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SPLITTING DENSITY FOR LIFTING ABOUT DISCRETE GROUPS

Dedicated to Ryoshi Hotta on the occasion of his sixty-fifth birthday

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Abstract. We study splitting densities of primitive elements of a discrete subgroup of a connected non-compact semisimple Lie group of real rank one with finite center in another larger such discrete subgroup. When the corresponding cover of such a locally symmetric negatively curved Riemannian manifold is regular, the densities can be easily obtained from the results due to Sarnak or Sunada. Our main interest is a case where the covering is not necessarily regular. Specifically, for the case of the modular group and its congruence subgroups, we determine the splitting densities explicitly. As an application, we study analytic properties of the zeta function defined by the Euler product over elements consisting of all primitive elements which satisfy a certain splitting law for a given lifting.

1. Introduction. Let $\boldsymbol{H}$ be the Poincaré upper half plane and Γ a discrete subgroup of $\mathrm{SL}_2(\boldsymbol{R})$ such that $\mathrm{vol}(\Gamma\backslash\boldsymbol{H}) < \infty$. Let $\mathrm{Prim}(\Gamma)$ be the set of primitive hyperbolic conjugacy classes of Γ, $N(\gamma)$ the square of the larger eigenvalue of $\gamma \in \mathrm{Prim}(\Gamma)$, and $\pi_\Gamma(x)$ the number of $\gamma \in \mathrm{Prim}(\Gamma)$ satisfying $N(\gamma) < x$. Then, the so-called prime geodesic theorem for $\Gamma\backslash\boldsymbol{H}$ or Γ was discovered by Selberg [Se] in the early 1950s. Indeed, it tells now (see also [Sa] and [He]) that

$$\pi_\Gamma(x) = \mathrm{li}(x) + O(x^\delta) \quad \text{as} \quad x \to \infty, \tag{1.1}$$

where $\mathrm{li}(x) := \int_2^x (1/\log t)dt$ and the constant δ, $0 < \delta < 1$, depends on Γ. Inspired by this Selberg's work, Sarnak [Sa] obtained, by using the one-to-one correspondence due to Gauss [G] between the primitive hyperbolic conjugacy classes of $\mathrm{SL}_2(\boldsymbol{Z})$ and the equivalence classes of the primitive indefinite binary quadratic forms, an asymptotic behavior of the sum of the class numbers of the quadratic forms from the prime geodesic theorem when $\Gamma = \mathrm{SL}_2(\boldsymbol{Z})$. A certain extension of the result in [Sa] for congruence subgroups of $\mathrm{SL}_2(\boldsymbol{Z})$ was recently obtained by the first author in [H].

The aim of this paper is to study various splitting densities of primitive elements of $\tilde{\Gamma}$ in Γ, where $\tilde{\Gamma}$ is a subgroup of Γ of finite index. Although each element of $\tilde{\Gamma}$ is in Γ, primitive elements of $\tilde{\Gamma}$ are not necessarily primitive in Γ. We consider a problem asking how many primitive elements of $\tilde{\Gamma}$ remain also primitive in Γ, and moreover, how many

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primitive elements of $\tilde{\Gamma}$ which are not primitive in Γ are equal to a given power of primitive elements of Γ.

Historically, this kind of branching problem is quite fundamental in algebraic number theory. Actually, for algebraic extensions of algebraic number fields, similar problems had been studied by, for example, Takagi [Ta1], Artin [Ar] and Tchebotarev [Tc] in the early 20th century. The problem for algebraic number fields can be drawn as follows; let k be an algebraic number field over $\boldsymbol{Q}$ and K an algebraic extension of k with $n := [K : k] < \infty$. We denote by $N_k(\mathfrak{a})$ the norm of an ideal $\mathfrak{a}$ in k. For a given prime ideal $\mathfrak{p}$ of k unramified in K, there exist a finite number of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of K and positive integers $e_1, \ldots, e_m$ ($e_1 \geq \cdots \geq e_m \geq 1$) such that $\mathfrak{p} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$ and $N_K(\mathfrak{p}_i) = N_k(\mathfrak{p})^{e_i}$. Since the sum $\sum_{i=1}^m e_i$ equals n, $(e_1, \ldots, e_m)$ is a partition of n. We call that a prime ideal $\mathfrak{p}$ of k is λ-type in K when $\lambda = (e_1, \ldots, e_m) \vdash n$. What is the main question is, for a given $\lambda \vdash n$, to count the number of prime ideals of k which are λ-type in K.

We now formulate our problem precisely in terms of the geometry of negatively curved locally symmetric Riemannian manifolds by use of the lifting of the primitive geodesics. First, we recall some notations for later use.

Let G be a connected non-compact semisimple Lie group of real rank one with finite center and $G = KA_{\mathfrak{p}}N$ be an Iwasawa decomposition of G. We denote by $\mathfrak{g}$, $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{n}$ the Lie algebras of G, $A_{\mathfrak{p}}$ and N, respectively. For the Cartan involution θ of G, $\mathfrak{a} \supset \mathfrak{a}_{\mathfrak{p}}$ is defined as a θ-stable Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^{\boldsymbol{C}}$, $\mathfrak{a}^{\boldsymbol{C}}$ be the complexifications of $\mathfrak{g}$, $\mathfrak{a}$, respectively. We denote by Φ^+ an $\mathfrak{a}_{\mathfrak{p}}$-compatible system of positive roots in the set of nonzero roots of $(\mathfrak{g}^{\boldsymbol{C}}, \mathfrak{a}^{\boldsymbol{C}})$, $P^+ = \{\alpha \in \Phi^+ \mid \alpha \not\equiv 0 \text{ on } \mathfrak{a}_{\mathfrak{p}}\}$ and Σ^+ the set of the restrictions of the elements of P^+ on $\mathfrak{a}_{\mathfrak{p}}$. Then Σ^+ is written as $\Sigma^+ = \{\beta\}$ or $\{\beta, 2\beta\}$ for some β. We choose $H_0 \in \mathfrak{a}_{\mathfrak{p}}$ such that $\beta(H_0) = 1$ and put $\rho = 1/2 \sum_{\alpha \in P^+} \alpha$ and $\rho_0 = \rho(H_0)$.

Let Γ be a discrete subgroup of G such that the volume of $X_\Gamma := \Gamma \backslash G/K$ is finite. We denote by $\mathrm{Prim}(\Gamma)$ a set of primitive hyperbolic conjugacy classes of Γ. For $\gamma \in \Gamma$, the norm $N(\gamma)$ is defined by

$$N(\gamma) = \max\{|\delta|^k \mid \delta \text{ is an eigenvalue of } \mathrm{Ad}(\gamma)\},$$

where Ad is the adjoint representation of $G_{\boldsymbol{C}}$, the analytic group with Lie algebra $\mathfrak{g}_{\boldsymbol{C}}$, and $k(= 1, 2)$ denotes the number of elements in Σ^+. Denote by $\pi_\Gamma(x)$ the number of $\gamma \in \mathrm{Prim}(\Gamma)$ satisfying $N(\gamma) < x$. Then $\pi_\Gamma(x)$ behaves

$$\pi_\Gamma(x) = \mathrm{li}(x^{2\rho_0}) + O(x^\delta) \quad \text{as} \quad x \to \infty,$$

where δ, $0 < \delta < 2\rho_0$, is a constant depending on Γ (see, e.g., [GW]).

PROBLEM 1.1. Let $\tilde{\Gamma}$ be a subgroup of Γ of finite index, and suppose that $X_{\tilde{\Gamma}}$ is a finite cover of X_Γ. We denote by p a natural projection from $X_{\tilde{\Gamma}}$ to X_Γ. Let C_γ be a closed primitive geodesic of X_Γ corresponding to $\gamma \in \mathrm{Prim}(\Gamma)$, and $l(\gamma)$ the length of C_γ ($N(\gamma) := e^{l(\gamma)}$). For a given $\gamma \in \mathrm{Prim}(\Gamma)$, there exists a finite number of elements $\gamma_1, \gamma_2, \ldots, \gamma_k$ of $\mathrm{Prim}(\tilde{\Gamma})$ and positive integers $m_1, \ldots, m_k$ such that $p(C_{\gamma_j}) = C_\gamma$ with $l(\gamma_j) = m_j l(\gamma)$. We may assume that $m_1 \geq \cdots \geq m_k$. Since $\sum_{j=1}^k m_j = [\Gamma : \tilde{\Gamma}](=: n)$, $(m_1, m_2, \ldots, m_k)$ is considered as a partition of n. We say that an element $\gamma \in \mathrm{Prim}(\Gamma)$ is λ-type in $\tilde{\Gamma}$ when $\lambda = (m_1, m_2, \dots, m_k) \vdash n$. We define $\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x)$ and its density $\mu^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x)$ relative to $\pi_{\Gamma}(x)$ as

$$\begin{aligned}\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) &:= \#\{\gamma \in \mathrm{Prim}(\Gamma) \mid \gamma \text{ is } \lambda\text{-type in } \tilde{\Gamma},\, N(\gamma) < x\}\,,\\ \mu^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) &:= \pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x)/\pi_{\Gamma}(x)\,.\end{aligned}$$

For a given $\lambda \vdash n$, study the asymptotic behavior of $\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x)$ and $\mu^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x)$ when $x \to \infty$.

If a covering $X_{\tilde{\Gamma}} \to X_{\Gamma}$ is regular, that is, $\tilde{\Gamma}$ is a normal subgroup of Γ, Problem 1.1 can be easily solved (Theorem 3.1) based on the results in [Sa] and [Su2]. Thus, the main focus of the present paper is a study of Problem 1.1 when $X_{\tilde{\Gamma}}$ is not necessarily a regular cover of X_{Γ}. In particular, in the case where $G = \mathrm{SL}_2(\boldsymbol{R})$, $\Gamma = \mathrm{SL}_2(\boldsymbol{Z})$ and $\tilde{\Gamma}$ is a congruence subgroup of $\mathrm{SL}_2(\boldsymbol{Z})$, the splitting densities can be obtained explicitly (see Section 4 and 5).

