

NON-REGULAR PSEUDO-DIFFERENTIAL OPERATORS ON THE WEIGHTED TRIEBEL-LIZORKIN SPACES

SHUICHI SATO

(Received March 28, 2005, revised May 14, 2007)

Abstract. We consider certain non-regular pseudo-differential operators and study the question of their boundedness on the weighted Triebel-Lizorkin and Besov spaces.

1. Introduction. Let $\Psi \in C^\infty(\mathbf{R}^n)$ satisfy $\text{supp}(\Psi) \subset \{1/2 \leq |\xi| \leq 2\}$, $|\Psi(\xi)| \geq c > 0$ for $3/5 \leq |\xi| \leq 5/3$ and

$$\sum_{j \in \mathbf{Z}} \Psi(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0,$$

where \mathbf{Z} denotes the set of all integers and $|\xi| = (\sum_{j=1}^n \xi_j^2)^{1/2}$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. Define $\Phi \in C_0^\infty(\mathbf{R}^n)$ by $\Phi(\xi) = 1 - \sum_{j \geq 1} \Psi(2^{-j}\xi)$. We note that Φ is supported in $\{|\xi| \leq 2\}$. In what follows, we also assume that Ψ is radial. We write $g_t(x) = t^{-n}g(x/t)$ for $t > 0$. Define $D_j(f) = f * (\hat{\Psi})_{2^{-j}}$ for $j \geq 1$ and $D_0(f) = f * \hat{\Phi}$, where \hat{g} denotes the Fourier transform:

$$\hat{g}(\xi) = \int_{\mathbf{R}^n} g(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \langle x, \xi \rangle = x_1 \xi_1 + x_2 \xi_2 + \dots + x_n \xi_n;$$

we also write $\hat{g} = \mathcal{F}(g)$.

Let A_r , $1 \leq r < \infty$, be the weight class of Muckenhoupt on \mathbf{R}^n . We recall that A_r , $1 < r < \infty$, is defined to be the class of all weight functions w on \mathbf{R}^n satisfying

$$\sup_Q \left(|Q|^{-1} \int_Q w(x) dx \right) \left(|Q|^{-1} \int_Q w(x)^{-1/(r-1)} dx \right)^{r-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$ and $|Q|$ denotes the Lebesgue measure of Q . Also, for a weight function w , we say that $w \in A_1$ if there exists a constant C such that $\mathcal{M}(w)(x) \leq Cw(x)$ for almost every x , where \mathcal{M} denotes the Hardy-Littlewood maximal operator.

Let $w \in A_\infty$, where $A_\infty = \bigcup_{1 \leq r < \infty} A_r$. The weighted (inhomogeneous) Triebel-Lizorkin space $F_p^{s,q}(w)$, with $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbf{R}$, is defined to be the space of all tempered distributions f on \mathbf{R}^n satisfying

$$\|f\|_{F_p^{s,q}(w)} = \left\| \left(\sum_{j=0}^{\infty} |2^{sj} D_j(f)|^q \right)^{1/q} \right\|_{L^p(w)} < \infty,$$

where $\|\cdot\|_{L^p(w)}$ denotes the weighted L^p norm: $\|g\|_{L^p(w)} = (\int |g(x)|^p w(x) dx)^{1/p}$. Also, the weighted (inhomogeneous) Besov space $B_p^{s,q}(w)$, with $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbf{R}$, is defined to be the space of all tempered distributions f on \mathbf{R}^n satisfying

$$\|f\|_{B_p^{s,q}(w)} = \left(\sum_{j=0}^{\infty} \|2^{sj} D_j(f)\|_{L^p(w)}^q \right)^{1/q} < \infty.$$

See [2], [4] and [13] for more details on these spaces.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, let $(\partial\xi)^\alpha$ denote a differential operator

$$(\partial/\partial\xi_1)^{\alpha_1} \dots (\partial/\partial\xi_n)^{\alpha_n}$$

of order $|\alpha| = \alpha_1 + \dots + \alpha_n$. For a function F on $\mathbf{R}^n \times \mathbf{R}^n$ and $x, \xi, h, \eta \in \mathbf{R}^n$, we write

$$(d_h F)(x, \xi) = F(x + h, \xi) - F(x, \xi),$$

$$(\delta_\eta F)(x, \xi) = F(x, \xi + \eta) - F(x, \xi).$$

We also define $(\delta_\eta d_h F)(x, \xi) = (d_h \delta_\eta F)(x, \xi) = (\delta_\eta (d_h F))(x, \xi)$.

Let $r > 0$ and $0 \leq \delta \leq 1$. Suppose that r is not an integer and that

$$|(\partial\xi)^\alpha (\partial x)^\beta \sigma(x, \xi)| \leq C(1 + |\xi|)^{-|\alpha| + \delta|\beta|} \quad \text{for } |\alpha| \leq N, |\beta| < r;$$

$$|(d_y (\partial\xi)^\alpha (\partial x)^\beta \sigma)(x, \xi)| \leq C|y|^{r-[\alpha]} (1 + |\xi|)^{-|\alpha| + \delta r} \quad \text{for } |\alpha| \leq N, |\beta| = [r],$$

where N is an even integer greater than $3n/2 + 1$ ($[a]$ denotes the integer such that $a - 1 < [a] \leq a$). Let T_σ be a pseudo-differential operator defined by

$$T_\sigma(f)(x) = \int_{\mathbf{R}^n} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i(x,\xi)} d\xi, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

where $\mathcal{S}(\mathbf{R}^n)$ denotes the Schwartz space. Then, Bourdaud [1] proved that T_σ is bounded on the Sobolev spaces $H_p^s (= F_p^{s,2})$ for $1 < p < \infty$, $s \in ((\delta - 1)r, r)$ and that the range of s is optimal (see [5] for further developments). Also, the boundedness of T_σ on the Besov spaces was studied. Related results can be found in [3], [6], [7], [8], [10], [11], [12], [14] and [15]. In particular, in [10] a weighted L^p , $p \geq 2$, norm inequality for T_σ was proved under a minimal regularity condition for the symbol σ .

