}_{j=1,2,\ldots}$ and $\{g_j(x)\}_{j=1,2,\ldots}$ are said to have the *combined homogeneity property* (which we denote by c. h. p.) provided that, for any Borel set A in $2m$-dimensional space, $\mu(\{x;\ (f_{n_1+h}(x), g_{n_1+h}(x), \ldots, f_{n_m+h}(x), g_{n_m+h}(x)) \in A\})$ is independent of h.

DEFINITION 3. Two sequences of measurable functions $\{f_j(x)\}_{j=1,2,\ldots}$ and $\{g_j(x)\}_{j=1,2,\ldots}$ are said to have the *weak combined homogeneity property* (which we denote by w. c. h. p.) provided that there exists a constant K such that, for any Borel set A in $2m$-dimensional space and for any positive integer k,

$$\limsup_{n} \frac{1}{n} \sum_{h=0}^{n-1} \mu(\{x;\ (f_{n_1+hk}(x), g_{n_1+hk}(x), \ldots, f_{n_m+hk}(x), g_{n_m+hk}(x)) \in A\})$$
$$\leqq K \cdot \mu(\{x;\ (f_{n_1}(x), g_{n_1}(x), \ldots, f_{n_m}(x), g_{n_m}(x)) \in A\}).$$

The following fundamental properties are easily seen.

1°. If the sequences $\{f_j(x)\}$ and $\{g_j(x)\}$ have c. h. p., they have w. c. h. p. and each of them has h. p.;

2°. If the sequence $\{f_j(x)\}$ has h. p., then two sequences $\{f_j(x)\}$ and $\{1\}$ have c. h. p.;

3°. If the sequences $\{f_j(x)\}$ and $\{g_j(x)\}$ have c. h. p., then the measure theoretic property of $f_1(x)[g_1(x)]$ implies the same property of every $f_j(x)$ [every $g_j(x)$]: for example, when $f_1(x)$ is μ-integrable, so does every $f_j(x)$, and when $g_1(x) > 0$ for μ-almost every $x \in X$, so does every $g_j(x)$;

4°. For a single valued, measurable, μ-measure preserving[2] transfor-

1) Numbers in square brackets refer to the references at the end of this paper.

2) A single valued transformation T of X into itself is called *measurable* if the inverse transformation T^{-1} transforms every set of $\mathfrak{B}$ to a set of $\mathfrak{B}$, and a measurable transformation T is called *μ-measure preserving* if $\mu(T^{-1}Y) = \mu(Y)$ for every $Y \in \mathfrak{B}$.

mation T of X into itself and for any measurable functions $f(x)$ and $g(x)$ we put

$$f_j(x) = f(T^{j-1}x), \quad g_j(x) = g(T^{j-1}x) \qquad (x \in X;\ j = 1, 2, \ldots.),$$

then the sequences $\{f_j(x)\}$ and $\{g_j(x)\}$ have c. h. p.

Our main theorem reads as follows:

THEOREM 1. *Let $\{f_j(x)\}$ and $\{g_j(x)\}$ be two sequences of measurable functions with c. h. p. such that $f_1(x)$ is μ-integrable and $g_1(x) > 0$, $\sum_{j=1}^{\infty} \min\{g_j(x), k\} = +\infty$ for μ-almost every $x \in X$ with respect to some positive k. Let $p(j)$ and $q(j)$ be two periodic functions defined on the positive integers such that $q(j) > 0$ for all j. Then the limit*

$$\lim_n \frac{\sum_{j=1}^{n} f_j(x)p(j)}{\sum_{j=1}^{n} g_j(x)q(j)}$$

exists and is finite for μ-almost every $x \in X$.

In particular, if $p(j) = q(j) = 1$ for all j, the assumption that $\sum_{j=1}^{\infty}\min\{g_j(x), k\} = +\infty$ is replaced by the assumption that $\sum_{j=1}^{\infty} g_j(x) = +\infty$.

The proof of Theorem 1 appears in section 3.

