Tohoku Math. J.
59 (2007), 441–454

SPECTRAL SYNTHESIS IN THE FOURIER ALGEBRA AND THE VAROPOULOS ALGEBRA

KRISHNAN PARTHASARATHY AND RAJENDRAN PRAKASH

(Received March 13, 2006, revised January 17, 2007)

Abstract. The objects of study in this paper are sets of spectral synthesis for the Fourier algebra $A(G)$ of a locally compact group and the Varopoulos algebra $V(G)$ of a compact group with respect to submodules of the dual space. Such sets of synthesis are characterized in terms of certain closed ideals. For a closed set in a closed subgroup H of G, the relations between these ideals in the Fourier algebras of G and H are obtained. The injection theorem for such sets of synthesis is then a consequence. For the Fourier algebra of the quotient modulo a compact subgroup, an inverse projection theorem is proved. For a compact group, a correspondence between submodules of the dual spaces of $A(G)$ and $V(G)$ is set up and this leads to a relation between the corresponding sets of synthesis.

Introduction. Spectral synthesis in the Fourier algebra of a locally compact abelian group is a vintage topic in harmonic analysis, with Malliavin's celebrated theorem on the failure of spectral synthesis going back to 1959. Although the study of spectral synthesis in the Fourier algebra $A(G)$ of an arbitrary locally compact group was intiated by Eymard himself in his original study ([2]) of $A(G)$, not many papers have appeared in the topic. In a recent work [6], Kaniuth and Lau introduce and study the concept of X-synthesis where X is an $A(G)$-submodule of the group von Neumann algebra $VN(G) = A(G)^*$. This concept is studied in some detail in this paper.

The concept of X-synthesis has been defined using supports of linear functionals. In Section 2, we define, in the general context of commutative, semisimple, regular Banach algebras, two closed ideals $I_A^X(E)$ and $J_A^X(E)$ and prove that E is of X-synthesis precisely when these two ideals are equal. When X is the full dual, this reduces to the usual definition of sets of synthesis.

Suppose, next, that H is a closed subgroup of G and $E \subseteq H$ is closed. If $r : A(G) \to A(H)$ is the restriction map, we show, in Section 3, that $I_{A(G)}^X(E) = r^{-1}(I_{A(H)}^{X_H}(E))$ and $J_{A(G)}^X(E) = r^{-1}(J_{A(H)}^{X_H}(E))$, where X_H is an $A(H)$-submodule of $VN(H)$ associated to X. An immediate consequence is the Injection Theorem for X-spectral sets due to Kaniuth and Lau [6].

For a compact subgroup K of G, Forrest [3] has defined and studied the Fourier algebra $A(G/K)$ on the homogeneous space G/K. With any $A(G)$-submodule X of $VN(G)$, we associate an $A(G/K)$-submodule X_K of $VN(G/K) = A(G/K)^*$ and prove that a closed set

2000 *Mathematics Subject Classification.* Primary 43A45; Secondary 43A77, 43A85.

Key words and phrases. Fourier algebra, Varopoulos algebra, spectral synthesis.

The second author gratefully acknowledges financial support from the National Board for Higher Mathematics in the form of a research fellowship during the period of this work.

$\tilde{E} \subseteq G/K$ is of X_K-synthesis for $A(G/K)$ if $\pi^{-1}(\tilde{E})$ is of X-synthesis for $A(G)$, $\pi : G \to G/K$ being the canonical map (Section 3). We recover a result on synthesis due to Forrest [3] when $X = VN(G)$.

For a compact abelian group G, Varopoulos [12] related synthesis in $A(G)$ to synthesis in the Varopoulos algebra $V(G)$. (He used this relation to give his famous tensor algebra proof of Malliavin's theorem.) This study has recently been carried over to nonabelian groups by Spronk and Turowska [10]. Our second main purpose is to investigate X-synthesis in this context. We set up (in Section 4) a correspondence between $A(G)$-submodules X of $VN(G)$ and $V(G)$-submodules Y of $V(G)^*$. Associated to an X we have a $Y = X_V$ and associated to a Y we give an $X = Y_A$. We show that this correspondence is a natural one by proving that $(X_V)_A = X$. The main result on synthesis in this context is: E is of X-synthesis for $A(G)$ if and only if $E^* := \{(x, y) \in G \times G; xy^{-1} \in E\}$ is of X_V-synthesis for $V(G)$. Some of the main ingredients in the proof are the facts that $Y = X_V$, $I_V(E^*)^\perp$ and $I_V^Y(E^*)$ are all $L^1(G)$-modules. These and other needed results are presented as a sequence of lemmas preceding the theorem in Section 5. When $X = VN(G)$, we recover the result of Varopoulos [12] and Spronk-Turowska [10].

1. Preliminaries. The Fourier algebra $A(G)$ of a locally compact abelian group G is just the algebra of Fourier transforms of integrable functions on the dual group $\hat{G}$. It is a commutative Banach algebra with the norm carried over from $L^1(\hat{G})$. When G is any arbitrary locally compact group, the *Fourier algebra* $A(G)$ as defined and studied by Eymard [2] consists of continuous functions on G of the form $u(x) = \langle \lambda(x) f, g\rangle$, $x \in G$, where $f, g \in L^2(G)$ and λ is the left regular representation of G. Thus $A(G)$ is the space of coefficient functions of the left regular representation. To describe the norm on $A(G)$, consider the group von Neumann algebra $VN(G)$ of G. Recall that $VN(G)$ is the closure in the weak operator topology of span $\{\lambda(x); x \in G\}$ in $\mathcal{B}(L^2(G))$. For $u \in A(G)$, with $u(x) = \langle \lambda(x) f, g\rangle$,

$$\|u\|_A = \sup\{|\langle Tf, g\rangle|; T \in VN(G), \|T\| \leq 1\}.$$

With this norm $A(G)$ is a Banach space, and with the pairing defined by $\langle T, u\rangle = \langle Tf, g\rangle$, it is the predual of $VN(G)$. Moreover, with pointwise multiplication, $A(G)$ is a commutative, semisimple, regular Banach algebra whose Gelfand structure space is identified with G (*via* point evaluations). All these and more can be found in Eymard [2], which is the basic reference for $A(G)$.

The second Banach algebra that would be considered is the *Varopoulos algebra* $V(G)$ of a compact group G. It is the completion of the algebraic tensor product $C(G) \otimes C(G)$ with respect to the norm defined by

$$\|v\|_V = \inf \|\textstyle\sum |\varphi_i|^2\|_\infty^{1/2} \|\textstyle\sum |\psi_i|^2\|_\infty^{1/2},$$

the infimum being taken over all (finite sum) representations $v = \sum \varphi_i \otimes \psi_i$ in $C(G) \otimes C(G)$. Thus $V(G) = C(G) \otimes^h C(G)$, the Haagerup tensor product. Every v in $V(G)$ can be represented as a norm convergent series $\sum \varphi_i \otimes \psi_i$ and $\|v\|_V$ is the infimum of

$\| \sum |\varphi_i|^2 \|_\infty^{1/2} \| \sum |\psi_i|^2 \|_\infty^{1/2}$ over all such representations, where $\sum |\varphi_i|^2$, $\sum |\psi_i|^2$ are (uniformly) convergent in $C(G)$. Varopoulos [12] used the projective tensor product, but the projective norm is equivalent to the Haagerup norm (see Spronk and Turowska [10]). $V(G)$ is a commutative, semisimple, regular Banach algebra with Gelfand structure space $G \times G$.