In [1], the case where r is an integer was also considered. In this note we confine ourselves to the case where r is not an integer and generalize results of [1] to the weighted (inhomogeneous) Triebel-Lizorkin spaces by using the idea of [10]. Moreover, by applying the results for the Triebel-Lizorkin spaces, we study the boundedness of T_σ on the weighted (inhomogeneous) Besov spaces. We refer to Sugimoto [12] for relevant results.

Let ω be a non-negative function on $[0, \infty) \times [0, \infty)$ satisfying the following:

($\omega.1$) there exist constants $C, M > 0$ such that

$$\omega(s, at) \leq C(1 + a)^M \omega(s, t) \quad \text{for all } a > 0;$$

($\omega.2$) there exists a constant $C > 0$ such that

$$\omega(s', t) \leq C\omega(s, t) \quad \text{for } s/2 \leq s' \leq 2s;$$

(ω.3) $0 < \omega(1, 1)$.

Let $0 \leq \delta \leq 1$. We assume that $r(> 0)$ is not an integer and the function ω satisfies

$$(1.1) \quad C(\omega) := \sup_{0 \leq j \leq h, 0 \leq h} \omega(2^j, 2^{-h})2^{-j\delta(r-[r])}2^{h(r-[r])} < \infty,$$

where $j, h \in \mathbf{Z}$. A prime example of ω satisfying (1.1), (ω.1)–(ω.3) is $\omega(s, t) = s^{\delta(r-[r])}t^{r-[r]}$ (see also Remark 3 below).

Let $\sigma(x, \xi)$ be a bounded function on $\mathbf{R}^n \times \mathbf{R}^n$. For the rest of this note, we assume that $\sigma \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ for the sake of simplicity. We write $\sigma^{(\alpha)}(x, \xi) = (\partial \xi)^\alpha \sigma(x, \xi)$, $\sigma_{(\beta)}(x, \xi) = (\partial x)^\beta \sigma(x, \xi)$ and $\sigma_{(\beta)}^{(\alpha)}(x, \xi) = (\partial \xi)^\alpha (\partial x)^\beta \sigma(x, \xi)$. Let L be a non-negative integer. We consider the following conditions:

$$(1.2) \quad |\sigma^{(\alpha)}(x, \xi)| \leq C_\alpha(1 + |\xi|)^{-|\alpha|} \quad \text{for } |\alpha| \leq L;$$

$$(1.3) \quad |(d_y \sigma_{(\beta)}^{(\alpha)})(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{-|\alpha| + \delta|\beta|} \omega(1 + |\xi|, |y|) \quad \text{for } |\alpha| \leq L, |\beta| = [r].$$

Then we have the following

THEOREM 1. *Let $1 < q \leq 2$. Suppose that $\sigma(x, \xi)$ satisfies (1.2) and (1.3) with $L = [n/q] + 1$. Let $s \in ((\delta - 1)r, r)$, $q \leq p < \infty$ and $w \in A_{p/q}$. Then the operator T_σ is bounded on $F_p^{s,q}(w)$. The operator norm is bounded by a constant which is independent of σ if the constants $C_\alpha, C_{\alpha, \beta}$ in (1.2) and (1.3) are fixed for $|\alpha| \leq [n/q] + 1, |\beta| = [r]$.*

Let L be a non-negative integer and $0 < a, b \leq 1$. We consider the following conditions:

$$(1.4) \quad |\sigma^{(\alpha)}(x, \xi)| \leq C_\alpha(1 + |\xi|)^{-|\alpha|} \quad \text{for } |\alpha| \leq L;$$

$$(1.5) \quad |(\delta_\eta \sigma^{(\alpha)})(x, \xi)| \leq C_\alpha(1 + |\xi|)^{-|\alpha| - a} |\eta|^a \quad \text{for } |\eta| < |\xi|/2, \quad |\alpha| = L;$$

$$(1.6) \quad |(d_y \sigma_{(\beta)}^{(\alpha)})(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{-|\alpha| + \delta|\beta|} \omega(1 + |\xi|, |y|) \\ \text{for } |\alpha| \leq L, \quad |\beta| = [r];$$

$$(1.7) \quad |(\delta_\eta d_y \sigma_{(\beta)}^{(\alpha)})(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{-|\alpha| - b + \delta|\beta|} |\eta|^b \omega(1 + |\xi|, |y|) \\ \text{for } |\eta| < |\xi|/2, \quad |\alpha| = L, \quad |\beta| = [r].$$

Then, Theorem 1 follows from the following

THEOREM 2. *Let $1 < q \leq 2, 0 < a \leq 1$ and $[n/q] + a > n/q$. Suppose that $\sigma(x, \xi)$ satisfies (1.4)–(1.7) with $L = [n/q], a = b$. Let $s \in ((\delta - 1)r, r)$, $q \leq p < \infty$ and $w \in A_{p/q}$. Then T_σ is bounded on $F_p^{s,q}(w)$. Moreover, the operator norm is bounded by a constant which is independent of σ if we fix the constants $C_\alpha, C_{\alpha, \beta}$ in (1.4)–(1.7) for $|\alpha| \leq [n/q], |\beta| = [r]$.*

This is a consequence of a more general result (Theorem 3). Let ρ be a non-negative function such that $\rho^{-1} \in L^1(\mathbf{R}^n)$. Let $1 < q \leq 2$. For an appropriate f , define

$$\|f\|_{A_\rho^q} = \left(\int_{\mathbf{R}^n} |\hat{f}(x)|^{q'} \rho(x)^{q'/q} dx \right)^{1/q'},$$

where q' is the exponent conjugate to q : $q' = q/(q - 1)$. We consider the following conditions:

$$(1.8) \quad C_1 := \sup_{j \geq 1} \sup_{x \in \mathbf{R}^n} \|\sigma(x, 2^j \cdot) \Psi(\cdot)\|_{A_p^q} < \infty;$$

$$(1.9) \quad \sup_{x \in \mathbf{R}^n} \|(d_y \sigma(\beta))(x, 2^j \cdot) \Psi(\cdot)\|_{A_p^q} \leq C_\beta 2^{j\delta[r]} \omega(2^j, |y|) \quad \text{for } |\beta| = [r], j \geq 1;$$

$$(1.10) \quad C_2 := \sup_{x \in \mathbf{R}^n} \|\sigma(x, \cdot) \Phi(\cdot)\|_{A_p^q} < \infty;$$

$$(1.11) \quad \sup_{x \in \mathbf{R}^n} \|(d_y \sigma(\beta))(x, \cdot) \Phi(\cdot)\|_{A_p^q} \leq C_\beta \omega(1, |y|) \quad \text{for } |\beta| = [r].$$