Applying our results in Sections 4 and 5 to Venkov-Zograf's formula [VZ] about the relation between the Selberg zeta functions for Γ and $\tilde{\Gamma}$, we can obtain an expression of the Selberg zeta function for the congruence subgroup as a product over elements of $\mathrm{Prim}(\mathrm{SL}_2(\boldsymbol{Z}))$. Then, in the last section, by taking a quotient of such expressions of Selberg's zeta functions for two congruence subgroups, we give a functional equation and an analytic continuation to the right half plane of the zeta function defined by the Euler product over elements consisting of all primitive elements of $\mathrm{Prim}(\mathrm{SL}_2(\boldsymbol{Z}))$ satisfying a certain splitting law for a given lifting in the congruence subgroup.

2. General cases. It is not true in general that $\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) > 0$ for $\lambda \vdash n = [\Gamma : \tilde{\Gamma}]$ as we see below. In fact, $\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) = 0$ may hold for many partitions $\lambda \vdash n$. Hence, in Problem 1.1, it is important to determine partitions λ of which the density $\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x)$ is positive. For a general pair $(\Gamma, \tilde{\Gamma})$ such that $\tilde{\Gamma} \subset \Gamma$, we have the following basic theorem.

THEOREM 2.1. *Let Γ' be the (unique) maximal normal subgroup of Γ contained in $\tilde{\Gamma}$. Let $\Xi := \Gamma/\Gamma'$ and* $\mathrm{Conj}(\Xi)$ *the set of conjugacy classes of Ξ. We denote $M(\gamma) :=$* $\min\{m \geq 1 \mid \gamma^m \in \Gamma'\}$ *for $\gamma \in \Gamma$ and $A_{\Gamma'\uparrow\Gamma} := \{M(\gamma) \mid \gamma \in \Gamma\} \subset \boldsymbol{N}$. Define*

$$\Lambda := \{(m_1, m_2, \dots, m_k) \vdash n \mid \text{there exists } M \in A_{\Gamma'\uparrow\Gamma} \text{ for all } m_i | M\}\,.$$

Then, for $\lambda \in \Lambda$, we have

$$\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) = \left(\sum_{\substack{[\gamma]\in\mathrm{Conj}(\Xi),\\ [\gamma] \text{ is } \lambda\text{-type in } \tilde{\Gamma}}} \frac{\#[\gamma]}{|\Xi|}\right) \mathrm{li}(x^{2\rho_0}) + O(x^{\delta}) \quad as \quad x \to \infty\,.$$

For $\lambda \notin \Lambda$, we have $\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) = 0$.

COROLLARY 2.2. *We have*

$$\lim_{x\to\infty}\mu^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x)=\begin{cases}\displaystyle\sum_{\substack{[\gamma]\in\mathrm{Conj}(\Xi),\\ [\gamma]\text{ is }\lambda\text{-type in }\tilde{\Gamma}}}\frac{\#[\gamma]}{|\Xi|} & \text{for } \lambda\in\Lambda\,,\\ 0 & \text{for } \lambda\notin\Lambda\,.\end{cases}$$

To prove Theorem 2.1, we need some preparations.

LEMMA 2.3. *Let $\Psi:=\tilde{\Gamma}/\Gamma'$ and $\lambda:=(m_1,\dots,m_k)\vdash n$. Denote by σ the permutation representation of Ξ on Ξ/Ψ ($\sigma\cong\mathrm{Ind}_{\tilde{\Gamma}}^{\Gamma}1$). For $\gamma\in\mathrm{Prim}(\Gamma)$, $\sigma(\gamma)$ is called λ-type when $\sigma(\gamma)$ is expressed as*

$$\sigma(\gamma)\sim\begin{pmatrix} S_{m_1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & S_{m_k}\end{pmatrix},\tag{2.1}$$

where S_{m_i} are $m_i\times m_i$-matrices given by

$$S_{m_i}=\begin{cases}1\,, & m_i=1\,,\\ \begin{pmatrix}0&1&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 0&0&\cdots&1\\ 1&0&\cdots&0\end{pmatrix}, & m_i\geq 2\,.\end{cases}$$

Then the following conditions are equivalent:

(i) *$\sigma(\gamma)$ is λ-type.*

(ii) *$\gamma\in\mathrm{Prim}(\Gamma)$ is λ-type in $\tilde{\Gamma}$.*

PROOF. Let $\mathrm{CSR}[\Gamma/\tilde{\Gamma}]$ be a complete system of representatives of $\Gamma/\tilde{\Gamma}$.

(I) If (i) holds, then there exist $A_1,\dots,A_k\subset\mathrm{CSR}[\Gamma/\tilde{\Gamma}]$ such that $\coprod_{i=1}^{k}A_i=\mathrm{CSR}[\Gamma/\tilde{\Gamma}]$ and

$$(a_1^{(i)})^{-1}\gamma a_2^{(i)},\dots,(a_{m_i-1}^{(i)})^{-1}\gamma a_{m_i}^{(i)},(a_{m_i}^{(i)})^{-1}\gamma a_1^{(i)}\in\tilde{\Gamma}$$

for $A_i=(a_1^{(i)},\dots,a_{m_i}^{(i)})$. Hence we have

$$(a_1^{(1)})^{-1}\gamma^{m_1}a_1^{(1)},\dots,(a_1^{(k)})^{-1}\gamma^{m_k}a_1^{(k)}\in\tilde{\Gamma}\,.$$

Putting $\gamma_i:=(a_1^{(i)})^{-1}\gamma^{m_i}a_1^{(i)}$, one can easily see that γ_i is primitive in $\tilde{\Gamma}$ and is Γ-conjugate to γ^{m_i}. Hence we have $p(C_{\gamma_j})=C_\gamma$ and $l(\gamma_j)=m_jl(\gamma)$.

(II) If (ii) holds, there exist $b_1,\dots,b_k\in\mathrm{CSR}[\Gamma/\tilde{\Gamma}]$ such that $\gamma_i=b_i^{-1}\gamma^{m_i}b_i$. Also, there exists $c_1^{(i)}\in\mathrm{CSR}[\Gamma/\tilde{\Gamma}]$ such that $(c_1^{(i)})^{-1}\gamma b_i\in\tilde{\Gamma}$. Since $b_i^{-1}\gamma^{m_i-1}c_1^{(i)}\in\tilde{\Gamma}$, it is easy to see that there exist $c_2^{(i)},\dots,c_{m_i-1}^{(i)}\in\mathrm{CSR}[\Gamma/\tilde{\Gamma}]$ such that

$$(c_1^{(i)})^{-1}\gamma b_i,(c_2^{(i)})^{-1}\gamma c_1^{(i)},\dots,b_i^{-1}\gamma c_{m_i-1}^{(i)}\in\tilde{\Gamma}$$

recursively. If there exist $1 \leq j_1 < j_2 \leq m_i - 1$ such that $c_{j_1}^{(i)} = c_{j_2}^{(i)}$, then $b_i^{-1}\gamma^{m_i - j_2} c_j^{(i)}$, $(c_{j_1}^{(i)})^{-1}\gamma^{j_2 - j_1} c_{j_1}^{(i)}$, $(c_{j_1}^{(i)})^{-1}\gamma^{j_1} b_i \in \tilde{\Gamma}$. Hence it follows that $b_i^{-1}\gamma^{m_i - j_2 + j_1} b_i \in \tilde{\Gamma}$. This contradicts, however, the fact that $\gamma_i = b_i^{-1}\gamma^{m_i} b_i$ is primitive in $\tilde{\Gamma}$. Therefore b_i's and $c_j^{(i)}$'s are mutually distinct and $\{b_i, c_j^{(i)}\} = \mathrm{CSR}[\Gamma/\tilde{\Gamma}]$. Hence $\sigma(\gamma)$ is $(m_1, \ldots, m_k)$-type. □

By using the trace formula, Sarnak has shown the following analytic distribution.

PROPOSITION 2.4 (Theorem 2.4 in [Sa], [Su1] or [Su2]). *For* $[g] \in \mathrm{Conj}(\Xi)$, *we have*

$$\#\{\gamma \in \mathrm{Prim}(\Gamma) \mid \gamma\Gamma' = [g], N(\gamma) < x\} \sim \frac{\#[g]}{|\Xi|}\mathrm{li}(x^{2\rho_0}) + O(x^{\delta}) \quad as \quad x \to \infty\,.$$

PROOF OF THEOREM 2.1. Assume that $\gamma \in \mathrm{Prim}(\Gamma)$ is $(1^{l_1}2^{l_2}\cdots n^{l_n})$-type in $\tilde{\Gamma}$. Then, because of Lemma 2.3, $\sigma(\gamma)$ is $(1^{l_1}2^{l_2}\cdots n^{l_n})$-type. Since $\sigma(\gamma^{M(\gamma)}) = \mathrm{Id}$, we have $l_j = 0$ for $j \nmid M(\gamma)$. Hence, for $\lambda \notin \Lambda$, we have $\mu^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) = 0$. Since the type of $\sigma(\gamma)$ is invariant under the Ξ-conjugation, by Proposition 2.4, the desired asymptotic law follows. □

3. Regular cover cases.

In Theorem 2.1, we see that $\mu^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) = 0$ holds for all $\lambda \notin \Lambda$. However, even if $\lambda \in \Lambda$, it is not necessarily that $\mu^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) > 0$. In fact, when $\tilde{\Gamma}$ is a normal subgroup of Γ ($\tilde{\Gamma} = \Gamma'$), we prove that only rectangle shape partitions can appear non-trivially.