By 2° and Theorem 1 we get

COROLLARY 1.1. (*Doob's ergodic theorem*). *Let $\{f_j(x)\}$ be a sequence of measurable functions with h.p. such that $f_1(x)$ is μ-integrable. Then the limit*

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} f_j(x)$$

exists and is finite for μ-almost every $x \in X$.

By 4° and Theorem 1 we get

COROLLARY 1.2. (*Hopf's ergodic theorem*). *Let T be a single valued, μ-measure preserving transformation of X into itself. Let $f(x)$ be any μ-integrable function, and let $g(x)$ be any measurable function such that $g(x) > 0$, $\sum_{j=0}^{\infty} g(T^j x) = +\infty$ for μ-almost every $x \in X$*[3]*. Then the limit*

3) If T is single valued, measurable, μ-measure preserving and μ-*incompressible* (that is, $Y \in \mathfrak{B}$ and $T^{-1}Y \supset Y$ imply $\mu(T^{-1}Y - Y) = 0$) and if $g(x)$ is any measurable function such that $g(x) > 0$ for μ-almost every $x \in X$, then $\sum_{j=0}^{\infty} g(T^j x) = +\infty$ for μ-almost every $x \in X$. (See [3, 2].)

We note here that if μ is finite and if T is single valued, measurable and μ-measure preserving, then T is μ-incompressible.

$$\lim_n \frac{\sum_{j=0}^{n-1} f(T^j x)}{\sum_{j=0}^{n-1} g(T^j x)}$$

exists and is finite for μ-almost every $x \in X$.

By 4° and Theorem 1 we get

COROLLARY 1.3. (*Khintchine's ergodic theorem*). *Let T be a single valued, measurable, μ-measure preserving transformation of X into itself. Let $f(x)$ be any μ-integrable function, and let $p(j)$ be any periodic function defined on the positive integers. Then the limit*

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} f(T^{j-1}x)p(j)$$

exists and is finite for μ-almost every $x \in X$.

In case the measure μ is finite, Theorem 1 may be moreover generalized as follows:

THEOREM 2. *Let μ be finite. Let $\{f_j(x)\}$ and $\{g_j(x)\}$ be two sequences of measurable functions with w.c.h.p. such that every $f_j(x)$ is μ-integrable and every $g_j(x)$ is positive for μ-almost every $x \in X$. Let $p(j)$ and $q(j)$ be two periodic functions defined on the positive integers such that $q(j) > 0$ for all j. Then the limit*

$$\lim_n \frac{\sum_{j=1}^{n} f_j(x)p(j)}{\sum_{j=1}^{n} g_j(x)q(j)}$$

exists and is finite for μ-almost every $x \in X$.

The proof of Theorem 2 appears in section 4.

It is easy to see that Theorem 2 contains the following result which is a part of the theorem of Ryll-Nardzewski [5, 6].

COROLLARY 2.1. *Let μ be finite. Let T be a single valued, measurable transformation of X into itself with respect to which there exists a constant K such that for any $Y \in \mathfrak{B}$*

$$\limsup_n \frac{1}{n} \sum_{h=0}^{n-1} \mu(T^{-h}Y) \leqq K \cdot \mu(Y).$$

Let $f(x)$ be any μ-integrable function. Then the limit

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$$

exists and is finite for μ-almost every $x \in X$.

3. In this section we shall prove Theorem 1.

If we put

$$f^{j+}(x) = \max \{f_j(x), 0\}, \quad f_j^-(x) = \max\{-f_j(x), 0\},$$

$$p^+(j) = \max\ \{p(j), 0\}, \quad p^-(j) = \max\{-p(j), 0\},$$

then $\{f_j^+(x)\}$ and $\{g_j(x)\}$ [$\{f_j^-(x)\}$ and $\{g_j(x)\}$] has c. h. p. and