We are concerned with spectral synthesis in $A(G)$ and $V(G)$. Here are the basic definitions . Let $\mathcal{A}$ be a commutative, semisimple, regular Banach algebra with Gelfand space $\Delta(\mathcal{A})$. For a closed set E in $\Delta(\mathcal{A})$, let

$$\begin{aligned} j_{\mathcal{A}}(E) &= \{a \in \mathcal{A};\, \hat{a} \text{ has compact support disjoint from } E\}\,, \\ J_{\mathcal{A}}(E) &= \overline{j_{\mathcal{A}}(E)}\,, \\ I_{\mathcal{A}}(E) &= \{a \in \mathcal{A};\, \hat{a} = 0 \text{ on } E\}\,. \end{aligned}$$

(When $\mathcal{A} = A(G)$, we write these as $j_A(E)$ etc; similarly, we use the notation $j_V(E)$ etc, in the case $\mathcal{A} = V(G)$.) All the three sets are ideals in $\mathcal{A}$ with zero set E and $j_{\mathcal{A}}(E) \subseteq I \subseteq I_A(E)$ for any ideal I with zero set E. E is said to be a *set of spectral synthesis* (or a *spectral set*) for $\mathcal{A}$ if $I_{\mathcal{A}}(E) = J_{\mathcal{A}}(E)$. This is equivalent to saying that there is a unique closed ideal with zero set E.

2. X-Synthesis. Kaniuth and Lau [6] have introduced the concept of sets of X-synthesis, where X is an $A(G)$-submodule of $VN(G)$. In this section we study this concept. With later use in mind, we formulate the definitions in a general context.

Let $\mathcal{A}$ be a commutative, semisimple, regular Banach algebra. The Banach space dual $\mathcal{A}^*$ has a natural $\mathcal{A}$-module structure. For $u \in \mathcal{A}$ and $T \in \mathcal{A}^*$, define $u.T$ by $\langle u.T, v\rangle = \langle T, uv\rangle$, $v \in \mathcal{A}$. The concept of the *support* of a linear functional $T \in \mathcal{A}^*$ is a much needed one in spectral synthesis. Of the different formulations, the one that would be convenient for our purposes is as follows:

$$\operatorname{supp} T = \{\chi \in \Delta(\mathcal{A});\; u.T \neq 0 \;\text{ whenever }\; \hat{u}(\chi) \neq 0\}\,.$$

It is a closed subset of $\Delta(\mathcal{A})$. Let X be an $\mathcal{A}$-submodule of $\mathcal{A}^*$. Then a closed set $E \subseteq \Delta(\mathcal{A})$ is a *set of X-synthesis* (or an *X-spectral set*) if $\langle T, u\rangle = 0$ for every $T \in X$ with $\operatorname{supp} T \subseteq E$ and every $u \in I_{\mathcal{A}}(E)$. When $X = \mathcal{A}^*$, we have the following result.

LEMMA 2.1. *A closed set $E \subseteq \Delta(\mathcal{A})$ is of spectral synthesis if and only if it is of $\mathcal{A}^*$-synthesis.*

PROOF. Using the regularity of $\mathcal{A}$, the proof is essentially the same as that given in [6] for the case $\mathcal{A} = A(G)$. □

We begin by looking at some examples in the case $\mathcal{A} = A(G)$ and $\mathcal{A}^* = VN(G)$.

EXAMPLE 2.2. (i) It is clear that if X, Y are two $A(G)$-submodules of $VN(G)$ such that $X \subseteq Y$, then any Y-spectral set is an X-spectral set. In particular, sets of synthesis are of X-synthesis for any choice of X.

(ii) If $X = UC_c(\hat{G}) := \{T \in VN(G);\ \text{supp}\, T \text{ is compact}\}$, then sets of X-synthesis are nothing but sets of local synthesis (see [6]). Recall that E is a set of local synthesis if every $u \in I_A(E)$ having compact support belongs to $J_A(E)$.

(iii) Let $x \in G$ be fixed and let $X = \{u.\lambda(x);\ u \in A(G)\}$. Then every closed set $E \subseteq G$ is of X-synthesis.

(iv) Let X be the set of all finite sums $\sum u_i.\lambda(x_i)$ with $u_i \in A(G)$ and $x_i \in G$. Consider $T = \sum_{i=1}^n u_i.\lambda(x_i) \in X$. If $x \notin \{x_1, \ldots, x_n\}$ and if $u \in A(G)$ is chosen such that $u(x_i) = 0$ for all i and $u(x) \neq 0$, then $u.T = 0$. It follows that supp $T \subseteq \{x_1, \ldots, x_n\}$. In fact, it is not difficult to see that supp $T = \{x_i;\ u_i(x_i) \neq 0\}$. From this it follows that for any closed set E, if supp $T \subseteq E$ and $u \in I_A(E)$, then $\langle T, u\rangle = \sum u(x_i)u_i(x_i) = 0$. In other words, every closed set is a set of X-synthesis.

(v) Let $F \subseteq G$ be closed. Consider

$$X = VN_F(G) := \{T \in VN(G); \text{supp}\, T \subseteq F\}\,.$$

It follows, by Eymard's results on supports of elements of $VN(G)$ of the form $S + T$ and $u.T$ ([2, Proposition 4.8]), that $VN_F(G)$ is an $A(G)$-submodule. Moreover, it is weak-$*$ closed: if $\{T_\alpha\}$ is a net converging weak-$*$ to T and if supp $T_\alpha \subseteq F$ for every α, then supp $T \subseteq F$ (Eymard [2]). In fact, it can be seen that

$$VN_F(G) = J_A(F)^\perp := \{T \in VN(G);\ \langle T, u\rangle = 0 \text{ for every } u \in J_A(F)\}\,.$$

Now it is easy to see that if $E \subseteq F$ is of X-synthesis, then it is actually a set of synthesis. Thus $E \subseteq F$ is of synthesis if and only if E is of $VN_F(G)$-synthesis. (When $F = G$, $VN_F(G) = VN(G)$ and we recover the result of [6] that $VN(G)$-synthesis is same as synthesis.) This is not true for sets $E \supset F$ as the next example shows.

(vi) Take $G = \boldsymbol{R}^n$, $F = S^{n-1}$, with $n \geq 3$, in the previous example. It is a classical result of L. Schwartz that F is of non-synthesis. Now let $E = E_1 \cup E_2$, where

$$E_1 = \{x \in \boldsymbol{R}^n; 1/2 \leq \|x\| \leq 3/2\}\,, \qquad E_2 = \{x \in \boldsymbol{R}^n; \|x\| = 1/4\}\,.$$

Then E_2 is a set of nonsynthesis, whereas E_1 is a set of synthesis (for instance, using the resuts on intersections of sets of synthesis in Muraleedharan and Parthasarathy [7]). Hence E is of nonsynthesis, because the union of two disjoint closed sets is of synthesis if and only if each of them is. On the other hand, it is easy to see that if $u \in I_A(E)$, then supp $u \cap F = \emptyset$ and so $\langle T, u\rangle = 0$ for $T \in VN_F(G)$. This means that E is of $VN_F(G)$-synthesis. Thus $E \supset F$ is of $VN_F(G)$-synthesis but is not of synthesis.