THEOREM 3.1. *If* $\tilde{\Gamma}$ *is a normal subgroup of* Γ ($\tilde{\Gamma} = \Gamma'$), *then, for* $\lambda = \lambda(m) = (m^{n/m})$ ($m \in A_{\Gamma'\uparrow\Gamma}$), *we have*

$$\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) = \left(\sum_{\substack{[\gamma]\in\mathrm{Conj}(\Xi),\\ M([\gamma])=m}} \frac{\#[\gamma]}{|\Xi|}\right)\mathrm{li}(x^{2\rho_0}) + O(x^{\delta}) \quad as \quad x \to \infty\,.$$

For $\lambda \vdash n$, *other than the shape above, we have* $\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) = 0$.

PROOF. Since $\tilde{\Gamma} = \Gamma'$, we have $\Xi/\Psi = \Xi/\{\mathrm{Id}\} = \Xi$. For $\gamma \in \Xi$, suppose that there exist $g \in \Xi$ and $l < M(\gamma)$ such that $\gamma^l g = g$. Then we have $\gamma^l = \mathrm{Id}$, but this contradicts the minimality of $M(\gamma)$. Hence, for any g, we see that the type of $\sigma(g)$ is given as $(m^{n/m})$ ($m \in A_{\Gamma'\uparrow\Gamma}$). Consequently, applying Proposition 2.4, we obtain the desired result. □

COROLLARY 3.2. *We have*

$$\lim_{x\to\infty} \mu^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x) = \begin{cases} \displaystyle\sum_{\substack{[\gamma]\in\mathrm{Conj}(\Xi),\\ M([\gamma])=m}} \frac{\#[\gamma]}{|\Xi|} & \textit{for } \lambda = \lambda(m)\ (m \in A_{\Gamma'\uparrow\Gamma})\,, \\ 0 & \textit{otherwise}\,. \end{cases}$$

REMARK 3.3. In the problem for an algebraic number field, results similar to Theorem 3.1 had been obtained. Actually, for the cases of unramified Galois extensions, the corresponding densities are non-zero only when the factorization is of rectangle type (see, e.g., [Ar, Tc, Ta2, Na]).

4. Cases of congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$**.** Let N be a positive integer. In this section, we consider the case when $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and $\tilde{\Gamma}$ is one of the following congruence subgroups:

$$\begin{aligned}
\Gamma_0(N) &:= \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) \mid \gamma_{21} \equiv 0 \bmod N\}\,,\\
\Gamma_1(N) &:= \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) \mid \gamma_{11} \equiv \gamma_{22} \equiv \pm 1,\, \gamma_{21} \equiv 0 \bmod N\}\,,\\
\Gamma(N) &:= \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) \mid \gamma_{11} \equiv \gamma_{22} \equiv \pm 1,\, \gamma_{12} \equiv \gamma_{21} \equiv 0 \bmod N\}\,.
\end{aligned}$$

Let p be a prime number and, for simplicity, assume that $p \geq 3$. First, we study the case of $N = p^r$. Note that the maximal normal subgroup of Γ contained in $\tilde{\Gamma}$ is $\Gamma' = \Gamma(p^r)$, whence $\Xi = \mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})/\{\pm\mathrm{Id}\}$ and

$$|\Xi| = \frac{1}{2}p^{3r-2}(p^2-1)\,,$$

$$n = \begin{cases} p^{r-1}(p+1)\,, & \tilde{\Gamma} = \Gamma_0(p^r)\,,\\ \dfrac{1}{2}p^{2r-2}(p^2-1)\,, & \tilde{\Gamma} = \Gamma_1(p^r)\,,\\ \dfrac{1}{2}p^{3r-2}(p^2-1)\,, & \tilde{\Gamma} = \Gamma(p^r)\,. \end{cases}$$

In these cases we have the following results.

THEOREM 4.1. *Let*

$$\lambda_0^{p^r}(1) := \left(1^{p^{r-1}(p+1)}\right),$$

$$\lambda_0^{p^r}(lp^k) := \begin{cases} \left((l)^{(p^r+p^{r-1}-2)/l}, (1)^2\right), & k=0\,,\ \ l|(p-1)/2\,,\ \ l>1\,,\\ \left((lp^k)^{p^{r-k-1}(p-1)/l}, (lp^{k-1})^{2p^{r-k-1}(p-1)/l}, \ldots, (l)^{2(p^{r-k}-1)/l}, (1)^2\right), & \\ & k>0\,,\ \ l|(p-1)/2\,,\ \ l>1\,,\\ \left((lp^k)^{p^{r-k-1}(p+1)/l}\right), & l|(p+1)/2\,,\ \ l>1\,, \end{cases}$$

$$\lambda_0^{p^r}(p^k, A) := \left((p^k)^{p^{r-k-1}(p-1)}, (p^{k-1})^{2p^{r-k-1}(p-1)}, \ldots, (p)^{2p^{r-k-1}(p-1)}, (1)^{2p^{r-k}}\right),$$

$$\lambda_0^{p^r}(p^k, B^{(k)}) := \begin{cases} \left((p^k)^{p^{r-k}}, (p^{k-2})^{p^{r-k}(p-1)}, \ldots, (p^2)^{p^{r-k/2-2}(p-1)}, (1)^{p^{r-k/2}}\right), & \\ & k \text{ is even}\,,\\ \left((p^k)^{p^{r-k}}, (p^{k-2})^{p^{r-k}(p-1)}, \ldots, (p)^{p^{r-(k+3)/2}(p-1)}, (1)^{p^{r-(k+1)/2}}\right), & \\ & k \text{ is odd}\,, \end{cases}$$

$$\begin{aligned}\lambda_0^{p^r}(p^k, B^{(m)}) := \Big((p^k)^{p^{r-k}}, (p^{k-2})^{p^{r-k}(p-1)}, \ldots,&\\ (p^{k-m+2})^{p^{r-k+(m-3)(p-1)/2}}, (p^{k-m})^{p^{r-k+(m-1)/2}}\Big)&\,,\\ 1 \leq m < k\,,\ \ m \text{ is odd}&\,,\end{aligned}$$

$$\begin{aligned}
\lambda_0^{p^r}(p^k, B^{(m,+)}) &:= \big((p^k)^{p^{r-k}}, (p^{k-2})^{p^{r-k}(p-1)}, \dots, (p^{k-m+2})^{p^{r-k+m/2-2}(p-1)},\\
&\qquad (p^{k-m})^{p^{r-k+m/2-1}(p-2)}, (p^{k-m-1})^{2p^{r-k+m/2-1}(p-1)}, \dots,\\
&\qquad (p)^{2p^{r-k+m/2-1}(p-1)}, (1)^{2p^{r-k+m/2}}\big), \quad 1 < m < k,\ m \textit{ is even},\\
\lambda_0^{p^r}(p^k, B^{(m,-)}) &:= \big((p^k)^{p^{r-k}}, (p^{k-2})^{p^{r-k}(p-1)}, \dots,\\
&\qquad (p^{k-m+3})^{p^{r-k+m/2-3}(p-1)}, (p^{k-m+1})^{p^{r-k+m/2-1}}\big), \quad 1<m<k,\ m \textit{ is even},\\
\lambda_0^{p^r}(p^k, C) &:= \big((p^k)^{p^{r-k-1}(p+1)}\big),\\
\lambda_1^{p^r}(lp^k) &:= \big((lp^k)^{p^{2r-k-2}(p^2-1)/2l}\big), \qquad l|(p\pm 1)/2, l \geq 1, 0 \leq k \leq r-1,\\
\lambda_1^{p^r}(p^k, B^{(m)}) &:= \big((p^k)^{p^{2r-k-1}(p-1)/2}, (p^{k-1})^{p^{2r-k-2}(p-1)^2/2}, \dots,\\
&\qquad (p^{k-m+1})^{p^{2r-k-2}(p-1)^2/2}, (p^{k-m})^{p^{2r-k-1}(p-1)/2}\big),\\
\lambda^{p^r}(l) &:= \big(l^{|\Xi|/l}\big).
\end{aligned}$$

Then we have

$$\begin{aligned}
\lim_{x\to\infty} \mu_{\Gamma_0(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda_0^{p^r}(1)}(x) = \lim_{x\to\infty} \mu_{\Gamma_1(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda_1^{p^r}(1)}(x) &= \lim_{x\to\infty} \mu_{\Gamma(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda^{p^r}(1)}(x)\\
&= \frac{2}{p^{3r-2}(p^2-1)},
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \mu_{\Gamma_0(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda_0^{p^r}(lp^k)}(x) = \lim_{x\to\infty} \mu_{\Gamma_1(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda_1^{p^r}(lp^k)}(x) &= \lim_{x\to\infty} \mu_{\Gamma(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda^{p^r}(lp^k)}(x)\\
&= \begin{cases}
\dfrac{\varphi(l)}{p^{r-1}(p-1)}, & k=0,\ l|(p-1)/2,\ l>1,\\
\dfrac{\varphi(l)}{p^{r-1}(p+1)}, & k=0,\ l|(p+1)/2,\ l>1,\\
\dfrac{\varphi(l)}{p^{r-k}}, & k>0,\ l|(p-1)/2,\ l>1,\\
\dfrac{\varphi(l)(p-1)}{p^{r-k}(p+1)}, & k>0,\ l|(p+1)/2,\ l>1,
\end{cases}\\
\lim_{x\to\infty} \mu_{\Gamma_0(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda_0^{p^r}(p^k,A)}(x) &= \frac{1}{p^{3(r-k)}},\\
\lim_{x\to\infty} \mu_{\Gamma_0(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda_0^{p^r}(p^k,C)}(x) &= \frac{p-1}{p^{3(r-k)}(p+1)},\\
\lim_{x\to\infty} \mu_{\Gamma_1(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda_1^{p^r}(p^k)}(x) &= \frac{2}{p^{3r-3k-1}(p+1)},\\
\lim_{x\to\infty} \mu_{\Gamma_0(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}^{\lambda_0^{p^r}(p^k,B^{(m,\pm)})}(x) &= \frac{(p-1)}{p^{3r-3k+m+1}},
\end{aligned}$$

$$\lim_{x\to\infty}\mu^{\lambda_0^{p^r}(p^k,B^{(m)})}_{\Gamma_0(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\lim_{x\to\infty}\mu^{\lambda_1^{p^r}(p^k,B^{(m)})}_{\Gamma_1(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\begin{cases}\dfrac{2}{p^{3r-2k}}\,, & m=k\,,\\ \dfrac{2(p-1)}{p^{3r-3k+m+1}}\,, & m<k\,,\end{cases}$$

$$\lim_{x\to\infty}\mu^{\lambda^{p^r}(p^k)}_{\Gamma(p^r)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\begin{cases}2/p\,, & k=r\,,\\ \dfrac{2(p^2+p+1)}{p^{3r-3k+1}(p+1)}\,, & k<r\,.\end{cases}$$

For any other $\lambda\vdash n$, *we have* $\pi^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(x)=0$.