$$\lim_n \frac{\sum_{j=1}^n f_j(x)p(j)}{\sum_{j=1}^n g_j(x)q(j)} = \lim_n \frac{\sum_{j=1}^n f_j^+(x)p^+(j)}{\sum_{j=1}^n g_j(x)q(j)} + \lim_n \frac{\sum_{j=1}^n f_j^-(x)p^-(j)}{\sum_{j=1}^n g_j(x)q(j)}$$

$$- \lim_n \frac{\sum_{j=1}^n f_j^+(x)p^-(j)}{\sum_{j=1}^n g_j(x)q(j)} - \lim_n \frac{\sum_{j=1}^n f_j^-(x)p^+(j)}{\sum_{j=1}^n g_j(x)q(j)},$$

whenever four limits in the right hand side exist, so that in the sequel we may assume that $f_j(x) \geqq 0$ for μ-almost every $x \in X$ and for all j and that $p(j) \geqq 0$ for all j.

Let Ω be the space of all one-sided infinite sequences of real numbers, where we denote a point of Ω by $\omega = (\xi_1, \xi_2 \ldots.)$. Let A be any Borel set in m-dimensional space. If we put for given $n_1, n_2, \ldots., n_m$

$$\Lambda = \{(\xi_1, \xi_2, \ldots.); (\xi_{n_1}, \xi_{n_2}, \ldots., \xi_{n_m}) \in A\},$$

then Λ is an ω-set. Let $\mathfrak{B}_\Omega(n_1, n_2, \ldots., n_m)$ be the Borel field determined by such ω-sets for given $n_1, n_2, \ldots., n_m$, let $\mathfrak{B}_\Omega^0$ denote the field $\bigcup_{(n_1,n_2,\ldots,n_m)} \mathfrak{B}_\Omega(n_1, n_2, \ldots., n_m)$, and let $\mathfrak{B}_\Omega$ be the Borel field generated by the sets of $\mathfrak{B}_\Omega^0$. Further, let $F_j(x)$'s be the functions defined by

$$F_{2j-1}(x) = f_j(x)p(j), \quad F_{2j}(x) = g_j(x)q(j) \qquad (x \in X;\ j = 1, 2, \ldots.),$$

and let φ be the transformation of X into Ω defined by

$$\varphi x = (F_1(x), F_2(x), \ldots.) \qquad (x \in X).$$

We now define α by

$$\alpha(\Lambda) = \mu(\varphi^{-1}\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega^0).$$

Then α is a non-negative, finitely additive set function on $\mathfrak{B}_\Omega^0$ such that α is a measure on each $\mathfrak{B}_\Omega(n_1, n_2, \ldots., n_m)$. Further, if

$$\Lambda \in \mathfrak{B}_\Omega^0,\ \Lambda_n \in \mathfrak{B}_\Omega^0,\ \Lambda_n \cap \Lambda_m = 0\ (n \neq m),\ \Lambda = \bigcup_{n=1}^\infty \Lambda_n,$$

then

$$\sum_{n=1}^\infty \alpha(\Lambda_n) = \sum_{n=1}^\infty \mu(\varphi^{-1}\Lambda_n) = \mu\left(\bigcup_{n=1}^\infty \varphi^{-1}\Lambda_n\right) = \mu\left[\varphi^{-1}\left(\bigcup_{n=1}^\infty \Lambda_n\right)\right] = \mu(\varphi^{-1}\Lambda) = \alpha(\Lambda).$$

Hence, by well known extension theorem[4], the definition of α can be extended on all sets of $\mathfrak{B}_\Omega$: precisely speaking, if we define $\bar{\alpha}$ by

4) It is usually proved under the assumption that $\alpha(\Omega) = 1$, but it remains true in case $\alpha(\Omega) = +\infty$.

$$\bar{\alpha}(\Lambda) = \inf\left\{\sum_{n=1}^{\infty}\alpha(\Lambda_n);\ \Lambda_n \in \mathfrak{B}_\Omega^0,\ \Lambda \subset \bigcup_{n=1}^{\infty}\Lambda_n\right\} \qquad (\Lambda \in \mathfrak{B}_\Omega), \tag{1}$$

then $\bar{\alpha}$ is a measure on $\mathfrak{B}_\Omega$ such that

$$\bar{\alpha}(\Lambda) = \alpha(\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega^0). \tag{2}$$