REMARK 2.3. (a) The $A(G)$-submodule X in example (iv) is weak-$*$ dense in $VN(G)$, yet every closed set is of X-synthesis. Kaniuth and Lau [6] have shown that every closed set is of $VN(G)$-synthesis if and only if G is discrete and $u \in \overline{uA(G)}$ for every $u \in A(G)$.

(b) The set $\lambda^{-1}(X) = \{x \in G;\ \lambda(x) \in X\}$ is closed if X is weak-$*$ closed. *Question*: When is it of X-synthesis? Observe that when $X = VN_F(G)$, $\lambda^{-1}(X) = F$ and so $\lambda^{-1}(X)$ is of X-synthesis if and only if it is of synthesis.

Recall that spectral synthesis has been defined in terms of ideals: E is of synthesis if $I_{\mathcal{A}}(E) = J_{\mathcal{A}}(E)$. So it is natural to try to reformulate the notion of X-synthesis in terms of ideals. This task, it turns out, is not difficult.

Let X be an $\mathcal{A}$-submodule of $\mathcal{A}^*$. For a closed set $E \subseteq \Delta(\mathcal{A})$, define

$$I^X_{\mathcal{A}}(E) = \{u \in \mathcal{A};\ \langle T, u\rangle = 0 \ \text{ for every } \ T \in X \cap I_{\mathcal{A}}(E)^{\perp}\},$$
$$J^X_{\mathcal{A}}(E) = \{u \in \mathcal{A};\ \langle T, u\rangle = 0 \ \text{ for every } \ T \in X \cap J_{\mathcal{A}}(E)^{\perp}\}.$$

These are clearly closed, and are ideals since X is an $\mathcal{A}$-submodule. Note that $J^X_{\mathcal{A}}(E) \subseteq I^X_{\mathcal{A}}(E)$. Observe also that when $X = \mathcal{A}^*$, $I^X_{\mathcal{A}}(E) = I_{\mathcal{A}}(E)$ and $J^X_{\mathcal{A}}(E) = J_{\mathcal{A}}(E)$. Here is the promised characterization of X-synthesis in terms of these ideals.

PROPOSITION 2.4. *Let $\mathcal{A}$ be a commutative, semisimple, regular Banach algebra and let X be an $\mathcal{A}$-submodule of $\mathcal{A}^*$. A closed set $E \subseteq \Delta(\mathcal{A})$ is of X-synthesis if and only if $I^X_{\mathcal{A}}(E) = J^X_{\mathcal{A}}(E)$.*

PROOF. Suppose E is of X-synthesis. Then

$$\begin{aligned} T \in X \cap J_{\mathcal{A}}(E)^{\perp} &\Rightarrow T \in X \ \text{ and } \ \operatorname{supp} T \subseteq E \\ &\Rightarrow \langle T, u\rangle = 0 \ \text{ for } \ u \in I_{\mathcal{A}}(E) \\ &\Rightarrow T \in X \cap I_{\mathcal{A}}(E)^{\perp}. \end{aligned}$$

Thus, $I^X_{\mathcal{A}}(E) \subseteq J^X_{\mathcal{A}}(E)$. So equality holds. Conversely, suppose $I^X_{\mathcal{A}}(E) = J^X_{\mathcal{A}}(E)$. Then

$$\begin{aligned} T \in X \ \text{ and } \ \operatorname{supp} T \subseteq E &\Rightarrow T \in X \cap J_{\mathcal{A}}(E)^{\perp} \\ &\Rightarrow \langle T, u\rangle = 0 \ \text{ for } \ u \in J^X_{\mathcal{A}}(E) = I^X_{\mathcal{A}}(E) \\ &\Rightarrow \langle T, u\rangle = 0 \ \text{ for } \ u \in I_{\mathcal{A}}(E) \subseteq I^X_{\mathcal{A}}(E). \end{aligned}$$

Thus E is of X-synthesis. □

The next result identifies the zero sets of the two ideals $I^X_A(E)$ and $J^X_A(E)$.

PROPOSITION 2.5. *Let X be a weak-$*$ closed $A(G)$-submodule of $VN(G)$ and let $E \subseteq G$ be closed. Consider the closed set $E_X := E \cap \lambda^{-1}(X)$. Then $Z(I^X_A(E)) = E_X = Z(J^X_A(E))$.*

PROOF. Suppose $x \in E_X$, so $x \in E$ and $\lambda(x) \in X$. For $u \in I_A(E)$, $\langle \lambda(x), u\rangle = u(x) = 0$ and $\lambda(x) \in I_A(E)^{\perp}$. Thus, if $v \in I^X_A(E)$, then $v(x) = \langle \lambda(x), v\rangle = 0$, since $\lambda(x) \in X \cap I_A(E)^{\perp}$. This means $x \in Z(I^X_A(E))$.

On the other hand, if $x \notin E$, there is an open set U with compact closure such that $x \in U \subset \bar{U} \subset E^c$. Then there is a $u \in A(G)$ with $u(x) = 1$ and $\operatorname{supp} u \subset U$. Thus $u(x) \neq 0$ and $u \in j_A(E) \subset J^X_A(E)$, and this, in turn, gives $x \notin Z(J^X_A(E))$. Further, if $\lambda(x) \notin X$, then there is a $u \in A(G)$ such that $\langle T, u\rangle = 0$ for all $T \in X$, while $u(x) = \langle \lambda(x), u\rangle \neq 0$, since X is weak-$*$ closed. This, in particular, gives that $u \in J^X_A(E)$, but $u(x) \neq 0$. This implies $x \notin Z(J^X_A(E))$. Thus, if $x \notin E_X$, then $x \notin Z(J^X_A(E))$.

We have therefore proved that $Z(J^X_A(E)) \subseteq E_X \subseteq Z(I^X_A(E)) \subseteq Z(J^X_A(E))$. The result follows. □

COROLLARY 2.6. *If X is a weak-$*$ closed $A(G)$-submodule of $VN(G)$, then* $J_A(E_X) \subseteq J_A^X(E) \subseteq I_A^X(E) \subseteq I_A(E_X)$.

PROOF. The first and the last inclusions are consequences of Proposition 2.5 and the fact that $J_A(E_X)$ is the smallest and $I_A(E_X)$ is the largest closed ideal, respectively, with zero set E_X. The middle inclusion being obvious, the corollary is proved.

COROLLARY 2.7. *With X as before, if E_X is of synthesis, then E is of X-synthesis.*

EXAMPLE 2.8. With notation as in Example 2.2 (vii), say, E_1 is of synthesis, $(E_1)_X = S^{n-1}$ is of nonsynthesis. Thus the reverse implication in Corollary 2.7 does not hold.