To prove Theorem 4.1, we first use the following classification of the conjugacy classes of $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})/\{\pm I\}$ (see [Di] for $r=1$ and [Kl] for larger r).

LEMMA 4.2. *Each element of* $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})/\{\pm I\}\setminus\{I\}$ *is conjugate to one of the following elements.*

γ	parameter	$M(\gamma)$	$\#[\gamma]$
$\begin{pmatrix}\delta & 0\\ 0 & \delta^{-1}\end{pmatrix}^{sp^{k-1}(p-1)/2l}$	$\begin{cases}1\le k\le r\,,\\ l\mid(p-1)/2(l>1)\,,\\ s\in(\mathbf{Z}/lp^{r-k}\mathbf{Z})^*/\{\pm1\}\end{cases}$	lp^{r-k}	$p^{2r-1}(p+1)$
$\begin{pmatrix}\delta & 0\\ 0 & \delta^{-1}\end{pmatrix}^{sp^{k-1}(p-1)}$	$\begin{cases}1\le k\le r-1\,,\\ s\in(\mathbf{Z}/p^{r-k}\mathbf{Z})^*/\{\pm1\}\end{cases}$	p^{r-k}	$p^{2r-2k-1}(p+1)$
$\begin{pmatrix}1+\alpha p^{2k+m} & p^k\\ \alpha p^{k+m} & 1\end{pmatrix}$	$\begin{cases}0\le k\le r-1\,,\\ 1\le m\le r-k\,,\\ \alpha\in(\mathbf{Z}/p^{r-k-m}\mathbf{Z})^*\end{cases}$	p^{r-k}	$p^{2r-2k-2}(p^2-1)/2$
$\begin{pmatrix}1+\nu\alpha p^{2k+m} & \nu p^k\\ \alpha p^{k+m} & 1\end{pmatrix}$	$\begin{cases}0\le k\le r-1\,,\\ 1\le m\le r-k\,,\\ \alpha\in(\mathbf{Z}/p^{r-k-m}\mathbf{Z})^*\end{cases}$	p^{r-k}	$p^{2r-2k-2}(p^2-1)/2$
$\Omega^{sp^{k-1}(p+1)/2l}$	$\begin{cases}1\le k\le r\,,\\ l\mid(p+1)/2\,,\ \ l>1\,,\\ s\in(\mathbf{Z}/lp^{r-k}\mathbf{Z})^*/\{\pm1\}\end{cases}$	lp^{r-k}	$p^{2r-1}(p-1)$
$\Omega^{sp^{k-1}(p+1)}$	$\begin{cases}1\le k\le r-1\,,\\ s\in(\mathbf{Z}/p^{r-k}\mathbf{Z})^*/\{\pm1\}\end{cases}$	p^{r-k}	$p^{2r-2k-1}(p-1)$

Here δ *is a generator of* $(\mathbf{Z}/p^r\mathbf{Z})^*/\{\pm1\}$, ν *is a non-quadratic residue of* p *and* $\Omega\in\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})/\{\pm I\}$ *is of order* $p^{r-1}(p+1)/2$. *Note that* $(\mathrm{tr}\Omega)^2-4$ *is a non-quadratic residue of* p.

The claims for $\tilde{\Gamma} = \Gamma(p^r)$ in Theorem 4.1 follow easily from Theorem 3.1 together with Lemma 4.2.

Now, we put

$$\begin{aligned}
A_0^{(k,l)} &:= \bigcup_{s\in(\mathbf{Z}/lp^{r-k}\mathbf{Z})^*/\{\pm1\}} \left[\begin{pmatrix}\delta & 0\\ 0 & \delta^{-1}\end{pmatrix}^{sp^{k-1}(p-1)/l}\right], \quad l\big|(p-1)/2\,,\\
A_k &:= \bigcup_{s\in(\mathbf{Z}/p^{r-k}\mathbf{Z})^*/\{\pm1\}} \left[\begin{pmatrix}\delta & 0\\ 0 & \delta^{-1}\end{pmatrix}^{sp^{k-1}(p-1)}\right],\\
B_k^{(m)} &:= \bigcup_{\alpha\in(\mathbf{Z}/p^{r-k-m}\mathbf{Z})^*} \left(\left[\begin{pmatrix}1+\alpha p^{2k+m} & p^k\\ \alpha p^{k+m} & 1\end{pmatrix}\right]\cup\left[\begin{pmatrix}1+\nu\alpha p^{2k+m} & \nu p^k\\ \alpha p^{k+m} & 1\end{pmatrix}\right]\right),\\
C_0^{(k,l)} &:= \bigcup_{s\in(\mathbf{Z}/lp^{r-k}\mathbf{Z})^*/\{\pm1\}} \left[\Omega^{sp^{k-1}(p+1)/l}\right], \quad l\big|(p+1)/2\,,\\
C_k &:= \bigcup_{s\in(\mathbf{Z}/p^{r-k}\mathbf{Z})^*/\{\pm1\}} \left[\Omega^{sp^{k-1}(p-1)}\right].
\end{aligned} \tag{4.1}$$

We divide $B_k^{(m)}$ for even $m < r-k$ by $B_k^{(m)} = B_k^{(m,+)} \cup B_k^{(m,-)}$ where

$$B_k^{(m,\pm)} := \bigcup_{\substack{\alpha\in(\mathbf{Z}/p^{r-k-m}\mathbf{Z})^*\\ \left(\frac{\alpha}{p}\right)=\pm1}} \left[\begin{pmatrix}1+\alpha p^{2k+m} & p^k\\ \alpha p^{k+m} & 1\end{pmatrix}\right] \cup \bigcup_{\substack{\alpha\in(\mathbf{Z}/p^{r-k-m}\mathbf{Z})^*\\ \left(\frac{\alpha}{p}\right)=\mp1}} \left[\begin{pmatrix}1+\nu\alpha p^{2k+m} & \nu p^k\\ \alpha p^{k+m} & 1\end{pmatrix}\right].$$

Note that

$$\begin{aligned}
\mathrm{SL}_2(\mathbf{Z})/\Gamma(p^r) - \Gamma(p)/\Gamma(p^r) &= \left(\bigcup_{l|(p-1)/2,\,l>1}\ \bigcup_{k=1}^{r} A_0^{(k,l)}\right) \cup \left(\bigcup_{m=1}^{r} B_0^{(m)}\right)\\
&\quad \cup \left(\bigcup_{l|(p+1)/2,\,l>1}\ \bigcup_{k=1}^{r} C_0^{(k,l)}\right),\\
\Gamma(p^k)/\Gamma(p^r) - \Gamma(p^{k+1})/\Gamma(p^r) &= A_k \cup \left(\bigcup_{m=1}^{r-k} B_k^{(m)}\right) \cup C_k\,,
\end{aligned}$$

and

$$\#A_0^{(k,l)} = \begin{cases} \frac{1}{2}\varphi(l)p^{3r-k-2}(p^2-1)\,, & k<r\,, \\ \frac{1}{2}\varphi(l)p^{2r-1}(p+1)\,, & k=r\,, \end{cases} \qquad \#A_k = \frac{1}{2}p^{3r-3k-2}(p^2-1)\,,$$

$$\#B_k^{(m)} = \begin{cases} p^{3r-3k-m-3}(p-1)^2(p+1)\,, & m<r-k\,, \\ p^{2r-2k-2}(p^2-1)\,, & m=r-k\,, \end{cases} \tag{4.2}$$

$$\#B_k^{(m,+)} = \#B_k^{(m,-)} = \frac{1}{2}\#B_k^{(m)}\,,$$

$$\#C_0^{(k,l)} = \begin{cases} \frac{1}{2}\varphi(l)p^{3r-k-2}(p-1)^2\,, & k<r\,, \\ \frac{1}{2}\varphi(l)p^{2r-1}(p-1)\,, & k=r\,, \end{cases} \qquad \#C_k = \frac{1}{2}p^{3r-3k-2}(p-1)^2\,,$$

where $\varphi(l)$ is the Euler function given by $\varphi(l) := \#(\mathbf{Z}/l\mathbf{Z})^*$. We also notice that the following relations hold among the sets $A_0^{(k,l)}, A_k, B_k^{(m)}, C_0^{(k,l)}, C_k$ for $1 \le M \le p$.