Let t be the least common multiple of both periods of the periodic functions $p(j)$ and $q(j)$, and let S be the transformation of Ω onto itself defined by

$$S\omega = \omega',$$

where

$$\omega = (\xi_1, \xi_2, \dots\dots),\ \omega' = (\xi_1', \xi_2', \dots\dots),\ \xi_j' = \xi_{j+2t}\ (j = 1, 2, \dots\dots).$$

Then S is single valued (but not one to one), measurable and satisfies

$$\alpha(S^{-1}\Lambda) = \alpha(\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega^0). \tag{3}$$

In fact, if we put for every $\Lambda \in \mathfrak{B}_\Omega(n_1, n_2, \dots\dots, n_m)$

$$A = \{(\xi_{n_1}, \xi_{n_2}, \dots\dots, \xi_{n_m});\ (\xi_1, \xi_2, \dots\dots, \xi_{n_1}, \dots\dots, \xi_{n_2}, \dots\dots, \xi_{n_m}, \dots\dots) \in \Lambda\}),$$

then A is a Borel set in m-dimensional space, and

$$\alpha(\Lambda) = \mu(\varphi^{-1}\Lambda) = \mu(\{x;\ (F_{n_1}(x), F_{n_2}(x), \dots\dots, F_{n_m}(x)) \in A\})$$
$$= \mu(\{x;\ (F_{n_1+2t}(x), F_{n_2+2t}(x), \dots\dots, F_{n_m+2t}(x)) \in A\}) = \mu[\varphi^{-1}(S^{-1}\Lambda)] = \alpha(S^{-1}\Lambda)$$

on account of c. h. p. of $\{f_j(x)\}$ and $\{g_j(x)\}$.

Next, note that to every covering of Λ with the sets Λ_n 's there corresponds a covering of $S^{-1}\Lambda$ with the sets $S^{-1}\Lambda_n$'s, but the converse is not necessarily valid, since S is not one to one. Thus, by (3), we have

$$\begin{aligned} \bar{\alpha}(\Lambda) &= \inf\left\{\sum_{n=1}^{\infty}\alpha(\Lambda_n);\ \Lambda_n \in \mathfrak{B}_\Omega^0,\ \ \Lambda \subset \bigcup_{n=1}^{\infty}\Lambda_n\right\} \\ &= \inf\left\{\sum_{n=1}^{\infty}\alpha(S^{-1}\Lambda_n);\ \Lambda_n \in \mathfrak{B}_\Omega^0,\ \ \Lambda \subset \bigcup_{n=1}^{\infty}\Lambda_n\right\} \\ &\geqq \inf\left\{\sum_{n=1}^{\infty}\alpha(\Lambda_n');\ \Lambda_n' \in \mathfrak{B}_\Omega^0,\ \ S^{-1}\Lambda \subset \bigcup_{n=1}^{\infty}\Lambda_n'\right\} = \bar{\alpha}(S^{-1}\Lambda) \quad (\Lambda \in \mathfrak{B}_\Omega). \end{aligned} \tag{4}$$

Let us now put

$$\beta(\Lambda) = \lim_n \bar{\alpha}(S^{-n}\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega).$$

Then β has the following properties:

$$\beta(\Lambda) \leqq \bar{\alpha}(\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega); \tag{5}$$

$$\beta(S^{-1}\Lambda) = \beta(\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega); \tag{6}$$

$$\beta(\Lambda) = \bar{\alpha}(\Lambda) = \alpha(\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega^0); \tag{7}$$

(8) If Λ^* is of finite $\bar{\alpha}$-measure, then $\beta(\Lambda \cap \Lambda^*)$ is a measure on $\mathfrak{B}_\Omega$ as the set function of Λ.