3. Subgroups and quotients. Let H be a closed subgroup of G. Relations between spectral synthesis in H and G/H with that in G are considered in this section.

Let $VN_H(G)$ denote the weak-$*$ closed span of $\{\lambda_G(h); h \in H\}$ in $VN(G)$, and let, as usual, $VN(H)$ be the group von Neumann algebra of H.

It is well known (Herz [5]) that the restriction map $r : u \mapsto ru = u|_H$ is a continuous linear surjection of $A(G)$ onto $A(H)$. It is shown in [6, Lemma 3.1] that the adjoint map $r^* : VN(H) \to VN(G)$ is an isomorphism of $VN(H)$ onto $VN_H(G)$.

For an $A(G)$-submodule X of $VN(G)$, write $X_H = r^{*-1}(X)$. It is easy to see that X_H is an $A(H)$-submodule of $VN(H)$. Note that $X_H = VN(H)$ when $X = VN(G)$. The next result relates the ideals $I^X_{A(G)}(E)$ and $J^X_{A(G)}(E)$ introduced earlier with the corresponding ideals in $A(H)$.

THEOREM 3.1. *Let H be a closed subgroup of G and let $E \subseteq H$ be a closed set. Then*
(i) $I^X_{A(G)}(E) = r^{-1}(I^{X_H}_{A(H)}(E))$,
(ii) $J^X_{A(G)}(E) = r^{-1}(J^{X_H}_{A(H)}(E))$.

PROOF. (i) Suppose $u \in A(G)$ and $ru \in I^{X_H}_{A(H)}(E)$. To show $u \in I^X_{A(G)}(E)$, let $T \in X \cap I_{A(G)}(E)^\perp$. Now $T \in I_{A(G)}(E)^\perp$ implies that $T = r^*(S)$ for a (unique) $S \in VN(H)$. We claim that $S \in X_H \cap I_{A(H)}(E)^\perp$. Since $T \in X$, $S \in X_H$ by definition, and $\langle T, u\rangle = \langle r^*S, u\rangle = \langle S, ru\rangle$. If $v \in I_{A(H)}(E)$, then $v = rw$ for some $w \in I_{A(G)}(E)$ and $\langle S, v\rangle = \langle S, rw\rangle = \langle r^*S, w\rangle = \langle T, w\rangle = 0$, since $T \in I_{A(G)}(E)^\perp$. This proves the claim that $S \in X_H \cap I_{A(H)}(E)^\perp$. But then $\langle T, u\rangle = \langle S, ru\rangle = 0$, proving that $u \in I^X_{A(G)}(E)$.

Conversely, let $u \in I^X_{A(G)}(E)$. Then $ru \in A(H)$. To show that $ru \in I^{X_H}_{A(H)}(E)$, let $S \in X_H \cap I_{A(H)}(E)^\perp$. Now $S \in X_H$ implies that $T = r^*S \in X$. We claim that $T \in I_{A(G)}(E)^\perp$. For, if $v \in I_{A(G)}(E)$, then clearly $rv \in I_{A(H)}(E)$ and $\langle T, v\rangle = \langle r^*S, v\rangle = \langle S, rv\rangle = 0$ since $S \in I_{A(H)}(E)^\perp$. Hence $T \in X \cap I_{A(G)}(E)^\perp$ and $\langle S, ru\rangle = \langle T, u\rangle = 0$. Thus $ru \in I^{X_H}_{A(H)}(E)$, so $u \in r^{-1}(I^{X_H}_{A(H)}(E))$.

(ii) Every closed subgroup is of synthesis, by [11, Theorem 3]. So $I_{A(G)}(H) = J_{A(G)}(H) \subseteq J_{A(G)}(E)$ and hence $J_{A(G)}(E)^\perp \subseteq I_{A(G)}(H)^\perp$. Thus $T \in X \cap J_{A(G)}(E)^\perp$ implies $T \in X \cap I_{A(G)}(H)^\perp$, and this in turn gives $T = r^*S$ with $S \in X_H$. On the other hand $T \in J_{A(G)}(E)^\perp$ also yields that supp $T \subseteq E$, and hence supp $S \subseteq E$ (by [6]). This means $S \in J_{A(H)}(E)^{\perp}$. All these observations combine to yield the implication that, for $T \in X \cap J_{A(G)}(E)^{\perp}$, $\langle T, u\rangle = \langle S, ru\rangle = 0$ if $u \in A(G)$ and $ru \in J^{X_H}_{A(H)}(E)$. Thus, any such u belongs to $J^{X}_{A(G)}(E)$.

To prove the converse part, first note that $v \in j_{A(G)}(E)$ implies $rv \in j_{A(H)}(E)$, and hence $v \in J_{A(G)}(E)$ implies $rv \in J_{A(H)}(E)$. With this observation, the proof of the converse part is similar to the one in (i). □

The injection theorem for sets of synthesis is a well known result due to Reiter (see [9]) in the abelian case. The next result is the injection theorem for sets of X-synthesis and is due to Kaniuth and Lau [6]. It is now an immediate consequence of Theorem 3.1 and Proposition 2.4.

COROLLARY 3.2 (Injection theorem for X-spectral sets). *A closed set $E \subseteq H$ is of X-synthesis in $A(G)$ if and only if it is of X_H-synthesis in $A(H)$.*

To consider quotients, let K be a compact subgroup of G. We consider the Fourier algebra on the homogeneous space G/K defined and studied by Forrest [3]. For $u \in A(G)$ define

$$Qu(x) = \int_K u(xk)dk\,,$$

where dk denotes the normalised Haar measure on K. Then Q maps $A(G)$ into itself and is, in fact, a projection. $A(G : K)$, the range of Q, consists of functions in $A(G)$ that are constant on left cosets of K. Its dual $VN(G : K)$ may be described as follows. Let $L^1(G : K)$ be the space of functions in $L^1(G)$ that are constant on cosets of K; it is the range of the projection defined on $L^1(G)$ as above. $VN(G : K)$ is the weak-$*$ closure of $L^1(G : K)$ in $VN(G)$. Functions u in $A(G : K)$ can be identified, in a natural way, with (continuous) functions $\tilde{u}$ on the quotient space $G/K : \tilde{u}(\pi(x)) = u(x)$, where $\pi : G \to G/K$ is the canonical map. Then $A(G/K)$ is defined as $\{\tilde{u} : u \in A(G : K)\}$ with $\|\tilde{u}\|_{A(G/K)} = \|u\|_{A(G:K)}$. In this way, $A(G/K)$ is a commutative, semisimple, regular Banach algebra with $\Delta(A(G/K)) = G/K$. We write $VN(G/K)$ for the dual of $A(G/K)$; it is identified with $VN(G : K)$ *via* the identification of $A(G/K)$ with $A(G : K)$.

If X is an $A(G)$-submodule of $VN(G)$, there is a naturally associated $A(G/K)$-submodule X_K of $VN(G/K)$. To see this, consider the projection $Q : A(G) \to A(G : K)$ and the isomorphism $\psi : A(G : K) \to A(G/K)$, $\psi(u) = \tilde{u}$. Thus $\psi \circ Q : A(G) \to A(G/K)$, so we can consider the adjoint $(\psi \circ Q)^* = Q^* \circ \psi^* : VN(G/K) \to VN(G)$. Let $X_K = (\psi \circ Q)^{*-1}(X)$.