Then we have the following:

THEOREM 3. *Suppose that the conditions (1.8)–(1.11) hold. Let $s \in ((\delta - 1)r, r)$, $q \leq p < \infty$ and $w \in A_{p/q}$. We further assume that*

$$(1.12) \quad \sup_{t > 0} \theta_t * f(x) \leq C\mathcal{M}(f)(x) \quad \text{a.e.}$$

for all non-negative bounded functions f , where $\theta(x) = \rho(x)^{-1}$. Then T_σ is bounded on $F_p^{s,q}(w)$. The operator norm is bounded by a constant which is independent of σ if the constants C_1, C_2, C_β ($|\beta| = [r]$) in (1.8)–(1.11) are fixed.

REMARK 1. Let $\omega(s, t) = s^{\delta(r-[r])} t^{r-[r]}$, $r > 0, 0 \leq \delta \leq 1$. By examples similar to those in [1], we can see the optimality of the range of s in Theorem 2 ($(\delta - 1)r < s < r$).

Indeed, for $k \geq 10$, let

$$\sigma_k(x, \xi) = \sum_{j=10}^k 2^{-jr} \exp(2\pi i 2^j x_1) \quad (x = (x_1, x_2, \dots, x_n)).$$

Then, $\sigma_k(x, \xi)$ uniformly satisfies (1.4)–(1.7) for all L and $a = b = 1$. We take Ψ in the definition of $F_p^{s,q}(w)$ such that $\Psi(\xi) = 1$ for $9/10 \leq |\xi| \leq 10/9$, $\Psi(2^{-m}\xi) = 0$ if $9/10 \leq 2^{-j}|\xi| \leq 10/9$ and $m \neq j$. Let $g \in \mathcal{S}(\mathbf{R}^n)$ satisfy $\text{supp}(\hat{g}) \subset \{|\xi| \leq 10^{-9}\}$, $0 < \|g\|_{L^p(w)} < \infty$. Then we can see $D_h T_{\sigma_k}(g)(x) = 2^{-hr} \exp(2\pi i 2^h x_1) g(x)$, $h = 10, 11, \dots, k$, and $D_h T_{\sigma_k}(g) = 0$ otherwise. Thus, if $s \geq r$,

$$\sup_{k \geq 10} \|T_{\sigma_k}(g)\|_{F_p^{s,q}(w)} = \sup_{k \geq 10} \left(\sum_{h=10}^k 2^{h(s-r)q} \right)^{1/q} \|g\|_{L^p(w)} = \infty.$$

Next, let

$$\sigma(x, \xi) = \sum_{j \geq 10} 2^{(\delta-1)rj} \exp(-2\pi i 2^j x_1) \psi(2^{-j}\xi),$$

where $\psi \in \mathcal{S}(\mathbf{R}^n)$ is supported in $\{7/8 \leq |\xi| \leq 8/7\}$ and $\psi(\xi) = 1$ for $9/10 \leq |\xi| \leq 10/9$. Then $\sigma(x, \xi)$ satisfies (1.4)–(1.7) for all L and $a = b = 1$. Put, for $m \geq 10$,

$$f_m(x) = \sum_{j=10}^m j^{-1} 2^{j(1-\delta)r} \exp(2\pi i 2^j x_1) g(x),$$

where g is as above. Then $T_\sigma(f_m)(x) = g(x) \sum_{j=10}^m j^{-1}$. On the other hand, $D_h(f_m)(x) = h^{-1}2^{h(1-\delta)r} \exp(2\pi i 2^h x_1)g(x)$ for $h = 10, 11, \dots, m$, $D_h(f_m)(x) = 0$ otherwise, where D_h is defined by Ψ specified above. Thus

$$\|f_m\|_{F_p^{s,q}(w)} = \left(\sum_{h=10}^m h^{-q} 2^{h(s+(1-\delta)r)q} \right)^{1/q} \|g\|_{L^p(w)},$$

and hence the sequence $\{f_m\}$ is bounded in $F_p^{s,q}(w)$ if $s < (\delta - 1)r$ or $s = (\delta - 1)r$ and $1 < q \leq \infty$, while $\{T_\sigma(f_m)\}$ is unbounded there.

REMARK 2. Let

$$\sigma_a(x, \xi) = \exp(-2\pi i \langle x, \xi \rangle - |x|^2)(1 + |\xi|^2)^{-n/a}, \quad a > 0.$$

Suppose that $n/2$ is not an integer and put $\varepsilon_0 = n/2 - [n/2] = 1/2$. If M is a non-negative integer and if $0 \leq M \leq [n/2] - 1$, then we can see that $\sigma_4(x, \xi)$ satisfies (1.4)–(1.7) with $L = M, a = b = 1, \varepsilon_0 \leq r < 1, \omega(s, t) = s^{\delta r} t^r, \delta = (r - \varepsilon_0)/r$. Although $0 \in ((\delta - 1)r, r)$, we can easily see that T_{σ_4} is not bounded on $L^2 = F_2^{0,2}$ (the unweighted Lebesgue space).

Also, we can see that $\sigma_4(x, \xi)$ satisfies (1.4)–(1.7) with $L = [n/2], a = b, \omega(s, t) = s^{\delta r} t^r, 0 < r < 1, \delta = (r - \varepsilon_0 + a)/r$, where $\varepsilon_0 - r \leq a \leq \varepsilon_0$ if $0 < r < \varepsilon_0$ and $0 < a \leq \varepsilon_0$ if $\varepsilon_0 \leq r < 1$. If $L + a < n/2$, then $0 \in ((\delta - 1)r, r)$; but as we mentioned above, T_{σ_4} is not bounded on L^2 (see Coifman-Meyer [3, p. 12] and Yabuta [14, Section 6]).