The properties (5) and (6) are evident, and (7) is an immediate consequence of (2) and (3). By use of (5), the property (8) follows from the facts that β is a non-negative, finitely additive set function on $\mathfrak{B}_\Omega$ and that

$\overline{\alpha}(\Lambda \cap \Lambda^*)$ is a finite measure on $\mathfrak{B}_\Omega$ as the set function of Λ.

Let Λ_m^n 's be the ω-sets defined by

$$\Lambda_m^n = \{(\xi_1, \xi_2, \ldots, \xi_{2m-1}, \ldots); \ |\xi_{2m-1}| \geqq 1/n\},$$

then

$$\Lambda_m^n \in \mathfrak{B}_\Omega^0, \quad \overline{\alpha}(\Lambda_m^n) = \alpha(\Lambda_m^n) < \infty,$$

since every $F_{2m-1}(x)$ is μ-integrable. Let us put

$$\Omega_0 = \Omega - \bigcup_{n,m=1}^{\infty} \Lambda_m^n, \quad \widetilde{\Omega}_0 = \bigcup_{h=0}^{\infty} S^{-h}\Omega_0, \quad \widetilde{\Omega} = \widetilde{\Omega} - \widetilde{\Omega}_0.$$

Then $\widetilde{\Omega}_0$ is an invariant set: that is, $S^{-1}\widetilde{\Omega}_0 = \widetilde{\Omega}_0$, and further $\widetilde{\Omega}$ is an invariant set which is the sum of countably many sets, of $\mathfrak{B}_\Omega$, of finite $\overline{\alpha}$-measure. Suppose that

$$\Omega_n \in \mathfrak{B}_\Omega, \quad \overline{\alpha}(\Omega_n) < \infty, \quad \Omega_n \cap \Omega_m = 0 \quad (n \neq m), \ \widetilde{\Omega} = \bigcup_{n=1}^{\infty} \Omega_n.$$

We define γ by

$$\gamma(\Lambda) = \sum_{n=1}^{\infty} \beta(\Lambda \cap \Omega_n) + \overline{\alpha}(\Lambda \cap \widetilde{\Omega}_0) \qquad (\Lambda \in \mathfrak{B}_\Omega).$$

Then γ has the following properties:

(9) $$\gamma(\Lambda) \leqq \overline{\alpha}(\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega);$$

(10) γ is a measure on $\mathfrak{B}_\Omega$ such that $\widetilde{\Omega}$ is the sum of countably many sets, of $\mathfrak{B}_\Omega$, of finite γ-measure:

(11) $$\gamma(S^{-1}\Lambda) = \gamma(\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega, \Lambda \subset \widetilde{\Omega});$$

(12) If $\Lambda \in \mathfrak{B}_\Omega$, $\gamma(\Lambda) = 0$, then $\overline{\alpha}(\Lambda) = 0$.

The property (9) is evident, and (10) follows immediately from (8) and (9). We shall now prove (11). Note that every $S^{-1}\Omega_n$, with Ω_n, is of finite $\overline{\alpha}$-measure, then by use of (6) and (8) we have

$$\begin{aligned}\gamma(\Lambda) &= \sum_{n=1}^{\infty} \beta(\Lambda \cap \Omega_n) = \sum_{n=1}^{\infty} \beta[S^{-1}(\Lambda \cap \Omega_n)] \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \beta(S^{-1}\Lambda \cap S^{-1}\Omega_n \cap \Omega_m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \beta(S^{-1}\Lambda \cap S^{-1}\Omega_n \cap \Omega_m) \\ &= \sum_{m=1}^{\infty} \beta(S^{-1}\Lambda \cap \Omega_m) = \gamma(S^{-1}\Lambda) \qquad (\Lambda \in \mathfrak{B}_\Omega, \Lambda \subset \widetilde{\Omega}).\end{aligned}$$