$$\{\gamma^M \mid \gamma \in A_0^{(k,l)}\} = \begin{cases} \Gamma(p^r)\,, & l|M\,,\ k=r\,, \\ A_k\,, & l|M\,,\ k\le r-1\,, \\ A_0^{(k+1,l)}\,, & M=p\,,\ k\le r-1\,, \\ A_0^{(k,l/\gcd(M,l))}\,, & \text{otherwise}\,, \end{cases}$$

$$\{\gamma^M \mid \gamma \in A_k\} = \begin{cases} \Gamma(p^r)\,, & M=p\,,\ k=r-1\,, \\ A_{k+1}\,, & M=p\,,\ k\le r-2\,, \\ A_k\,, & \text{otherwise}\,, \end{cases}$$

$$\{\gamma^M \mid \gamma \in B_k^{(m,\pm)}\} = \begin{cases} \Gamma(p^r)\,, & M=p\,,\ k=r-1\,,\ m=1\,, \\ B_{k+1}^{(r-k-1)}\,, & M=p\,,\ k\le r-2\,,\ m=r-k\,, \\ B_{k+1}^{(m,\pm)}\,, & M=p\,,\ k\le r-2\,,\ m\le r-k-1\,, \\ B_k^{(m,\pm)}\,, & \text{otherwise}\,, \end{cases} \tag{4.3}$$

$$\{\gamma^M \mid \gamma \in C_0^{(k,l)}\} = \begin{cases} \Gamma(p^r)\,, & l|M\,,\ k=r\,, \\ C_k\,, & l|M\,,\ m\le r-1\,, \\ C_0^{(k+1,l)}\,, & M=p\,,\ k\le r-1\,, \\ C_0^{(k,l/\gcd(M,l))}\,, & \text{otherwise}\,, \end{cases}$$

$$\{\gamma^M \mid \gamma \in C_k\} = \begin{cases} \Gamma(p^r)\,, & M=p\,,\ k=r-1\,, \\ C_{k+1}\,, & M=p\,,\ k\le r-2\,, \\ C_k\,, & \text{otherwise}\,. \end{cases}$$

To calculate the type of the representatives of conjugacy classes of Ξ, the following lemma (when $r = 1$, see also [H]) is an inevitable step.

LEMMA 4.3. *Let* $\sigma(\gamma, \Gamma) := \mathrm{tr}\big(\mathrm{Ind}_{\Gamma}^{\mathrm{SL}_2(\mathbf{Z})} 1\big)(\gamma)$. *Then we have*

$$\sigma(\gamma, \Gamma_0(p^r)) = \begin{cases} p^{r-1}(p+1)\,, & \gamma \in \Gamma(p^r)\,, \\ 2p^k\,, & \gamma \in A_k\,, \\ 2\,, & \gamma \in A_0^{(k,l)}\,, \\ p^{[(r+k)/2]}\,, & \gamma \in B_k^{(r-k)}\,, \\ 2p^{k+m/2}\,, & \gamma \in B_k^{(m,+)}\,, \quad m \text{ is even}\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

$$\sigma(\gamma, \Gamma_1(p^r)) = \begin{cases} \dfrac{1}{2}p^{2r-2}(p^2-1)\,, & \gamma \in \Gamma(p^r)\,, \\ \dfrac{1}{2}p^{r+k-1}(p-1)\,, & \gamma \in B_k^{(r-k)}\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

PROOF. It is easy to see that the complete system of representatives of $\mathrm{SL}_2(\mathbf{Z})/\Gamma_0(p^r)$ can be chosen as

$$\left\{ \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \begin{pmatrix} lp & -1 \\ 1 & 0 \end{pmatrix} \;\middle|\; m \in \mathbf{Z}/p^r\mathbf{Z}, l \in \mathbf{Z}/p^{r-1}\mathbf{Z} \right\}. \tag{4.4}$$

Hence we have

$$\begin{aligned} \sigma(\gamma, \Gamma_0(p^r)) = \#\{m \in \mathbf{Z}/p^r\mathbf{Z} \mid \gamma_{12}m^2 + (\gamma_{11} - \gamma_{22})m - \gamma_{21} \equiv 0 \bmod p^r\} \\ + \#\{l \in \mathbf{Z}/p^{r-1}\mathbf{Z} \mid \gamma_{21}p^2l^2 + p(\gamma_{11} - \gamma_{22})l - \gamma_{12} \equiv 0 \bmod p^r\}. \end{aligned}$$

Calculating the terms above for each element in the table of Lemma 4.2, we get the claims for $\Gamma = \Gamma_0(p^r)$. Moreover, since $\Gamma_1(p^r)$ is a normal subgroup of $\Gamma_0(p^r)$, we have

$$\begin{aligned} \sigma(\gamma, \Gamma_1(p^r)) &= \sum_{\substack{g \in \mathrm{CSR}[\mathrm{SL}_2(\mathbf{Z})/\Gamma_0(p^r)] \\ g^{-1}\gamma g^{-1} \in \Gamma_0(p^r)}} \mathrm{tr}\big(\mathrm{Ind}_{\Gamma_1(p^r)}^{\Gamma_0(p^r)} 1\big)(g^{-1}\gamma g) \\ &= \sum_{\substack{g \in \mathrm{CSR}[\mathrm{SL}_2(\mathbf{Z})/\Gamma_0(p^r)] \\ g^{-1}\gamma g^{-1} \in \Gamma_1(p^r)}} [\Gamma_0(p^r) : \Gamma_1(p^r)] \\ &= \frac{1}{2}p^{r-1}(p-1)\#\{g \in \mathrm{CSR}[\mathrm{SL}_2(\mathbf{Z})/\Gamma_0(p^r)] \bigm| g^{-1}\gamma g \in \Gamma_1(p^r)\}\,. \end{aligned}$$

Hence the assertions for $\Gamma = \Gamma_1(p^r)$ follows from (4.4). $\square$

By using (4.3) together with the lemma above, we may determine the type of each element of Ξ as follows.

LEMMA 4.4. *We have*

$$\lambda_0^{p^r}(1) = \textit{type of } \gamma \in \varGamma(p^r) \textit{ in } \varGamma_0(p^r)\,,$$

$$\lambda_0^{p^r}(lp^k) = \begin{cases} \textit{type of } \gamma \in A_0^{(r-k,l)} \textit{ in } \varGamma_0(p^r)\,, & l|(p-1)/2\,,\ l > 1\,, \\ \textit{type of } \gamma \in C_0^{(r-k,l)} \textit{ in } \varGamma_0(p^r)\,, & l|(p+1)/2\,,\ l > 1\,, \end{cases}$$

$$\lambda_0^{p^r}(p^k, A) = \textit{type of } \gamma \in A_{r-k} \textit{ in } \varGamma_0(p^r)\,,$$

$$\lambda_0^{p^r}(p^k, B^{(m,\pm)}) = \textit{type of } \gamma \in B_{r-k}^{(m,\pm)} \textit{ in } \varGamma_0(p^r)\,,$$

$$\lambda_0^{p^r}(p^k, C) = \textit{type of } \gamma \in C_{r-k} \textit{ in } \varGamma_0(p^r)\,,$$

and

$$\lambda_1^{p^r}(1) = \textit{type of } \gamma \in \varGamma(p^r) \textit{ in } \varGamma_1(p^r)\,,$$

$$\lambda_1^{p^r}(lp^k) = \begin{cases} \textit{type of } \gamma \in A_0^{(r-k,l)} \textit{ in } \varGamma_1(p^r)\,, & l|(p-1)/2\,,\ l > 1\,, \\ \textit{type of } \gamma \in C_0^{(r-k,l)} \textit{ in } \varGamma_1(p^r)\,, & l|(p+1)/2\,,\ l > 1\,, \end{cases}$$

$$\lambda_1^{p^r}(p^k) = \textit{type of } \gamma \in A_{r-k} \cup C_{r-k} \textit{ in } \varGamma_1(p^r)\,,$$

$$\lambda_1^{p^r}(p^k, B^{(m)}) = \textit{type of } \gamma \in B_{r-k}^{(m)} \textit{ in } \varGamma_1(p^r)\,.$$

PROOF. By Lemma 2.3, if γ is $(1^{l_1}2^{l_2}\cdots n^{l_n})$-type in $\tilde{\varGamma}$, then we have $\mathrm{tr}\sigma(\gamma^m) = \sum_{j|m} jl_j$. Hence, l_j's are calculated recursively by

$$ml_m = \sum_{j|m} \mu(m/j)\mathrm{tr}\sigma(\gamma^{m/j})\,, \tag{4.5}$$

where μ is the Möbius function. Hence, using (4.3), (4.5) and Lemma 4.3, we obtain the desired results recursively. □

Applying Lemma 4.4 and (4.2) to Corollary 2.2, we can calculate the densities for $N = p^r$ when p is an odd prime. The densities for $N = 2^r$ can be calculated similarly. For a general integer $N > 1$, by virtue of the following proposition, we may determine $\pi^{\lambda}_{\tilde{\varGamma}\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)$ recursively.