REMARK 3. Let $\varepsilon = r - [r], \beta \geq \varepsilon, \gamma \geq 0, 0 \leq \delta \leq 1$. For $s \geq 1, t \geq 0$, let

$$\omega_0(s, t) = \begin{cases} s^{\delta\varepsilon} t^\varepsilon & \text{if } st < 1, \\ s^{\delta\varepsilon} t^\varepsilon (st)^{\beta-\varepsilon} & \text{if } st \geq 1, \end{cases}$$

and for $0 \leq s < 1, t \geq 0$, let $\omega_0(s, t) = s^\gamma t^\varepsilon$ ($t < 1$), $\omega_0(s, t) = s^\gamma t^\beta$ ($t \geq 1$). Then, ω_0 satisfies (1.1), $(\omega.1)$ – $(\omega.3)$.

In Section 2, we recall results relevant to the proof of Theorem 3. In Section 3, we prove Theorem 3 by applying these results. Theorem 2 is proved in Section 4 by using Theorem 3. In Section 5, we state results on the boundedness of T_σ on $B_p^{s,q}(w)$ as applications of the results for $F_p^{s,q}(w)$.

2. Results for the proof of Theorem 3. Take radial functions $\psi, \varphi \in C_0^\infty(\mathbf{R}^n)$ such that $\text{supp}(\psi) \subset \{1/4 < |\xi| < 4\}, \psi(\xi) = 1$ if $1/2 \leq |\xi| \leq 2$ and $\text{supp}(\varphi) \subset \{|\xi| < 4\}, \varphi(\xi) = 1$ if $|\xi| \leq 2$. Decompose

$$\begin{aligned} \sigma(x, \xi) &= \sigma(x, \xi)\Phi(\xi) + \sum_{j \geq 1} \sigma(x, \xi)\Psi(2^{-j}\xi) \\ &= \sigma(x, \xi)\Phi(\xi)\varphi(\xi)^2 + \sum_{j \geq 1} \sigma(x, \xi)\Psi(2^{-j}\xi)\psi(2^{-j}\xi)^2 \\ &= \left(\int_{\mathbf{R}^n} A_0(x, k) \exp(2\pi i \langle k, \xi \rangle) dk \right) \varphi(\xi)^2 \end{aligned}$$

$$+ \sum_{j \geq 1} \left(\int_{\mathbb{R}^n} A_j(x, k) \exp(2\pi i \langle 2^{-j}k, \xi \rangle) dk \right) \psi(2^{-j}\xi)^2,$$

where

$$A_j(x, k) = \int_{\mathbb{R}^n} \sigma(x, 2^j\xi) \Psi(\xi) \exp(-2\pi i \langle k, \xi \rangle) d\xi, \quad j \geq 1,$$

$$A_0(x, k) = \int_{\mathbb{R}^n} \sigma(x, \xi) \Phi(\xi) \exp(-2\pi i \langle k, \xi \rangle) d\xi.$$

For $j \geq 1$, put

$$A_j^{(2)}(x, k) = \int_{\mathbb{R}^n} ((\hat{\varphi})_{2^{-j+12}} * \sigma(\cdot, 2^j\xi))(x) \Psi(\xi) \exp(-2\pi i \langle k, \xi \rangle) d\xi,$$

where $((\hat{\varphi})_{2^{-j+12}} * \sigma(\cdot, 2^j\xi))(x) = \int (\hat{\varphi})_{2^{-j+12}}(y) \sigma(x - y, 2^j\xi) dy$.

LEMMA 1. *Suppose that the conditions (1.8) and (1.9) hold. Put $A_j^{(1)}(x, k) = A_j(x, k) - A_j^{(2)}(x, k)$, $j \geq 1$. Then*

$$(2.1) \quad \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |A_j^{(1)}(x, k)|^{q'} \rho(k)^{q'/q} dk \leq C 2^{-q'j[r]} 2^{q'j\delta[r]} \omega(2^j, 2^{-j})^{q'},$$

$$(2.2) \quad \sup_{j \geq 1} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |A_j^{(2)}(x, k)|^{q'} \rho(k)^{q'/q} dk < \infty.$$

Furthermore, the Fourier transform of $A_j^{(2)}(x, k)$ with respect to the x -variable is supported in $\{|\xi| \leq 2^{j-10}\}$ for all k .

PROOF. First we see that

$$\begin{aligned} \int |A_j^{(2)}(x, k)|^{q'} \rho(k)^{q'/q} dk &\leq C \int |(\hat{\varphi})_{2^{-j+12}}(y)| \|\sigma(x + y, 2^j\cdot) \Psi(\cdot)\|_{A_p^q}^{q'} dy \\ &\leq C \sup_{x \in \mathbb{R}^n} \|\sigma(x, 2^j\cdot) \Psi(\cdot)\|_{A_p^q}^{q'}. \end{aligned}$$

Therefore by (1.8) we get (2.2). The support condition for the Fourier transform of $A_j^{(2)}$ is easily seen.

Next, we prove (2.1). Put

$$H(x, y, 2^j\xi) = \sigma(x + y, 2^j\xi) - \sum_{|\beta| \leq [r]} \frac{1}{\beta!} y^\beta \sigma_{(\beta)}(x, 2^j\xi),$$

where $\beta! = \beta_1! \cdots \beta_n!$, $\beta = (\beta_1, \dots, \beta_n)$, $y^\beta = y_1^{\beta_1} \cdots y_n^{\beta_n}$. Since

$$\int \hat{\varphi}(y) dy = 1, \quad \int \hat{\varphi}(y) y^\alpha dy = 0 \quad \text{if } |\alpha| > 0,$$

then we have

$$\int (\hat{\varphi})_{2^{-j+12}}(y) \sigma(x + y, 2^j\xi) dy - \sigma(x, 2^j\xi) = \int (\hat{\varphi})_{2^{-j+12}}(y) H(x, y, 2^j\xi) dy,$$

and hence

$$\int |A_j^{(1)}(x, k)|^{q'} \rho(k)^{q'/q} dk \leq C \int |(\hat{\varphi})_{2^{-j+12}}(y)| \|H(x, y, 2^j \cdot) \Psi(\cdot)\|_{A_p^q}^{q'} dy.$$