For the proof of (12) it is sufficient to show that if $\Lambda \in \mathfrak{B}_\Omega$, $\Lambda \subset \widetilde{\Omega}$ and $\overline{\alpha}(\Lambda) > 0$, then $\gamma(\Lambda) > 0$. It is no loss of generality to assume that $\overline{\alpha}(\Lambda) < \infty$ and that, in the right hand side of (1), $\Lambda_n \cap \Lambda_m = 0$ $(n \neq m)$. Hence, by the definition of $\overline{\alpha}$, there exist the sets Λ_n's such that

$$\Lambda_n \in \mathfrak{B}_\Omega^0, \ \Lambda_n \cap \Lambda_m = 0 \ (n \neq m), \ \Lambda \subset \bigcup_{n=1}^{\infty} \Lambda_n, \ \overline{\alpha}\Big(\bigcup_{n=1}^{\infty} \Lambda_n\Big) = \sum_{n=1}^{\infty} \alpha(\Lambda_n) < \infty.$$

By (5), (8) and (7)

$$\gamma(\Lambda)=\sum_{n=1}^{\infty}\beta(\Lambda\cap\Omega_n)=\beta(\Lambda)=\beta\Big(\bigcup_{n=1}^{\infty}\Lambda_n\Big)-\beta\Big(\bigcup_{n=1}^{\infty}\Lambda_n-\Lambda\Big)$$
$$\geqq\beta\Big(\bigcup_{n=1}^{\infty}\Lambda_n\Big)-\bar{\alpha}\Big(\bigcup_{n=1}^{\infty}\Lambda_n-\Lambda\Big)=\sum_{n=1}^{\infty}\beta(\Lambda_n)-\Big(\sum_{n=1}^{\infty}\alpha(\Lambda_n)-\bar{\alpha}(\Lambda)\Big)$$
$$=\bar{\alpha}(\Lambda)>0,$$

as was to be proved.

Let $\xi_j(\omega)$'s be the functions defined by

$$\xi_j(\omega)=\xi_j,$$

where ξ_j is the j-th coordinate of ω. Then it is easy to see that if ψ is any Borel function defined in m-dimensional space, then

$$\bar{\alpha}(\{\omega\,;\psi(\xi_{n_1}(\omega),\xi_{n_2}(\omega),\ldots,\xi_{n_m}(\omega))<k\})=\mu(\{x\,;\psi(F_{n_1}(x),F_{n_2}(x),\ldots,F_{n_m}(x))<k\})\tag{13}$$

for any real k.

By the assumption of the theorem, $F_{2j-1}(x)$ is μ-integrable and $F_{2j}(x)>0$, $\sum_{j=1}^{\infty}F_{2j}(x)\geqq\min q(j)\cdot\sum_{j=1}^{\infty}g_j(x)=+\infty$ for μ-almost every $x\in X$, so that by (13) and (9) we have that $(\xi_1(\omega)+\xi_3(\omega)+\ldots.+\xi_{2t-1}(\omega))$ is γ-integrable and $(\xi_2(\omega)+\xi_4(\omega)+\ldots.+\xi_{2t}(\omega))>0$, $\sum_{j=1}^{\infty}\xi_{2j}(\omega)=+\infty$ for γ-almost every $\omega\in\Omega$. Here we note that $\xi_j(S^h\omega)=\xi_{j+2ht}(\omega)$ for all $\omega\in\Omega$ and for all j, h and that S is γ-measure preserving as the transformation of $\tilde{\Omega}$ onto itself on account of (11). Hence we have, by Hopf's ergodic theorem,[5] that the limit

$$\lim_n\frac{\sum_{j=1}^{nt}\xi_{2j-1}(\omega)}{\sum_{j=1}^{nt}\xi_{2j}(\omega)}=\lim_n\frac{\sum_{h=0}^{n-1}[\xi_1(S^h\omega)+\xi_3(S^h\omega)+\ldots.+\xi_{2t-1}(S^h\omega)]}{\sum_{h=0}^{n-1}[\xi_2(S^h\omega)+\xi_4(S^h\omega)+\ldots.+\xi_{2t}(S^h\omega)]}\tag{14}$$

exists and is finite for γ-almost every $\omega\in\tilde{\Omega}$. On the other hand, it is easy to see that the limit (14) exists and vanishes for all $\omega\in\tilde{\Omega}_0$, so that the limit (14) exists and is finite for γ-almost every $\omega\in\Omega$. By use of (12) and (13) we get that the limit

$$\lim_n\frac{\sum_{j=1}^{nt}f_j(x)p(j)}{\sum_{j=1}^{nt}g_j(x)q(j)}=\lim_n\frac{\sum_{j=1}^{nt}F_{2j-1}(x)}{\sum_{j=1}^{nt}F_{2j}(x)}\tag{15}$$

exists and is finite for μ-almost every $x\in X$.