LEMMA 3.3. *Let the notation be as given above. Then*

(i) $Q^*(u.T) = u.Q^*(T)$ *for* $u \in A(G : K)$ *and* $T \in VN(G : K)$,

(ii) $\psi^*(\tilde{u}.\tilde{T}) = u.\psi^*(\tilde{T})$ *for* $\tilde{u} \in A(G/K)$ *and* $\tilde{T} \in VN(G/K)$,

(iii) X_K *is an* $A(G/K)$*-submodule of* $VN(G/K)$.

PROOF. (i) Let $u \in A(G : K)$ and $T \in VN(G : K)$. For $v \in A(G)$,

$$\begin{aligned}\langle Q^*(u.T), v\rangle = \langle u.T, Qv\rangle &= \langle T, u.Qv\rangle \\ &= \langle T, Q(uv)\rangle = \langle Q^*(T), uv\rangle \\ &= \langle u.Q^*(T), v\rangle\,,\end{aligned}$$

where we have used the fact that $Q(uv) = u.Q(v)$ if $u = Qu$.

(ii) Let $\tilde{u} \in A(G/K)$ and $\tilde{T} \in VN(G/K)$. For $v \in A(G : K)$, an easy computation shows that $\langle \psi^*(\tilde{u}.\tilde{T}), v\rangle = \langle u.\psi^*(\tilde{T}), v\rangle$.

(iii) It suffices to prove that X_K is $A(G/K)$-invariant. Let $\tilde{u} \in A(G/K)$ and $\tilde{T} \in X_K$, so $(\psi \circ Q)^*(\tilde{T}) \in X$. But then, a little calculation shows that $(\psi \circ Q)^*(\tilde{u}.\tilde{T}) = u.(\psi \circ Q)^*(\tilde{T}) \in X$. Hence $\tilde{u}.\tilde{T} \in X_K$. □

LEMMA 3.4. *Let $\tilde{E}$ be a closed set in G/K and let $\tilde{T} \in VN(G/K)$. If* supp $\tilde{T} \subseteq \tilde{E}$, *then* $\operatorname{supp}(\psi \circ Q)^*(\tilde{T}) \subseteq \pi^{-1}(\tilde{E})$.

PROOF. Let $x \in \operatorname{supp}(\psi \circ Q)^*(\tilde{T})$. Suppose $\tilde{u} \in A(G/K)$ and $\tilde{u}(\pi(x)) \neq 0$, i.e., $\tilde{u} \circ \pi(x) \neq 0$. Then $\tilde{u} \circ \pi.(\psi \circ Q)^*(\tilde{T}) \neq 0$. For some $v \in A(G)$

$$\begin{aligned}0 &\neq \langle \tilde{u} \circ \pi.(\psi \circ Q)^*(\tilde{T}), v\rangle = \langle (\psi \circ Q)^*(\tilde{T}), \tilde{u} \circ \pi.v\rangle \\ &= \langle \tilde{T}, \psi(Q(\tilde{u} \circ \pi.v))\rangle = \langle \tilde{T}, \psi(\tilde{u} \circ \pi.Qv)\rangle \\ &= \langle \tilde{T}, \psi(\tilde{u} \circ \pi)\psi(Qv)\rangle = \langle \tilde{T}, \tilde{u}\psi(Qv)\rangle \\ &= \langle \tilde{u}.\tilde{T}, \psi(Qv)\rangle\,.\end{aligned}$$

Thus $\tilde{u}\,.\,\tilde{T} \neq 0$, and so $\pi(x) \in \operatorname{supp} \tilde{T} \subseteq \tilde{E}$. □

We can now relate sets of synthesis for $A(G/K)$ and $A(G)$.

THEOREM 3.5. *If $\pi^{-1}(\tilde{E})$ is a set of X-synthesis for $A(G)$, then $\tilde{E}$ is a set of X_K-synthesis for $A(G/K)$.*

PROOF. In view of the lemmas, the proof is now easy. Suppose $\tilde{T} \in X_K$ and supp $\tilde{T} \subseteq \tilde{E}$. If $\tilde{u} \in I_{A(G/K)}(\tilde{E})$, then $u = \tilde{u} \circ \pi \in I_{A(G)}(\pi^{-1}(\tilde{E}))$. If $\pi^{-1}(\tilde{E})$ is of X-synthesis, the definition of X_K and Lemma 3.4 now give

$$0 = \langle (\psi \circ Q)^*(\tilde{T}), u\rangle = \langle \tilde{T}, \psi(Qu)\rangle = \langle \tilde{T}, \tilde{u}\rangle\,,$$

completing the proof.

When $X = VN(G)$, $X_K = VN(G/K)$ and we get the following result of Forrest [3].

COROLLARY 3.6. *If $\pi^{-1}(\tilde{E})$ is a set of synthesis for $A(G)$, then $\tilde{E}$ is a set of synthesis for $A(G/K)$.*

The question whether, conversely, $\pi^{-1}(\tilde{E})$ is a set of X-synthesis for $A(G)$ whenever $\tilde{E}$ is a set of X_K-synthesis for $A(G/K)$ is open even for the case $X = VN(G)$.

4. Submodules of $A(G)^*$ and $V(G)^*$. In this section, assuming that G is compact, we give a correspondence between $A(G)$-submodules of $A(G)^* = VN(G)$ and $V(G)$-submodules of $V(G)^*$. Here $V(G)$ is the Varopoulos algebra on G as defined in Section 1. The G-invariant functions in $V(G)$ form a closed subalgebra of $V(G)$:

$V_{\text{inv}}(G) = \{w \in V(G); w(xt, yt) = w(x, y) \textit{ for } x, y, t \in G\}$. Spronk and Turowska [10] have proved that the map

$$N : A(G) \to V_{\text{inv}}(G)$$

defined by $Nu(x, y) = u(xy^{-1})$ is an isometric isomorphism of $A(G)$ onto $V_{\text{inv}}(G)$. This imbedding of $A(G)$ in $V(G)$ and the projection of $V(G)$ on $V_{\text{inv}}(G)$ described below go back to Varopoulos (see [12]) in the abelian case. $V_{\text{inv}}(G)$ is complemented in $V(G)$ and P defined, for $w \in V(G)$, by

$$Pw(x, y) = \int_G w(xt, yt)dt$$

is a contractive projection $V(G) \to V_{\text{inv}}(G)$ (see [10, Proposition 2.3]).

For an $A(G)$-submodule X of $VN(G)$, define

$$X_V = \{S \in V(G)^*; (w.S) \circ N \in X \text{ for all } w \in V(G)\}.$$

It is clear that X_V is a $V(G)$-submodule of $V(G)^*$. Further X_V is weak-$*$ closed if X is.

Conversely, for a $V(G)$-submodule Y of $V(G)^*$, define

$$Y_A = \{T \in VN(G); (u.T) \circ N^{-1} \circ P \in Y \text{ for all } u \in A(G)\}.$$

Y_A is an $A(G)$-submodule of $VN(G)$, which is weak-$*$ closed if Y is.