By Taylor's formula we see that

$$\|H(x, y, 2^j \cdot) \Psi(\cdot)\|_{A_p^q} \leq C \sum_{|\beta|=[r]} |y|^{[r]} \sup_{0 \leq t \leq 1} \|\sigma_{(\beta)}(x + ty, 2^j \cdot) \Psi(\cdot) - \sigma_{(\beta)}(x, 2^j \cdot) \Psi(\cdot)\|_{A_p^q}.$$

Thus by (1.9)

$$\begin{aligned} \int |A_j^{(1)}(x, k)|^{q'} \rho(k)^{q'/q} dk &\leq C \int |(\hat{\varphi})_{2^{-j+12}}(y)| |y|^{q'[r]} 2^{q'j\delta[r]} \sup_{0 \leq t \leq 1} \omega(2^j, t|y|)^{q'} dy \\ &\leq C \int |\hat{\varphi}(y)| |y|^{q'[r]} 2^{-q'j[r]} 2^{q'j\delta[r]} \sup_{0 \leq t \leq 1} \omega(2^j, 2^{-j+12}t|y|)^{q'} dy \\ &\leq C 2^{-q'j[r]} 2^{q'j\delta[r]} \omega(2^j, 2^{-j})^{q'} \int |\hat{\varphi}(y)| |y|^{q'[r]} (1 + |y|)^{q'M} dy \\ &\leq C 2^{-q'j[r]} 2^{q'j\delta[r]} \omega(2^j, 2^{-j})^{q'}, \end{aligned}$$

where we have used $(\omega.1)$. This proves (2.1), which completes the proof of Lemma 1.

LEMMA 2. *Suppose that the conditions (1.8) through (1.11) hold. Let $A_0^{(1)} = A_0$, and for $j \geq 1$ let $A_j^{(1)}$ be as in Lemma 1. Define $A_{j,h}^{(1)}(x, k) = D_h(A_j^{(1)}(\cdot, k))(x)$ for $j, h \geq 0$. Then*

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} \left(\int |A_{j,h}^{(1)}(x, k)|^{q'} \rho(k)^{q'/q} dk \right)^{1/q'} \\ \leq C \min(2^{-j[r]} 2^{j\delta[r]} \omega(2^j, 2^{-j}), 2^{-h[r]} 2^{j\delta[r]} \omega(2^j, 2^{-h})). \end{aligned}$$

PROOF. Let $j \geq 1$ and $h \geq 1$. We note that

$$A_{j,h}^{(1)}(x, k) = (\hat{\Psi})_{2^{-h}} * A_j(\cdot, k)(x) - (\hat{\varphi})_{2^{-j+12}} * (\hat{\Psi})_{2^{-h}} * A_j(\cdot, k)(x).$$

Since $\int \hat{\Psi}(y) y^\alpha dy = 0$ for all α , we have

$$(\hat{\Psi})_{2^{-h}} * \sigma(\cdot, 2^j \xi)(x) = \int (\hat{\Psi})_{2^{-h}}(y) H(x, y, 2^j \xi) dy,$$

where H is as in the proof of Lemma 1. Thus, as in the proof of Lemma 1, we have

$$(2.3) \quad \int |D_h(A_j(\cdot, k))(x)|^{q'} \rho(k)^{q'/q} dk \leq C 2^{-q'h[r]} 2^{q'j\delta[r]} \omega(2^j, 2^{-h})^{q'}$$

and hence

$$\begin{aligned} (2.4) \quad &\int |(\hat{\varphi})_{2^{-j+12}} * (\hat{\Psi})_{2^{-h}} * A_j(\cdot, k)(x)|^{q'} \rho(k)^{q'/q} dk \\ &\leq C \int |(\hat{\varphi})_{2^{-j+12}}(y)| \left[\int |(\hat{\Psi})_{2^{-h}} * A_j(\cdot, k)(x - y)|^{q'} \rho(k)^{q'/q} dk \right] dy \\ &\leq C 2^{-q'h[r]} 2^{q'j\delta[r]} \omega(2^j, 2^{-h})^{q'}. \end{aligned}$$

Also, by using Lemma 1, we get

$$(2.5) \quad \int |A_{j,h}^{(1)}(x, k)|^{q'} \rho(k)^{q'/q} dk \leq C 2^{-q'j|r|} 2^{q'j\delta|r|} \omega(2^j, 2^{-j})^{q'} \quad \text{for } h \geq 0, j \geq 1.$$

Combining (2.3)–(2.5), we obtain the conclusion of Lemma 2 for $j \geq 1$. The proof for the case $j = 0$ can be done similarly by using (1.10) and (1.11). \square

We can prove the following result by applying Hölder’s inequality.

LEMMA 3. *Let $a > 1, 1 \leq r < \infty$, and let $\{x_k\}_{k=0}^\infty$ be a sequence of complex numbers such that $\sum_{k=0}^\infty |x_k|^r < \infty$. Then*

$$\sum_{j=0}^\infty \left| a^{-j} \sum_{k=0}^j a^k x_k \right|^r \leq (a/(a-1))^r \sum_{k=0}^\infty |x_k|^r.$$

The following lemma generalizes a result stated in [1].