PROPOSITION 4.5. *Let $\varGamma_1$ and $\varGamma_2$ be subgroups of $\varGamma$ of finite index, and $\varGamma_1'$ and $\varGamma_2'$ the maximal normal subgroups of $\varGamma$ contained in $\varGamma_1$ and $\varGamma_2$, respectively. Put $\varXi_1 := \varGamma/\varGamma_1'$, $\varXi_2 := \varGamma/\varGamma_2'$, $n_1 := [\varGamma : \varGamma_1]$ and $n_2 := [\varGamma : \varGamma_2]$. Assume that $\varGamma_1'$ and $\varGamma_2'$ are relatively prime in $\varGamma$, i.e., the index of $\varGamma_1' \cap \varGamma_2'$ in $\varGamma$ is finite and $\varGamma_1'\varGamma_2' = \varGamma$ (note that $\varGamma_1$ and $\varGamma_2$ are also relatively prime in $\varGamma$). Then we have*

$$\lim_{x\to\infty} \mu^{\lambda}_{\varGamma_1\cap\varGamma_2\uparrow\varGamma}(x) = \sum_{\substack{\lambda_1\vdash n_1,\lambda_2\vdash n_2 \\ \lambda=\lambda_1\otimes\lambda_2}} \lim_{x\to\infty} \mu^{\lambda_1}_{\varGamma_1\uparrow\varGamma}(x) \lim_{x\to\infty} \mu^{\lambda_2}_{\varGamma_2\uparrow\varGamma}(x)\,,$$

where $\lambda_1 \otimes \lambda_2 := (m_1l_1, \ldots, m_1l_{k_2}, m_2l_1, \ldots, m_{k_1}l_{k_2}) \vdash n_1n_2$ for $\lambda_1 = (m_1, \ldots, m_{k_1}) \vdash n_1$ and $\lambda_2 = (l_1, \ldots, l_{k_2}) \vdash n_2$. For a partition λ which can not be expressed as $\lambda_1 \otimes \lambda_2$, we have $\mu^{\lambda}_{\varGamma_1\cap\varGamma_2\uparrow\varGamma}(x) = 0$.

REMARK 4.6. It is clear that, if Γ_1' and Γ_2' are relatively prime in Γ, then Γ_1 and Γ_2 are also relatively prime in Γ. The converse is true in the case where $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and Γ_1, Γ_2 are congruence subgroups of Γ with relatively prime levels. In general, however, this is not true; in fact, if we take $\Gamma = \Gamma_0(p)$, $\Gamma_1 = \Gamma_1(p)$ and

$$\Gamma_2 = \left\{ \gamma \in \mathrm{SL}_2(\mathbf{Z}) \,\middle|\, \gamma \equiv \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} \bmod p, \delta \in (\mathbf{Z}/p\mathbf{Z})^* \right\},$$

we have $\Gamma_1\Gamma_2 = \Gamma$ but $\Gamma_1'\Gamma_2' \neq \Gamma$ because $\Gamma_1' = \Gamma_1(p)$ and $\Gamma_2' = \Gamma(p)$.

To show Proposition 4.5, we first note the following lemmas.

LEMMA 4.7. *Let Γ_1 and Γ_2 be relatively prime subgroups of Γ. Then $[\Gamma : \Gamma_1 \cap \Gamma_2] = [\Gamma : \Gamma_1][\Gamma : \Gamma_2]$ holds, and $\mathrm{CSR}[\Gamma_2/(\Gamma_1 \cap \Gamma_2)]$ gives a complete system of representatives of Γ/Γ_1.*

PROOF. Let $\mathrm{CSR}[\Gamma_2/(\Gamma_1 \cap \Gamma_2)] = \{a_1, \dots, a_k\}$. It is easy to see that

$$\Gamma_2 = \bigcup_{i=1}^{k} a_i(\Gamma_1 \cap \Gamma_2) = \bigcup_{i=1}^{k} (a_i\Gamma_1 \cap \Gamma_2) = \left(\bigcup_{i=1}^{k} a_i\Gamma_1 \right) \cap \Gamma_2 .$$

Hence, we have $\bigcup_{i=1}^{k} a_i\Gamma_1 \supset \Gamma_2$. Since Γ_1 and Γ_2 are relatively prime, and $\bigcup_{i=1}^{k} a_i\Gamma_1$ contains both Γ_1 and Γ_2, we have $\bigcup_{i=1}^{k} a_i\Gamma_1 = \Gamma$. Hence $\mathrm{CSR}[\Gamma/\Gamma_1]$ can be chosen as a subset of $\mathrm{CSR}[\Gamma_2/(\Gamma_1 \cap \Gamma_2)]$. Now, we choose $\mathrm{CSR}[\Gamma/\Gamma_1] := \{b_1, \dots, b_l\} \subset \mathrm{CSR}[\Gamma_2/(\Gamma_1 \cap \Gamma_2)] \subset \Gamma_2$. Then it is easy to see that

$$\Gamma_2 = \Gamma \cap \Gamma_2 = \left(\bigcup_{j=1}^{l} b_j\Gamma_1 \right) \cap \Gamma_2 = \bigcup_{j=1}^{l} (b_j\Gamma_1 \cap \Gamma_2) = \bigcup_{j=1}^{l} b_j(\Gamma_1 \cap \Gamma_2) .$$

Therefore, we conclude that $\mathrm{CSR}[\Gamma/\Gamma_1] = \mathrm{CSR}[\Gamma_2/(\Gamma_1 \cap \Gamma_2)]$. □

LEMMA 4.8. *Let Γ_1 and Γ_2 be relatively prime subgroups of Γ. If $\gamma \in \mathrm{Prim}(\Gamma)$ is λ_1-type in Γ_1 and is also λ_2-type in Γ_2, then γ is $\lambda_1 \otimes \lambda_2$-type in $\Gamma_1 \cap \Gamma_2$. Furthermore, if γ is λ-type in $\Gamma_1 \cap \Gamma_2$, then there exist $\lambda_1 \vdash n_1$, $\lambda_2 \vdash n_2$ such that $\lambda = \lambda_1 \otimes \lambda_2$, and γ is λ_1-type in Γ_1 and λ_2-type in Γ_2 simultaneously.*

PROOF. By Lemma 4.7, it is easy to see that

$$\mathrm{Ind}_{\Gamma_1 \cap \Gamma_2}^{\Gamma} 1 = \mathrm{Ind}_{\Gamma_2}^{\Gamma}(\mathrm{Ind}_{\Gamma_1 \cap \Gamma_2}^{\Gamma_2} 1) = \mathrm{Ind}_{\Gamma_2}^{\Gamma}(\mathrm{Ind}_{\Gamma_1}^{\Gamma} 1\big|_{\Gamma_2}) .$$

Hence, if we put $\mathrm{CSR}[\Gamma/\Gamma_2]=\{a_1,\dots,a_{n_2}\}$, we have

$$\begin{aligned}\mathrm{tr}(\mathrm{Ind}_{\Gamma_1\cap\Gamma_2}^{\Gamma}1)(\gamma)&=\mathrm{tr}(\mathrm{Ind}_{\Gamma_2}^{\Gamma}(\mathrm{Ind}_{\Gamma_1}^{\Gamma}1|_{\Gamma_2}))(\gamma)\\&=\sum_{\substack{1\le i\le n_2\\ a_i^{-1}\gamma a_i\in\Gamma_2}}\mathrm{tr}(\mathrm{Ind}_{\Gamma_1}^{\Gamma}1|_{\Gamma_2})(a_i^{-1}\gamma a_i)\\&=\sum_{\substack{1\le i\le n_2\\ a_i^{-1}\gamma a_i\in\Gamma_2}}\mathrm{tr}(\mathrm{Ind}_{\Gamma_1}^{\Gamma}1)(\gamma)=\mathrm{tr}(\mathrm{Ind}_{\Gamma_2}^{\Gamma}1)(\gamma)\times\mathrm{tr}(\mathrm{Ind}_{\Gamma_1}^{\Gamma}1)(\gamma)\,.\end{aligned}$$

Similarly, for $k\ge 1$, we also have

$$\mathrm{tr}(\mathrm{Ind}_{\Gamma_1\cap\Gamma_2}^{\Gamma}1)(\gamma^k)=\mathrm{tr}(\mathrm{Ind}_{\Gamma_2}^{\Gamma}1)(\gamma^k)\times\mathrm{tr}(\mathrm{Ind}_{\Gamma_1}^{\Gamma}1)(\gamma^k)\,.$$

Hence, according to (4.5), we see that the type of $(\mathrm{Ind}_{\Gamma_1\cap\Gamma_2}^{\Gamma}1)(\gamma)$ coincides that of $(\mathrm{Ind}_{\Gamma_1}^{\Gamma}1)(\gamma)\otimes(\mathrm{Ind}_{\Gamma_2}^{\Gamma}1)(\gamma)$. □

PROOF OF PROPOSITION 4.5. Since Γ_1' and Γ_2' are relatively prime, according to Lemma 4.7, we can choose $\mathrm{CSR}[\Gamma/\Gamma_1']$ and $\mathrm{CSR}[\Gamma/\Gamma_2']$ as subsets of Γ_2' and Γ_1', respectively. Now, we put $\mathrm{CSR}[\Gamma/\Gamma_1']=\{a_1,\dots,a_{|\Xi_1|}\}\subset\Gamma_2'$ and $\mathrm{CSR}[\Gamma/\Gamma_2']:=\{b_1,\dots,b_{|\Xi_2|}\}\subset\Gamma_1'$. Then $\mathrm{CSR}[\Gamma/(\Gamma_1'\cap\Gamma_2')]$ can be chosen as $\{a_ib_j\}_{1\le i\le|\Xi_1|,1\le j\le|\Xi_2|}$. For $\lambda_1\vdash n_1$ and $\lambda_2\vdash n_2$, we denote by $a_1',\dots,a_{k_1}'$ (resp. $b_1',\dots,b_{k_2}'$) the elements of $\mathrm{CSR}[\Gamma/\Gamma_1']$ (resp. $\mathrm{CSR}[\Gamma/\Gamma_2']$) which are λ_1-type in Γ_1 (resp. λ_2-type in Γ_2). It is easy to see that $a_i'b_j'$, $1\le i\le k_1$, $1\le j\le k_2$, is λ_1-type in Γ_1 and is λ_2-type in Γ_2. Hence, it follows from Theorem 2.1 that

$$\begin{aligned}&\#\{\gamma\in\mathrm{CSR}[\Gamma/(\Gamma_1'\cap\Gamma_2')]\mid\gamma\text{ is }\lambda_1\text{-type in }\Gamma_1\text{ and is }\lambda_2\text{-type in }\Gamma_2\}\\&\quad=\#\{\gamma\in\mathrm{CSR}[\Gamma/\Gamma_1']\mid\gamma\text{ is }\lambda_1\text{-type in }\Gamma_1\}\#\{\gamma\in\mathrm{CSR}[\Gamma/\Gamma_2']\mid\gamma\text{ is }\lambda_2\text{-type in }\Gamma_2\}\\&\quad=|\Xi_1|\lim_{x\to\infty}\mu_{\Gamma_1}^{\lambda_1}(x)\times|\Xi_2|\lim_{x\to\infty}\mu_{\Gamma_2}^{\lambda_2}(x).\end{aligned}$$

Therefore, by Lemma 4.8, we have the proposition. □

5. Examples for congruence subgroups.

THE CASE OF $\tilde{\Gamma}=\Gamma_0(3)$.