Let $g_j^k(x)$ denote the $\min\{g_j(x),k\}$. Then, by virtue of c. h. p. of $\{f_j(x)\}$

5) It is originally proved for the one to one transformation (see [3]), but it holds generally even for the single valued transformation. (See [2: Theorem 2].) We use here the general case.

and $\{g_j(x)\}$, two sequences $\{f_j(x)\}$ and $\{g_j^k(x)\}$ have c. h. p.. Further, by the assumption of the theorem, $f_1(x)$ is μ-integrable and $g_1^k(x) > 0$, $\sum_{j=1}^{\infty} g_j^k(x) = +\infty$ for μ-almost every $x \in X$. Particularly, let $p(j) = q(j) = 1$ for all j, then $t = 1$. Thus, by (15), we get that the limit

$$\lim_n \frac{\sum_{j=1}^{n} f_j(x)}{\sum_{j=1}^{n} g_j^k(x)} \tag{16}$$

exists and is finite for μ-almost every $x \in X$. Let C denote the $\max_j p(j)/\min_j q(j)$ with respect to $p(j)$ and $q(j)$ given in the theorem, then by (15) and (16) we have that for any integer l with $1 \leqq l < t$

$$\limsup_n \frac{\sum_{j=nt}^{nt+l} f_j(x)p(j)}{\sum_{j=1}^{nt} g_j(x)q(j)} \leqq C \cdot \limsup_n \frac{\sum_{j=nt}^{(n+1)t} f_j(x)}{\sum_{j=1}^{nt} g_j^k(x)} \tag{17}$$

$$\leqq C \cdot \left[\limsup_n \frac{\sum_{j=1}^{(n+1)t} f_j(x)}{\sum_{j=1}^{(n+1)t} g_j^k(x)} \cdot \frac{\sum_{j=1}^{nt} g_j^k(x) + kt}{\sum_{j=1}^{nt} g_j^k(x)} - \lim_n \frac{\sum_{j=1}^{nt} f_j(x)}{\sum_{j=1}^{nt} g_j^k(x)} \right]$$

$$= C \cdot \left[\lim_n \frac{\sum_{j=1}^{(n+1)t} f_j(x)}{\sum_{j=1}^{(n+1)t} g_j^k(x)} - \lim \frac{\sum_{j=1}^{nt} f_j(x)}{\sum_{j=1}^{nt} g_j^k(x)} \right] = 0$$

for μ-almost every $x \in X$.

Thus, by (15) and (17), we get that the limit

$$\lim_n \frac{\sum_{j=1}^{n} f_j(x)p(j)}{\sum_{j=1}^{n} g_j(x)q(j)}$$

exists and is finite for μ-almost every $x \in X$.

Finally it is to be noted that if $p(j) = q(j) = 1$ for all j, (15) is the required one, and that in the course of the proof of (15) it was superfluous to assume that $\sum_{j=1}^{\infty} \min \{g_j(x), k\} = +\infty$, since, in fact, we used only that $\sum_{j=1}^{\infty} g_j(x) = +\infty$. Hence we complete the proof.