LEMMA 4. *For $j = 0, 1, 2, \dots$, let f_j be a tempered distribution whose Fourier transform is supported in $\{|\xi| < c2^j\}$ for some constant $c > 0$ (note that f_j is a function by the support condition). We assume that $f_j = 0$ for all but a finite number of values of j . Let $s > 0, 1 < p < \infty, 1 < q < \infty$ and $w \in A_p$. Then we have*

$$\left\| \sum_{j=0}^\infty f_j \right\|_{F_p^{s,q}(w)} \leq C \left\| \left(\sum_{j=0}^\infty 2^{qjs} |f_j|^q \right)^{1/q} \right\|_{L^p(w)}.$$

PROOF. There exists a positive integer N such that

$$(2.6) \quad \sum_{j=0}^\infty f_j = \sum_{j=0}^\infty \sum_{l=0}^{j+N} D_l(f_j) = \sum_{l=0}^\infty \sum_{j=\max(l-N,0)}^\infty D_l(f_j) = \sum_{l=0}^\infty D_l \left(\sum_{j=\max(l-N,0)}^\infty f_j \right).$$

Now, by Hölder’s inequality we have, for appropriate functions g_l ,

$$(2.7) \quad \begin{aligned} \left| \sum_{l=0}^\infty \int D_l \left(\sum_{j=\max(l-N,0)}^\infty f_j \right) g_l dx \right| &= \left| \sum_{j=0}^\infty \int f_j \left(\sum_{l=0}^{j+N} D_l(g_l) \right) dx \right| \\ &\leq \left(\int \left(\sum_{j \geq 0} 2^{qjs} |f_j(x)|^q \right)^{p/q} w(x) dx \right)^{1/p} I^{1/p'}, \end{aligned}$$

where

$$I = \int \left(\sum_{j \geq 0} \left| 2^{-js} \left(\sum_{l=0}^{j+N} D_l(g_l) \right) \right|^{q'} \right)^{p'/q'} w(x)^{-p'/p} dx.$$

By Lemma 3 with $a = 2^s$ and $r = q'$, we have

$$\begin{aligned} \sum_{j \geq 0} \left| 2^{-js} \left(\sum_{l=0}^{j+N} D_l(g_l) \right) \right|^{q'} &= \sum_{j \geq 0} \left| 2^{-js} \left(\sum_{l=0}^{j+N} 2^{sl} D_l(2^{-sl} g_l) \right) \right|^{q'} \\ &\leq C \sum_{l=0}^{\infty} 2^{-q'sl} |D_l(g_l)|^{q'}. \end{aligned}$$

Therefore

$$\begin{aligned} I &\leq C \int \left(\sum_{l=0}^{\infty} 2^{-q'sl} |D_l(g_l)|^{q'} \right)^{p'/q'} w(x)^{-p'/p} dx \\ &\leq C \int \left(\sum_{l=0}^{\infty} 2^{-q'sl} |g_l|^{q'} \right)^{p'/q'} w(x)^{-p'/p} dx, \end{aligned}$$

where the last inequality follows from a well-known vector valued inequality, since $w^{-p'/p} \in A_{p'}$ (see [9]). By a duality argument using this estimate in (2.7), we have

$$\begin{aligned} \int \left(\sum_{l=0}^{\infty} 2^{qsl} \left| D_l \left(\sum_{j=\max(l-N,0)}^{\infty} f_j \right) \right|^q \right)^{p/q} w(x) dx \\ \leq C \int \left(\sum_{j \geq 0} 2^{qjs} |f_j(x)|^q \right)^{p/q} w(x) dx. \end{aligned}$$

From this and (2.6) we can easily get the conclusion. □

3. Proof of Theorem 3. Put, for $j \geq 1$,

$$\begin{aligned} E_j(f)(x, k) &= \int_{\mathbf{R}^n} \exp(2\pi i \langle 2^{-j}k, \xi \rangle) \psi(2^{-j}\xi)^2 \hat{f}(\xi) \exp(2\pi i \langle x, \xi \rangle) d\xi \\ &= (\tau_{-k} \mathcal{F}^{-1}(\psi))_{2^{-j}} * \Delta_j(f)(x), \end{aligned}$$

where $\tau_k f(x) = f(x - k)$ and

$$\Delta_j(f)(x) = \int_{\mathbf{R}^n} \psi(2^{-j}\xi) \hat{f}(\xi) \exp(2\pi i \langle x, \xi \rangle) d\xi.$$

Also put $E_0(f)(x, k) = (\tau_{-k} \mathcal{F}^{-1}(\varphi)) * \Delta_0(f)(x)$, where

$$\Delta_0(f)(x) = \int_{\mathbf{R}^n} \varphi(\xi) \hat{f}(\xi) \exp(2\pi i \langle x, \xi \rangle) d\xi.$$

Then we can see that

$$T_{\sigma}(f)(x) = \sum_{j \geq 0} \int A_j(x, k) E_j(f)(x, k) dk, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

Decompose $A_j(x, k) = A_j^{(1)}(x, k) + A_j^{(2)}(x, k)$ for $j \geq 0$, where $A_j^{(1)}$ and $A_j^{(2)}$ are as in Lemmas 1 and 2 ($A_0^{(2)} \equiv 0$). Put

$$B_i(f)(x) = \sum_{j \geq 0} \int A_j^{(i)}(x, k) E_j(f)(x, k) dk, \quad i = 1, 2.$$

Then $T_\sigma(f) = B_1(f) + B_2(f)$. We note the following. For a positive integer N , let

$$S_N^{(i)}(f)(x) = \sum_{j=0}^N \int A_j^{(i)}(x, k) E_j(f)(x, k) dk, \quad i = 1, 2.$$

Then we can easily see that $|S_N^{(i)}(f)(x)| \leq C$ for some $C > 0$ independent of x and N . Also, we can see that $S_N^{(i)}(f)(x) \rightarrow B_i(f)(x)$ as $N \rightarrow \infty$ for all x .

We estimate $B_i(f)$, $i = 1, 2$, separately under the hypotheses of Theorem 3. We begin with the estimation of $B_2(f)$. First, observing that the Fourier transform of $\int A_j^{(2)}(x, k) E_j(f)(x, k) dk$, $j \geq 1$, is supported in an annulus of the form $\{c_1 2^j < |\xi| < c_2 2^j\}$ with $c_1, c_2 > 0$, by a vector valued inequality (see [9], [2]) we have, for $s \in \mathbf{R}$,

$$\|B_2(f)\|_{F_p^{s,q}(w)}^p \leq C \int \left(\sum_{j=1}^{\infty} 2^{qjs} \left| \int A_j^{(2)}(x, k) E_j(f)(x, k) dk \right|^q \right)^{p/q} w(x) dx.$$

By Hölder’s inequality and Lemma 1 (2.2), the right hand side is bounded by, up to a constant factor,

$$\int \left(\sum_{j \geq 1} \int \rho(k)^{-1} 2^{qjs} |E_j(f)(x, k)|^q dk \right)^{p/q} w(x) dx.$$