Using this correspondence, we shall, in the next section, explore a relation between spectral synthesis in $A(G)$ and in $V(G)$. But for now, we show that the correspondence is a nicely behaved one. We need the following lemma that will also be used later in the proof of Lemma 5.3.

LEMMA 4.1. *For $w \in V(G)$ and $T \in VN(G)$, we have $w.(T \circ N^{-1} \circ P) \circ N = u.T$, where $u = N^{-1}(Pw)$.*

PROOF. For $v \in A(G)$

$$\begin{aligned}\langle w.(T \circ N^{-1} \circ P) \circ N, v\rangle &= \langle w.(T \circ N^{-1} \circ P), Nv\rangle \\ &= \langle T \circ N^{-1} \circ P, wNv\rangle = \langle T \circ N^{-1}, P(wNv)\rangle \\ &= \langle T \circ N^{-1}, PwNv\rangle = \langle T \circ N^{-1}, NuNv\rangle \\ &= \langle T \circ N^{-1}, N(uv)\rangle = \langle T, uv\rangle \\ &= \langle u.T, v\rangle.\end{aligned}$$

Observe that we have made use of the fact that $P(ww') = Pw.w'$ if $w' \in V_{inv}(G)$. □

PROPOSITION 4.2. *Let X be an $A(G)$-submodule of $VN(G)$. Then $(X_V)_A = X$.*

PROOF. Suppose $T \in (X_V)_A$. Then $u.T \circ N^{-1} \circ P \in X_V$ for all $u \in A(G)$. This, in turn, means that $w.(u.T \circ N^{-1} \circ P) \circ N \in X$ for all $w \in V(G)$. For $u, v \in A(G)$ and $w \in$

$V(G)$ applying Lemma 4.1 with T replaced by $v.T$ we have $w.(v.T \circ N^{-1} \circ P) \circ N = uv.T$. Thus, $uv.T = (w.(u.T \circ N^{-1} \circ P)) \circ N \in X$. In particular, taking $u = 1$ and $w = 1 \otimes 1$, so that $N^{-1}(Pw) = 1$, we get that $T \in X$.

Conversely, suppose $T \in X$. Let $u \in A(G)$ and $S = (u.T) \circ N^{-1} \circ P$. We check that $S \in X_V$. For $w \in V(G)$ and $v \in A(G)$, $\langle w.S \circ N, v\rangle = \langle w.S, Nv\rangle = \langle u.T \circ N^{-1} \circ P, wNv\rangle = \langle u'u.T, v\rangle$ as before, where $u' = N^{-1}(Pw)$. This means that $w.S \circ N = u'u.T \in X$. So $S \in X_V$, i.e., $(u.T) \circ N^{-1} \circ P \in X_V$, for all $u \in A(G)$. Thus $T \in (X_V)_A$, and the proof is complete.

Here are some examples of X and the corresponding X_V.

EXAMPLE 4.3. (i) If $X = VN(G)$, then $X_V = V(G)^*$.

(ii) This example is motivated by the results on synthesis that are discussed in the next section. Consider the map $\theta : G \times G \to G$, $\theta(x, y) = xy^{-1}$. For a closed set $E \subseteq G$, consider the closed set

$$E^* := \theta^{-1}(E) = \{(x, y) \in G \times G; xy^{-1} \in E\}.$$

Then it is known that $u \in I_A(E) \Leftrightarrow Nu \in I_V(E^*)$ and $u \in J_A(E) \Leftrightarrow Nu \in J_V(E^*)$(see [12], [10]). Let $X = \{T \in VN(G); \text{supp } T \subseteq E\}$. Then $X_V = \{S \in V(G)^*; \text{supp } S \subseteq E^*\}$. To see this, let $S \in V(G)^*$ with supp $S \subseteq E^*$. We show that supp $w.S \circ N \subseteq E$ for $w \in V(G)$. For this, observe that

$$\begin{aligned} u \in J_A(E) &\Rightarrow Nu \in J_V(E^*) \\ &\Rightarrow w.Nu \in J_V(E^*) \text{ for all } w \in V(G) \\ &\Rightarrow 0 = \langle S, w.Nu\rangle = \langle w.S \circ N, u\rangle. \end{aligned}$$

This means that $w.S \circ N \in J_A(E)^\perp = X$. Thus $S \in X_V$. Conversely, suppose $S \in X_V$. This means that $w.S \circ N \in X$ for all $w \in V(G)$, i.e., supp $w.S \circ N \subseteq E$. To prove supp $S \subseteq E^*$, we have to show that if $(x, y) \in$ supp S then $xy^{-1} \in E$. Let $(x, y) \in$ supp S. Then

$$\begin{aligned} u \in A(G),\ u(xy^{-1}) \neq 0 &\Rightarrow Nu(x, y) \neq 0 \Rightarrow Nu.S \neq 0 \\ &\Rightarrow \text{there is a } w \in V(G) \text{ with } 0 \neq \langle Nu.S, w\rangle = \langle w.S, Nu\rangle \\ &\quad = \langle w.S \circ N, u\rangle = \langle u.(w.S) \circ N, 1\rangle \\ &\Rightarrow u.(w.S \circ N) \neq 0. \end{aligned}$$

Thus $xy^{-1} \in \text{supp}(wS \circ N) \subseteq E$. Another way of stating this example is: if $X = J_A(E)^\perp$, then $X_V = J_V(E^*)^\perp$.

(iii) If $X = \{\sum_1^n u_i.\lambda(x_i);\ u_i \in A(G),\ x_i \in G,\ n \in \mathbf{N}\}$, then $X_V = \{S \in V(G)^*;\ \text{supp } S \subseteq F^*,\ F \subset G \text{ is finite}\}$.

(iv) Consider the circle group $G = \boldsymbol{T}$. In this case $VN(G) = \ell^\infty(\boldsymbol{Z})$. If $X = c_0(\boldsymbol{Z})$, then $X_V = \{S \in V(G)^*;\ \hat{S}(n, -n) \to 0 \text{ as } |n| \to \infty\}$, where $\hat{S}(m, n); = \langle S, e_m \otimes e_n\rangle$ and $e_m(t) = \exp(2\pi imt)$.

5. Synthesis in $A(G)$ and $V(G)$. The setting in this section is as in the previous section. In particular, G is a compact group and $V(G)$ is the Varopoulos algebra of G. We look for a relation between synthesis in $A(G)$ and in $V(G)$. More specifically, we prove, with the notation of Section 4, that a closed set $E \subseteq G$ is a set of X-synthesis for $A(G)$ if and only if E^* is a set of X_V-synthesis for $V(G)$. For the case when $X = VN(G)$, this result goes back to Varopoulos [12] for abelian G and the nonabelian case is given by Spronk and Turowska [10]. We begin with a couple of lemmas.

LEMMA 5.1. *Let $E \subseteq G$ be a closed set. Then $I_V(E^*)$ and $J_V(E^*)$ are both invariant under the projection $P : V(G) \to V_{\text{inv}}(G)$.*

PROOF. For $I_V(E^*)$, the result is obvious: if $(x, y) \in E^*$, then $(xt, yt) \in E^*$ for all $t \in G$. So $w \in I_V(E^*)$ implies $w(xt, yt) = 0$ for all $t \in G$, whence $Pw(x, y) = 0$.