4. In this section we shall sketch the proof of Theorem 2.

We define Ω, $\mathfrak{B}_\Omega(n_1, n_2, \dots n_m)$, $\mathfrak{B}_\Omega^0$, $\mathfrak{B}_\Omega$, $F_j(x)$, φ, $\alpha, \bar{\alpha}, t$ and S as in the proof of Theorem 1. Then $\overline{\alpha}$ is a *finite* measure of $\mathfrak{B}_\Omega$, since

$$\overline{\alpha}(\Omega) = \alpha(\Omega) = \mu(\varphi^{-1}\Omega) = \mu(X) < \infty.$$

By w. c. h. p. of $\{f_j(x)\}$ and $\{g_j(x)\}$ there exists a constant K such that

$$\text{(1)} \qquad \limsup_n \frac{1}{n}\sum_{h=0}^{n-1}\overline{\alpha}(S^{-h}\Lambda) \leqq K\cdot\overline{\alpha}(\Lambda) \qquad (\Lambda \in \mathfrak{B}^0_\Omega).$$

In fact, if we define a Borel set A for every $\Lambda \in \mathfrak{B}_\Omega(n_1, n_2, \ldots, n_m)$ as in the proof of Theorem 1, then

$$\limsup_n \frac{1}{n}\sum_{h=0}^{n-1}\overline{\alpha}(S^{-h}\Lambda) = \limsup_n \frac{1}{n}\sum_{h=0}^{n-1}\alpha(S^{-h}\Lambda)$$

$$= \limsup_n \frac{1}{n}\sum_{h=0}^{n-1}\mu(\{x\,;(F_{n_1+2ht}(x), F_{n_2+2ht}(x), \ldots, F_{n_m+2ht}(x)) \in A\})$$

$$\leqq K\cdot\mu(\{x\,;(F_{n_1}(x), F_{n_2}(x), \ldots, F_{n_m}(x)) \in A\}) = K\cdot\alpha(\Lambda) = K\cdot\overline{\alpha}(\Lambda).$$

Next, let p_m be the transformation of Ω onto m-dimensional space defined by

$$p_m\omega = (\xi_1, \xi_2, \ldots, \xi_m),$$

where $\omega = (\xi_1, \xi_2, \ldots .)$. Let Λ be any set of $\mathfrak{B}_\Omega$. If we put $\Lambda_m = p_m^{-1}p_m\Lambda$, then

$$\text{(2)} \qquad \Lambda_m \in \mathfrak{B}^0_\Omega,\ \ \Lambda_1 \supset \Lambda_2 \supset \ldots,\ \ \Lambda = \bigcap_{m=1}^{\infty}\Lambda_m,$$

so that

$$\text{(3)} \qquad \lim_m \overline{\alpha}(\Lambda_m) = \overline{\alpha}(\Lambda).$$

By (1) and (2)

$$\limsup_n \frac{1}{n}\sum_{h=0}^{n-1}\overline{\alpha}(S^{-h}\Lambda) \leqq \limsup_n \frac{1}{n}\sum_{h=0}^{n-1}\overline{\alpha}(S^{-h}\Lambda_m) \leqq K\cdot\overline{\alpha}(\Lambda_m),$$

so that, by (3),

$$\limsup_n \frac{1}{n}\sum_{h=0}^{n-1}\overline{\alpha}(S^{-h}\Lambda) \leqq K\cdot\overline{\alpha}(\Lambda).$$

Then there exists a finite invariant measure γ defined on $\mathfrak{B}_\Omega$ such that $\gamma(\Lambda) \leqq \overline{\alpha}(\Lambda)$ for every $\Lambda \in \mathfrak{B}_\Omega$, and $\overline{\alpha}(\Lambda) = 0$ for every $\Lambda \in \mathfrak{B}_\Omega$ with $\gamma(\Lambda) = 0$. (See[5, 6].) Thus S is γ-measure preserving and further, γ-incompressible, so that if we define $\xi_j(\omega)$ as in the proof of Theorem 1, then $\sum_{j=1}^{\infty}\xi_{2j}(\omega) = +\infty$ for γ-almost every $\omega \in \Omega$. (See footnote 3).) Hence we may prove the remaining part of the proof by the same way as in the proof of Theorem 1.

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MATHEMATICAL INSTITUTE, TOKYO METROPOLITAN UNIVERSITY, TOKYO.