Let g be a non-negative function on \mathbf{R}^n . Then, a direct computation yields

$$\begin{aligned} & \int \sum_{j \geq 1} \int \rho(k)^{-1} 2^{qjs} |E_j(f)(x, k)|^q dk g(x) dx \\ & \leq C \sum_{j \geq 1} \int \left(\int \rho(k)^{-1} \int 2^{jn} |\mathcal{F}^{-1}(\psi)(2^j(x - y) + k)| g(x) dx dk \right) 2^{qjs} |\Delta_j(f)(y)|^q dy \\ & \leq C \sum_{j \geq 1} \int \left(\int \rho(k)^{-1} \mathcal{M}(g)(y - 2^{-j}k) dk \right) 2^{qjs} |\Delta_j(f)(y)|^q dy \\ & \leq C \sum_{j \geq 1} \int \mathcal{M}\mathcal{M}(g)(y) 2^{qjs} |\Delta_j(f)(y)|^q dy, \end{aligned}$$

where the last inequality follows from (1.12). Put

$$I = \int \left(\sum_{j \geq 1} 2^{qjs} |\Delta_j(f)(y)|^q \right)^{p/q} w(y) dy.$$

Now we assume that $p > q$. Then by Hölder's inequality

$$\begin{aligned} & \sum_{j \geq 1} \int \mathcal{M}\mathcal{M}(g)(y) 2^{qjs} |\Delta_j(f)(y)|^q dy \\ & \leq C \left(\int \mathcal{M}\mathcal{M}(g)^{(p/q)'}(y) w(y)^{-q(p/q)'/p} dy \right)^{1/(p/q)'} I^{q/p} \\ & \leq C \|g\|_{L^{(p/q)'}(w^{-q(p/q)'/p})} I^{q/p}, \end{aligned}$$

if $g \in L^{(p/q)'}(w^{-q(p/q)'/p})$ and $w \in A_{p/q}$. Therefore by the converse of Hölder's inequality we have

$$(3.1) \quad \int \left(\sum_{j \geq 1} \int \rho(k)^{-1} 2^{qjs} |E_j(f)(x, k)|^q dk \right)^{p/q} w(x) dx \leq CI.$$

The case $p = q$, $w \in A_1$ can be treated similarly and we also have this inequality. Thus

$$(3.2) \quad \|B_2(f)\|_{F_p^{s,q}(w)}^p \leq C \int \left(\sum_{j \geq 1} 2^{qjs} |\Delta_j(f)(y)|^q \right)^{p/q} w(y) dy.$$

Next, we estimate $B_1(f)$. For positive integers N, M , put

$$U_{N,M}(f)(x) = \sum_{j=0}^N \sum_{h=0}^M \int A_{j,h}^{(1)}(x, k) E_j(f)(x, k) dk,$$

where $A_{j,h}^{(1)}$ is as in Lemma 2. We estimate $U_{N,M}(f)$ on $F_p^{s,q}(w)$. The estimate will be uniform in N and M . Put $\tilde{A}_{j,h}^{(1)} = A_{j,h}^{(1)}$ if $0 \leq h \leq M$ and $\tilde{A}_{j,h}^{(1)} = 0$ if $h > M$; also $\tilde{E}_j(f) = E_j(f)$ if $0 \leq j \leq N$ and $\tilde{E}_j(f) = 0$ if $j > N$. Then

$$U_{N,M}(f)(x) = \sum_{j \geq 0} \sum_{h \geq 0} \int \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk = G_1(x) + G_2(x) + G_3(x),$$

where

$$\begin{aligned} G_1(x) &= \sum_{j=10}^{\infty} \sum_{h=0}^{j-10} \int \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk, \\ G_2(x) &= \sum_{j=10}^{\infty} \sum_{h=j-9}^{\infty} \int \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk = \sum_{h=1}^{\infty} \sum_{j=10}^{h+9} \int \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk, \\ G_3(x) &= \sum_{h=0}^{\infty} \sum_{j=0}^9 \int \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk. \end{aligned}$$

We estimate G_1, G_2 and G_3 separately.

Observing that the Fourier transform of $\int \sum_{0 \leq h \leq j-10} \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk$, $j \geq 10$, is supported in an annulus of the form $\{c_1 2^j < |\xi| < c_2 2^j\}$, $c_1, c_2 > 0$, via Hölder's inequality

we have

$$\begin{aligned} \|G_1\|_{F_p^{s,q}(w)}^p &\leq C \int \left(\sum_{j=10}^{\infty} 2^{qjs} \left| \int \sum_{h=0}^{j-10} \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk \right|^q \right)^{p/q} w(x) dx \\ &\leq C \int \left(\sum_{j=10}^{\infty} \left(\int \left| \sum_{h=0}^{j-10} \tilde{A}_{j,h}^{(1)}(x, k) \right|^{q'} \rho(k)^{q'/q} dk \right)^{q/q'} F_j(s, q, x) \right)^{p/q} w(x) dx, \end{aligned}$$

where

$$F_j(s, q, x) = \int \rho(k)^{-1} 2^{qjs} |\tilde{E}_j(f)(x, k)|^q dk.$$

We note that

$$\sum_{0 \leq h \leq j-10} \tilde{A}_{j,h}^{(1)}(x, k) = g^{(j,M)} * A_j^{(1)}(\cdot, k)(x), \quad j \geq 10$$

for some $g^{(j,M)} \in \mathcal{S}(\mathbf{R}^n)$ such that $\|g^{(j,M)}\|_{L^1} \leq c$, where c is a constant independent of j and M . Therefore by Lemma 1 (2.1) we have

$$\int \left| \sum_{h=0}^{j-10} \tilde{A}_{j,h}^{(1)}(x, k) \right|^{q'} \rho(k)^{q'/q} dk \leq C 2^{-q'j[r]} 2^{q'j\delta[r]} \omega(2^j, 2^{-j})^{q'}.$$

Thus for any $s \in \mathbf{R}$

$$\begin{aligned} (3.3) \quad \|G_1\|_{F_p^{s,q}(w)}^p &\leq CC(\omega)^p \int \left(\sum_{j \geq 10} 2^{qjr(\delta-1)} F_j(s, q, x) \right)^{p/q} w(x) dx \\ &\leq CC(\omega)^p \int \left(\sum_{j \geq 10} 2^{qjs} 2^{qjr(\delta-1)} |\Delta_j(f)(y)|^q \right)^{p/q} w(y) dy, \end{aligned}$$

where the second inequality can be proved as above (see (3.1)).