To prove the result for $J_V(E^*)$, it suffices, by continuity of P, to show that $Pw \in J_V(E^*)$ whenever $w \in j_V(E^*)$. It is, in fact, true that supp $Pw \cap E^* = \emptyset$ for $w \in j_V(E^*)$. To see this, let

$$U = \{(x, y) \in G \times G;\ Pw(x, y) \neq 0\},$$
$$W = \{(x, y) \in G \times G;\ w(x, y) \neq 0\}.$$

Thus supp $Pw = \bar{U}$ and supp $w = \bar{W}$. Since $Pw(x, y) \neq 0$ implies $w(xt, yt) \neq 0$ for some $t \in G$, it follows that $\theta(U) \subseteq \theta(W)$. Hence

$$\theta(\bar{U}) \subseteq \overline{\theta(U)} \subseteq \overline{\theta(W)} \subseteq \overline{\theta(\bar{W})} = \theta(\bar{W}).$$

Recalling that G is compact, the last equality holds because of the compactness of $\bar{W}$, hence of $\theta(\bar{W})$. Suppose there is a point $(x, y) \in$ supp $Pw \cap E^*$, i.e., $(x, y) \in \bar{U} \cap E^*$. Then

$$\theta(x, y) \in \theta(\bar{U}) \cap E \subseteq \theta(\bar{W}) \cap E,$$

and so $\theta(x, y) = \theta(s, t)$ for some $(s, t) \in$ supp $w \cap E^*$, a contradiction, since supp $w \cap E^* = \emptyset$ as $w \in j_V(E^*)$. □

REMARK 5.2. The following shorter proof of the second part of Lemma 5.1 has been kindly suggested to us by the referee: Using vector-valued integration, write $Pw = \int_G t.w\, dt$. For $w \in J_V(E^*)$ and $S \in J_V(E^*)^{\perp}$, $\langle S, Pw\rangle = \int_G \langle S, t.w\rangle dt = 0$, whence $Pw \in J_V(E^*)$.

The case $X = VN(G)$ of the next lemma has already been mentioned in Example 4.3 (ii). This special case is made use of in the proof below. For a closed set $F \subseteq G \times G$ and a $V(G)$-submodule Y of $V(G)^*$, recall, from Section 2, the definition of the closed ideals $I_V^Y(F)$ and $J_V^Y(F)$.

LEMMA 5.3. *Let X be an $A(G)$-submodule of $VN(G)$ and let $Y = X_V$ be the associated $V(G)$-submodule of $V(G)^*$. Let E be a closed subset of G. Then, for $u \in A(G)$,*

(i) $u \in I_A^X(E) \Leftrightarrow Nu \in I_V^Y(E^*)$,

(ii) $u \in J_A^X(E) \Leftrightarrow Nu \in J_V^Y(E^*)$.

PROOF. (i) Suppose $u \in I_A^X(E)$. To prove $Nu \in I_V^Y(E^*)$, let $S \in Y \cap I_V(E^*)^\perp$. Then $w.S \circ N \in X$ for $w \in V(G)$; in particular, $S \circ N \in X$. Further, if $v \in I_A(E)$, then $Nv \in I_V(E^*)$ by the special case mentioned above and so $\langle S \circ N, v\rangle = \langle S, Nv\rangle = 0$. This means that $S \circ N \in I_A(E)^\perp$ and hence $\langle S, Nu\rangle = \langle S \circ N, u\rangle = 0$. This proves the forward implication in (i).

For the converse, let $Nu \in I_V^Y(E^*)$ and $T \in X \cap I_A(E)^\perp$. We claim that $T \circ N^{-1} \circ P \in Y \cap I_V(E^*)^\perp$. Now, for $w_0 \in V(G)$, $w_0.(T \circ N^{-1} P) \circ N = u_0 T \in X$, by Lemma 4.1, where $u_0 = N^{-1}(Pw_0)$. So by definition $T \circ N^{-1} \circ P \in Y$. To see that $T \circ N^{-1} \circ P \in I_V(E^*)^\perp$, let $w' \in I_V(E^*)$. Then $Pw' \in I_V(E^*)$ by Lemma 5.1 and so $N^{-1}(Pw') \in I_A(E)$. Hence $\langle T \circ N^{-1} \circ P, w'\rangle = \langle T \circ N^{-1}, Pw'\rangle = \langle T, N^{-1}(Pw')\rangle = 0$. This completes the proof of the claim. It is now easy to finish the proof of (i):

$$0 = \langle T \circ N^{-1} \circ P, Nu\rangle = \langle T \circ N^{-1}, Nu\rangle = \langle T, u\rangle .$$

We have thus proved that $\langle T, u\rangle = 0$ for $T \in X \cap I_A(E)^\perp$, i.e., $u \in I_A^X(E)$.

(ii) The proof of the first part of (ii) is just a repetition of that of the first part of (i) with J in place of I. In view of the second part of Lemma 5.1, the previous sentence may be repeated with ‘second part’ replacing ‘first part’. The lemma is thus proved.

Next, observe that G acts continuously on $V(G)$ as a group of isometries: for $t \in G$ and $w \in V(G)$, $t.w \in V(G)$ is given by $t.w(x, y) = w(xt, yt)$, for $x, y \in G$. Further, this action of G induces an action of $L^1(G)$ on $V(G)$: for $f \in L^1(G)$ and $w \in V(G)$

$$f.w = \int_G f(t)t.wdt .$$

As noted in [10], this vector valued integral makes sense and this action turns $V(G)$ into an essential Banach $L^1(G)$-module. We also need the dual action of $L^1(G)$ on $V(G)^*$: For $f \in L^1(G)$ and $S \in V(G)^*$, $f.S$ is defined by

$$\langle f.S, w\rangle = \langle S, f.w\rangle , \quad w \in V(G) .$$

We need a few lemmas on these actions of $L^1(G)$ on $V(G)$ and on $V(G)^*$.