In this case, $\Gamma'=\Gamma(3)$, $\Xi=\mathrm{SL}_2(\mathbf{Z}/3\mathbf{Z})/\{\pm\mathrm{Id}\}$, $|\Xi|=12$ and $n=1$. We have

$$\lim_{x\to\infty}\mu_{\Gamma_0(3)\uparrow\mathrm{SL}_2(\mathbf{Z})}^{\lambda_0^3(1)}(x)=\frac{1}{12}\,,\qquad\lim_{x\to\infty}\mu_{\Gamma_0(3)\uparrow\mathrm{SL}_2(\mathbf{Z})}^{\lambda_0^3(2)}(x)=\frac{1}{4}\,,$$
$$\lim_{x\to\infty}\mu_{\Gamma_0(3)\uparrow\mathrm{SL}_2(\mathbf{Z})}^{\lambda_0^3(3)}(x)=\frac{2}{3}\,,$$

where

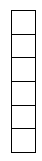
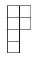
$$\lambda_0^3(1)=(1^4)\quad(=\ )\,,\qquad\lambda_0^3(2)=(2^2)\quad(=\ )\,,$$
$$\lambda_0^3(3)=(3,1)\quad(=\ )\,.$$

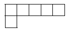
THE CASE OF $\tilde{\Gamma} = \Gamma_0(5)$.

In this case, $\Gamma' = \Gamma(5)$, $\Xi = \mathrm{SL}_2(\mathbf{Z}/5\mathbf{Z})/\{\pm \mathrm{Id}\}$, $|\Xi| = 60$ and $n = 6$. We have

$$\lim_{x\to\infty} \mu^{\lambda_0^5(1)}_{\Gamma_0(5)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{60}\,, \qquad \lim_{x\to\infty} \mu^{\lambda_0^5(2)}_{\Gamma_0(5)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{4}\,,$$
$$\lim_{x\to\infty} \mu^{\lambda_0^5(3)}_{\Gamma_0(5)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{3}\,, \qquad \lim_{x\to\infty} \mu^{\lambda_0^5(5)}_{\Gamma_0(5)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{2}{5}\,,$$

where

$\lambda_0^5(1) = (1^6)$ (= ), $\qquad \lambda_0^5(2) = (2^2, 1^2)$ (= ),

$\lambda_0^5(3) = (3^2)$ (= ), $\qquad \lambda_0^5(5) = (5, 1)$ (= ).

THE CASE OF $\tilde{\Gamma} = \Gamma_0(5^2)$.

In this case, $\Gamma' = \Gamma(5^2)$, $\Xi = \mathrm{SL}_2(\mathbf{Z}/5^2\mathbf{Z})/\{\pm \mathrm{Id}\}$, $|\Xi| = 7500$ and $n = 30$. We have

$$\lim_{x\to\infty} \mu^{\lambda_0^{25}(1)}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{7500}\,, \qquad \lim_{x\to\infty} \mu^{\lambda_0^{25}(2)}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{20}\,,$$
$$\lim_{x\to\infty} \mu^{\lambda_0^{25}(3)}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{15}\,, \qquad \lim_{x\to\infty} \mu^{\lambda_0^{25}(5,A)}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{125}\,,$$
$$\lim_{x\to\infty} \mu^{\lambda_0^{25}(5,B^{(1)})}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{2}{625}\,, \qquad \lim_{x\to\infty} \mu^{\lambda_0^{25}(5,C)}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{2}{375}\,,$$
$$\lim_{x\to\infty} \mu^{\lambda_0^{25}(10)}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{5}\,, \qquad \lim_{x\to\infty} \mu^{\lambda_0^{25}(15)}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{4}{15}\,,$$
$$\lim_{x\to\infty} \mu^{\lambda_0^{25}(25,B^{(1)})}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{8}{25}\,, \qquad \lim_{x\to\infty} \mu^{\lambda_0^{25}(25,B^{(2)})}_{\Gamma_0(25)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{2}{25}\,,$$

where

$$\lambda_0^{25}(1) = (1^{30})\,, \qquad \lambda_0^{25}(2) = (2^{14}, 1^2)\,,$$
$$\lambda_0^{25}(3) = (3^{10})\,, \qquad \lambda_0^{25}(5, A) = (5^4, 1^{10})\,,$$
$$\lambda_0^{25}(5, B^{(1)}) = (5^5, 1^5)\,, \qquad \lambda_0^{25}(5, C) = (5^6)\,,$$
$$\lambda_0^{25}(10) = (10^2, 2^4, 1^2)\,, \qquad \lambda_0^{25}(15) = (15^2)\,,$$
$$\lambda_0^{25}(25, B^{(1)}) = (25, 1^5)\,, \qquad \lambda_0^{25}(25, B^{(2)}) = (25, 5)\,.$$

THE CASE OF $\tilde{\Gamma} = \Gamma_0(3 \times 5^2)$.

Since $\Gamma_0(3 \times 5^2) = \Gamma_0(3) \cap \Gamma_0(5^2)$ and $\Gamma' = \Gamma(3) \cap \Gamma(5^2)$, by employing Proposition 4.5, we have the following results ($|\Xi| = 90000$, $n = 120$).

$$\lim_{x\to\infty} \mu^{(1^{120})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{90000}\,, \qquad \lim_{x\to\infty} \mu^{(2^{56},1^8)}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{240}\,,$$
$$\lim_{x\to\infty} \mu^{(10^8,2^{16},1^8)}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{60}\,, \qquad \lim_{x\to\infty} \mu^{(3^{40})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{180}\,,$$
$$\lim_{x\to\infty} \mu^{(15^8)}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{45}\,, \qquad \lim_{x\to\infty} \mu^{(5^{16},1^{40})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x) = \frac{1}{1500}\,,$$

$$\lim_{x\to\infty}\mu^{(5^{20},1^{20})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{3750}\,,\qquad \lim_{x\to\infty}\mu^{(25^{4},1^{20})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{2}{75}\,,$$
$$\lim_{x\to\infty}\mu^{(25^{4},5^{2})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{150}\,,\qquad \lim_{x\to\infty}\mu^{(5^{24})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{2250}\,,$$
$$\lim_{x\to\infty}\mu^{(2^{60})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{30000}\,,\qquad \lim_{x\to\infty}\mu^{(4^{28},2^{4})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{80}\,,$$
$$\lim_{x\to\infty}\mu^{(20^{4},4^{8},2^{4})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{20}\,,\qquad \lim_{x\to\infty}\mu^{(6^{20})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{60}\,,$$
$$\lim_{x\to\infty}\mu^{(30^{4})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{15}\,,\qquad \lim_{x\to\infty}\mu^{(10^{8},2^{10})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{500}\,,$$
$$\lim_{x\to\infty}\mu^{(10^{10},2^{10})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{1250}\,,\qquad \lim_{x\to\infty}\mu^{(50^{2},2^{10})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{2}{25}\,,$$
$$\lim_{x\to\infty}\mu^{(50^{2},10^{2})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{50}\,,\qquad \lim_{x\to\infty}\mu^{(10^{12})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{750}\,,$$
$$\lim_{x\to\infty}\mu^{(3^{30},1^{30})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{11250}\,,\qquad \lim_{x\to\infty}\mu^{(6^{14},3^{2},2^{14},1^{2})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{1}{30}\,,$$
$$\lim_{x\to\infty}\mu^{(30^{2},10^{2},6^{4},3^{2},2^{4},1^{2})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{2}{15}\,,\qquad \lim_{x\to\infty}\mu^{(9^{10},3^{10})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{2}{45}\,,$$
$$\lim_{x\to\infty}\mu^{(45^{2},15^{2})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{8}{45}\,,\qquad \lim_{x\to\infty}\mu^{(15^{4},5^{5},3^{10},1^{10})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{2}{375}\,,$$
$$\lim_{x\to\infty}\mu^{(15^{5},5^{5},3^{5},1^{5})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{4}{1875}\,,\qquad \lim_{x\to\infty}\mu^{(75,25,3^{5},1^{5})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{16}{75}\,,$$
$$\lim_{x\to\infty}\mu^{(75,25,15,5)}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{4}{75}\,,\qquad \lim_{x\to\infty}\mu^{(15^{6},5^{6})}_{\Gamma_0(75)\uparrow \mathrm{SL}_2(\mathbf{Z})}(x)=\frac{4}{1125}\,.$$

REMARK 5.1. In the case of $\tilde{\Gamma}=\Gamma_0(3\times 5^2)$, $\lambda_0^3(m_1)\otimes\lambda_0^{5^2}(l_1)\neq\lambda_0^3(m_2)\otimes\lambda_0^{5^2}(l_2)$ holds if $(m_1,l_1)\neq(m_2,l_2)$. For a general pair of relatively prime Γ_1 and Γ_2, we do not know whether there are partitions $\lambda_1,\lambda_1'\vdash n_1$ and $\lambda_2,\lambda_2'\vdash n_2$ with $\lambda_1\neq\lambda_1'$, $\lambda_2\neq\lambda_2'$ such that $\lambda_1\otimes\lambda_2=\lambda_1'\otimes\lambda_2'$ provided that $\mu^{\lambda_1}_{\Gamma_1\uparrow\Gamma}(x)$, $\mu^{\lambda_1'}_{\Gamma_1\uparrow\Gamma}(x)$, $\mu^{\lambda_2}_{\Gamma_2\uparrow\Gamma}(x)$ and $\mu^{\lambda_2'}_{\Gamma_2\uparrow\Gamma}(x)$ have non-zero densities.