LEMMA 5.4. *For a closed subset E of G, $I_V(E^*)^\perp$ is an $L^1(G)$-submodule of $V(G)^*$.*

PROOF. This is easy. First, it is clear from the definition that if $w \in I_V(E^*)$ and $f \in L^1(G)$, then $f.w \in I_V(E^*)$. Hence, for $w \in I_V(E^*)$, $S \in I_V(E^*)^\perp$ and $f \in L^1(G)$, $\langle f.S, w\rangle = \langle S, f.w\rangle = 0$. □

LEMMA 5.5. *Let X be an $A(G)$-submodule of $VN(G)$ and let X_V be the associated $V(G)$-submodule of $V(G)^*$. Then X_V is an $L^1(G)$-submodule of $V(G)^*$.*

PROOF. Recall that $S \in X_V$ if and only if $w.S \circ N \in X$ for all $w \in V(G)$. Let $S \in X_V$ and $f \in L^1(G)$. For $w \in V(G)$ and $u \in A(G)$

$$\begin{aligned}\langle (w.(f.S)) \circ N, u\rangle &= \langle w.(f.S), Nu\rangle = \langle f.S, wNu\rangle \\ &= \langle S, f.(wNu)\rangle = \langle S, f.wNu\rangle \\ &= \langle ((f.w).S) \circ N, u\rangle \,.\end{aligned}$$

Along the way, we have used the easily verified fact that, for $f \in L^1(G)$, $w \in V(G)$ and $v \in V_{\text{inv}}(G)$, $f.(wv) = (f.w)v$. We have thus proved that the $L^1(G)$-action and the $V(G)$-action on $V(G)^*$ commute when restricted to $V_{\text{inv}}(G) : (w.(f.S)) \circ N = ((f.w).S) \circ N$, and this last object belongs to X since $S \in X_V$. This yields the required result that $f.S \in X_V$, completing the proof. □

LEMMA 5.6. *Let $E \subseteq G$ be closed, let X be an $A(G)$-submodule of $VN(G)$ and let $Y = X_V$ be the associated $V(G)$-submodule of $V(G)^*$. Then $I_V^Y(E^*)$ is an $L^1(G)$-submodule of $V(G)$.*

PROOF. We have to show that if $w \in I_V^Y(E^*)$ and $f \in L^1(G)$, then $f.w \in I_V^Y(E^*)$. This is an immediate consequence of Lemmas 5.4 and 5.5: For $S \in X_V \cap I_V(E^*)^{\perp}$, we have $\langle S, f.w\rangle = \langle f.S, w\rangle = 0$. □

We are now ready to prove the main result of the section. In addition to the preceding lemmas, we also make use of the case $\mathcal{A} = V(G)$ of Proposition 2.4.

THEOREM 5.7. *Let X be an $A(G)$-submodule of $VN(G)$ and let $Y = X_V$ be the associated $V(G)$-submodule of $V(G)^*$. Then a closed subset E of G is a set of X-synthesis for $A(G)$ if and only if E^* is a set of X_V-synthesis for $V(G)$.*

PROOF. One part is immediate from Lemma 5.3: If E^* is of X_V-synthesis, then

$$u \in I_A^X(E) \Rightarrow Nu \in I_V^Y(E^*) \Rightarrow Nu \in J_V^Y(E^*) \Rightarrow u \in J_A^X(E)\,.$$

The converse is more involved. Armed with our array of lemmas, we can easily mimic the proof of [10, Theorem 3.1], where Spronk and Turowska prove the result for the case $X = VN(G)$, $X_V = V(G)^*$. For the convenience of the readers, here is a brief summary of the arguments.

Suppose E is of X-synthesis and $w \in I_V^Y(E^*)$. It suffices to show that $w \in J_V^Y(E^*)$. For each $\pi \in \hat{G}$, the unitary dual of G, define the matrix functions w^π and $\tilde{w}^\pi$ by

$$\begin{aligned}w^\pi(x, y) &= \int_G w(xt, yt)\pi(t)dt\,, \\ \tilde{w}^\pi(x, y) &= w^\pi(x, y)\pi(x)\,.\end{aligned}$$

If u_{ij}^π, $i, j = 1, \dots, d_\pi$, are the matrix coefficients of π, consider

$$w_{ij}^\pi = u_{ij}^\pi.w \quad \text{and} \quad \tilde{w}_{ij}^\pi = \sum_k u_{ik}^\pi \otimes 1\, w_{kj}^\pi\,.$$

Observe that $w_{ij}^{\pi} \in I_V^Y(E^*)$ and $\tilde{w}_{ij}^{\pi} \in I_V^Y(E^*) \cap V_{\rm inv}(G)$. Hence Lemma 5.3 implies $N^{-1}(\tilde{w}_{ij}^{\pi}) \in I_A^X(E) = J_A^X(E)$, whence $\tilde{w}_{ij}^{\pi} \in J_V^Y(E^*)$. But

$$w_{ij}^{\pi} = \sum \check{u}_{ik}^{\pi} \otimes 1\ \tilde{w}_{kj}^{\pi}\,,$$

so $w_{ij}^{\pi} \in J_V^Y(E^*)$. Thus, we have proved that if $w \in I_V^Y(E^*)$, then $w_{ij}^{\pi} \in J_V^Y(E^*)$ for all i, j. Moreover, as observed in [10], $L^1(G)$ has a bounded approximate identity (u_α) such that

$$u_\alpha \in \text{ span}\{u_{ij}^{\pi}; i, j = 1, \dots, d_\pi,\ \pi \in \hat{G}\}$$

for all α. So $u_\alpha.w \in \text{span}\{w_{ij}^{\pi}; i, j = 1, \dots, d_\pi, \pi \in \hat{G}\} \subset J_V^Y(E^*)$. But then $w = \lim u_\alpha.w \in J_V^Y(E^*)$. □

CONCLUDING REMARKS. Froelich [4] has studied the relation between spectral synthesis on abelian groups and the concept of operator synthesis introduced by Arveson [1]. Spronk and Turowska [10] investigate this for compact (nonabelian) groups. In a paper that has just appeared ([8]), we have defined a version of operator synthesis analogous to X-synthesis and have studied the relation between these two. Our results on weak X-synthesis are to be included in a separate communication.

We heartily thank the referee for his perspicuous suggestions.

REFERENCES

[1] W. ARVESON, Operator algebras and invariant subspaces, Ann. of Math. (2) 100 (1974), 433–532.
[2] P. EYMARD, L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. France 92 (1964), 181–236.
[3] B. FORREST, Fourier analysis on coset spaces, Rocky Mountain J. Math. 28 (1998), 173–190.
[4] J. FROELICH, Compact operators, invariant subspaces and spectral synthesis, J. Functional Analysis 81 (1988), 1–37.
[5] C. HERZ, Harmonic synthesis for subgroups, Ann. Inst. Fourier (Grenoble) 23 (1973), 91–123.
[6] E. KANIUTH AND A. T. LAU, Spectral synthesis for $A(G)$ and subspaces of $VN(G)$, Proc. Amer. Math. Soc. 129 (2001), 3253–3263.
[7] T. K. MURALEEDHARAN AND K. PARTHASARATHY, Difference spectrum and spectral synthesis, Tohoku Math. J. (2) 51 (1999), 65–73.
[8] K. PARTHASARATHY AND R. PRAKASH, Spectral synthesis and operator synthesis, Studia Math. 177 (2006), 173–181.
[9] H. REITER, Classical harmonic analysis and locally compact groups, Clarendon Press, Oxford, 1968.
[10] N. SPRONK AND L. TUROWSKA, Spectral synthesis and operator synthesis for compact groups, J. London Math. Soc. (2) 66 (2002), 361–376.
[11] M. TAKESAKI AND N. TATSUUMA, Duality and subgroups II, J. Functional Analysis 11 (1972), 184–190.
[12] N. TH. VAROPOULOS, Tensor algebras and harmonic analysis, Acta Math. 119 (1967), 51–112.

RAMANUJAN INSTITUTE FOR ADVANCED
STUDY IN MATHEMATICS
UNIVERSITY OF MADRAS
CHENNAI 600 005
INDIA