REMARK 5.2. All elements γ of Γ are $(\dots,M(\gamma)^l)$-type in $\tilde{\Gamma}$, $l>0$, for examples discussed in this section (see Theorem 2.1 for the definition of $M(\gamma)$). In general, we may expect that there is some γ whose type is of the form $(\dots,M)$ for $M<M(\gamma)$, but we have never found such examples unfortunately. It is interesting to study the density $\hat{\mu}_{\tilde{\Gamma}\uparrow\Gamma}(x):=\hat{\pi}_{\tilde{\Gamma}\uparrow\Gamma}(x)/\pi_\Gamma(x)$ as $x\to\infty$, where $\hat{\pi}_{\tilde{\Gamma}\uparrow\Gamma}(x):=\{\gamma\in\mathrm{Prim}(\Gamma)\mid N(\gamma)<x,\gamma\text{ is }(\dots,M)\text{-type in }\tilde{\Gamma}\text{ for some }M<M(\gamma)\}$.

6. A Remark on Selberg's zeta functions. Let $\zeta_\Gamma(s)$ be a Selberg zeta function of Γ defined by

$$\zeta_\Gamma(s):=\prod_{\gamma\in\mathrm{Prim}(\Gamma)}(1-N(\gamma)^{-s})^{-1}\,,\quad \Re s>1\,.$$

For $\tilde{\Gamma} \subset \Gamma$ and $\lambda \vdash n$, we define a Selberg type zeta function attached to this data by

$$\zeta^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(s) := \prod_{\substack{\gamma\in\mathrm{Prim}(\Gamma)\\ \gamma \text{ is } \lambda\text{-type in } \tilde{\Gamma}}} (1 - N(\gamma)^{-s})^{-1}\,, \quad \Re s > 1\,.$$

By using Venkov-Zograf's formula [VZ], we have

$$\begin{aligned}
\zeta_{\tilde{\Gamma}}(s) &= \zeta_{\Gamma}(s,\sigma)\\
&= \prod_{\gamma\in\mathrm{Prim}(\Gamma)} \det\left(\mathrm{Id} - \sigma(\gamma)N(\gamma)^{-s}\right)^{-1}\\
&= \prod_{\lambda\vdash n} \prod_{\substack{\gamma\in\mathrm{Prim}(\Gamma)\\ \gamma \text{ is } \lambda\text{-type in } \tilde{\Gamma}}} \det\left(\mathrm{Id} - \sigma(\gamma)N(\gamma)^{-s}\right)^{-1}\\
&= \prod_{\lambda=(m_1,m_2,\dots,m_k)\vdash n} \zeta^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(m_1 s)\cdots\zeta^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(m_k s)\,,
\end{aligned} \tag{6.1}$$

where $\sigma := \mathrm{Ind}_{\tilde{\Gamma}}^{\Gamma} 1$. Although detailed investigation of $\zeta^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(s)$ remains in the future, in this section, we study $\zeta^{\lambda}_{\tilde{\Gamma}\uparrow\Gamma}(s)$ for a particular case where $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, $\tilde{\Gamma} = \Gamma_1(p)$ and $\Gamma(p)$ as follows.

In these cases, from the discussion in Section 4, it is easy to see that

$$\gamma \text{ is } \lambda_1^p(m)\text{-type in } \Gamma_1(p) \Leftrightarrow \gamma \text{ is } \lambda^p(m)\text{-type in } \Gamma(p) \Leftrightarrow M(\gamma) = m\,.$$

Then we have

$$\zeta^{\lambda_1^p(m)}_{\Gamma_1(p)\uparrow\mathrm{SL}_2(\mathbf{Z})}(s) = \zeta^{\lambda^p(m)}_{\Gamma(p)\uparrow\mathrm{SL}_2(\mathbf{Z})}(s) = \prod_{\substack{\gamma\in\mathrm{Prim}(\Gamma)\\ M(\gamma)=m}} (1 - N(\gamma)^{-s})^{-1}\,.$$

For simplicity, we denote this function by $\zeta^{(p,m)}_{\mathrm{SL}_2(\mathbf{Z})}(s)$. The functions $\zeta^{(p,m)}_{\mathrm{SL}_2(\mathbf{Z})}(s)$ have the following properties.

PROPOSITION 6.1. *Let p be an odd prime. Then we have*

$$\left\{\frac{(\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s))^p}{\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(ps)}\right\}^{(p-1)/2} = \frac{(\zeta_{\Gamma_1(p)}(s))^p}{\zeta_{\Gamma(p)}(s)}\,. \tag{6.2}$$

Furthermore, $(\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s))^{p^r(p-1)/2}$ can be analytically continued to $\Re s > 1/p^r$ as a meromorphic function and $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s)$ has infinitely many singular points near $s = 0$.

PROOF. By using (6.1) together with results in Section 4, we have

$$\begin{aligned}\zeta_{\Gamma(p)}(s) &= (\zeta^{(p,1)}_{\mathrm{SL}_2(\mathbf{Z})}(s))^{-p(p^2-1)/2}(\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(ps))^{-(p^2-1)/2}\\ &\quad\times \prod_{m|\frac{p\pm1}{2},m>1}(\zeta^{(p,m)}_{\mathrm{SL}_2(\mathbf{Z})}(ms))^{-p(p^2-1)/2m},\\ \zeta_{\Gamma_1(p)}(s) &= (\zeta^{(p,1)}_{\mathrm{SL}_2(\mathbf{Z})}(s))^{-(p^2-1)/2}(\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s)\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(ps))^{-(p-1)/2}\\ &\quad\times \prod_{m|\frac{p\pm1}{2},m>1}(\zeta^{(p,m)}_{\mathrm{SL}_2(\mathbf{Z})}(ms))^{-(p^2-1)/2m}.\end{aligned}$$

Hence the formula (6.2) follows immediately.

Since $\zeta_{\Gamma_1(p)}(s)$ and $\zeta_{\Gamma(p)}(s)$ are meromorphic in the whole $\boldsymbol{C}$ and $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(ps)$ is non-zero and holomorphic in $\Re s > 1/p$, we see that $(\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s))^{p(p-1)/2}$ is analytically continued to $\Re s > 1/p$ as a meromorphic function. Now, we take p^{r-1}-powers of the both sides of (6.2). Then we have

$$\begin{aligned}(\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s))^{p^r(p-1)/2} &= (\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(ps))^{p^{r-1}(p-1)/2}\left\{\frac{(\zeta_{\Gamma_1(p)}(s))^p}{\zeta_{\Gamma(p)}(s)}\right\}^{p^{r-1}}\\ &= (\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(p^2s))^{p^{r-2}(p-1)/2}\left\{\frac{(\zeta_{\Gamma_1(p)}(ps))^p}{\zeta_{\Gamma(p)}(ps)}\right\}^{p^{r-2}}\left\{\frac{(\zeta_{\Gamma_1(p)}(s))^p}{\zeta_{\Gamma(p)}(s)}\right\}^{p^{r-1}}\\ &= \cdots\\ &= (\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(p^rs))^{(p-1)/2}\prod_{k=1}^{r}\left\{\frac{(\zeta_{\Gamma_1(p)}(p^ks))^p}{\zeta_{\Gamma(p)}(p^ks)}\right\}^{p^{r-k}}. \end{aligned} \tag{6.3}$$

Since $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(p^rs)$ is non-zero holomorphic in $\Re s > 1/p^r$, we can obtain the meromorphic continuation of $(\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s))^{p^r(p-1)/2}$ to the half plane $\Re s > 1/p^r$.

Both the functions $\zeta_{\Gamma_1(p)}(s)$ and $\zeta_{\Gamma(p)}(s)$ have simple poles at $s = 1$ (see [He]). Hence, $(\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s))^p$ has a double pole at $s = 1$. Thus $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(ps)$ has a branch point at $s = 1/p$. Since $\zeta_{\Gamma_1(p)}(s)$ and $\zeta_{\Gamma(p)}(s)$ are meromorphic at $s = 1/p$, by (6.2), $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s)$ should have a branch point at $s = 1/p$. Then $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(ps)$ has a branch point at $s = 1/p^2$. Successively, we see that $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s)$ has branch points at $s = 1, 1/p, 1/p^2, 1/p^3, \ldots$. This shows that the series of the branch points $\{1, 1/p, 1/p^2, 1/p^3, \ldots\}$ has an accumulation point at 0. Similarly, it is easy to see that $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s)$ has branch points at $s = 1/2 + ir_j$, $(1/2 + ir_j)/p$, $(1/2 + ir_j)/p^2, \ldots$, where $1/4 + r_j^2$ is the j-th non-trivial eigenvalue of the Laplacian on $X_{\Gamma(p)}$ but not the spectrum of $X_{\Gamma_1(p)}$. Hence $\zeta^{(p,p)}_{\mathrm{SL}_2(\mathbf{Z})}(s)$ has infinitely many branch points near $s = 0$. $\square$

REMARK 6.2. In Proposition 6.1, we obtain the analytic continuation of $\zeta_{\mathrm{SL}_2(\mathbf{Z})}^{(p,p)}(s)$ to the half plane $\Re s > 0$. We do not know, however, whether $\zeta_{\mathrm{SL}_2(\mathbf{Z})}^{(p,p)}(s)$ can be analytically continued to a region contained in $\Re s \leq 0$ or has a natural boundary $\Re s = 0$.

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