Next we estimate G_2 . Since the Fourier transform of $\int \sum_{j=10}^{h+9} \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk$, $h \geq 1$, is supported in $\{|\xi| < c2^h\}$, by Lemma 4 we have, for $0 < s < r$,

$$\|G_2\|_{F_p^{s,q}(w)}^p \leq C \int \left(\sum_{h=1}^{\infty} 2^{qhs} \left| \int \sum_{j=10}^{h+9} \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk \right|^q \right)^{p/q} w(x) dx.$$

By Hölder's inequality, the right hand side is bounded by, up to a constant factor,

$$\int \left(\sum_{h=1}^{\infty} 2^{h(s-r)} \sum_{j=10}^{h+9} 2^{(r-s)j} \left| \int (2^{-rj} 2^{hr} \tilde{A}_{j,h}^{(1)}(x, k)) (2^{sj} \tilde{E}_j(f)(x, k)) dk \right|^q \right)^{p/q} w(x) dx.$$

By Hölder's inequality and Lemma 2, the inner integral is bounded by

$$\begin{aligned} &\left(\int |2^{-rj} 2^{hr} \tilde{A}_{j,h}^{(1)}(x, k)|^{q'} \rho(k)^{q'/q} dk \right)^{1/q'} \left(\int |2^{sj} \tilde{E}_j(f)(x, k)|^q \rho(k)^{-1} dk \right)^{1/q} \\ &\leq C 2^{-(r-\delta[r])j} 2^{h(r-[r])} \omega(2^j, 2^{-h}) \left(\int |2^{sj} \tilde{E}_j(f)(x, k)|^q \rho(k)^{-1} dk \right)^{1/q}. \end{aligned}$$

Therefore, $\|G_2\|_{F_p^{s,q}(w)}^p$ is bounded by

$$\begin{aligned} & CC(\omega)^p \int \left(\sum_{h=1}^{\infty} 2^{h(s-r)} \sum_{j=10}^{h+9} 2^{(r-s)j} \int |2^{-(1-\delta)rj} 2^{sj} \tilde{E}_j(f)(x, k)|^q \rho(k)^{-1} dk \right)^{p/q} w(x) dx \\ & \leq CC(\omega)^p \int \left(\sum_{j=10}^{\infty} \int |2^{-(1-\delta)rj} 2^{sj} \tilde{E}_j(f)(x, k)|^q \rho(k)^{-1} dk \right)^{p/q} w(x) dx . \end{aligned}$$

Thus, we have (see (3.1))

$$(3.4) \quad \|G_2\|_{F_p^{s,q}(w)}^p \leq CC(\omega)^p \int \left(\sum_{j \geq 10} 2^{-q(1-\delta)rj} 2^{qjs} |\Delta_j(f)(y)|^q \right)^{p/q} w(y) dy .$$

Finally, in the same way as in the case of G_2 , if $0 < s < r$, we have

$$\begin{aligned} (3.5) \quad \|G_3\|_{F_p^{s,q}(w)}^p & \leq C \int \left(\sum_{h=0}^{\infty} 2^{qhs} \left| \int \sum_{j=0}^9 \tilde{A}_{j,h}^{(1)}(x, k) \tilde{E}_j(f)(x, k) dk \right|^q \right)^{p/q} w(x) dx \\ & \leq C_{\eta} C(\omega)^p \int \left(\sum_{j=0}^9 2^{-\eta j} 2^{qjs} |\Delta_j(f)(y)|^q \right)^{p/q} w(y) dy \end{aligned}$$

for any $\eta \geq 0$.

By (3.3) through (3.5) we have

$$(3.6) \quad \|U_{N,M}(f)\|_{F_p^{s,q}(w)}^p \leq C \int \left(\sum_{j=0}^{\infty} 2^{qjs} |\Delta_j(f)(y)|^q \right)^{p/q} w(y) dy ,$$

where the constant C is independent of N and M . Fix j and put

$$T_M(f)(x) = \sum_{h=0}^M \int A_{j,h}^{(1)}(x, k) E_j(f)(x, k) dk .$$

Then we can see that $|T_M(f)(x)| \leq C$ for some $C > 0$ independent of x and M and that $T_M(f)(x) \rightarrow \int A_j^{(1)}(x, k) E_j(f)(x, k) dk$ as $M \rightarrow \infty$ for all x . Therefore, letting $M \rightarrow \infty$ then $N \rightarrow \infty$ in (3.6), we have

$$(3.7) \quad \|B_1(f)\|_{F_p^{s,q}(w)}^p \leq C \int \left(\sum_{j=0}^{\infty} 2^{qjs} |\Delta_j(f)(y)|^q \right)^{p/q} w(y) dy .$$

By the estimates (3.2) and (3.7), we can get the conclusion of Theorem 3, since

$$\int \left(\sum_{j=0}^{\infty} 2^{qjs} |\Delta_j(f)(y)|^q \right)^{p/q} w(y) dy \leq C \|f\|_{F_p^{s,q}(w)}^p$$

for $s \in \mathbf{R}$, $0 < q < \infty$, $0 < p < \infty$ and $w \in A_{\infty}$ (see, e.g., [2]).

4. Proof of Theorem 2. Under the hypotheses of Theorem 2 we prove the validity of the conditions (1.8) through (1.11) with $\rho(k) = (1 + |k|^q)^s$, $s = [n/q] + d$, where d is chosen so that $a > d$ and $[n/q] + d > n/q$.

Let $j \geq 1$. Then, integration by parts gives

$$A_j(x, k) = (2\pi i k_m)^{-[n/q]} \int_{\mathbf{R}^n} [(\partial/\partial \xi_m)^{[n/q]}(\sigma(x, 2^j \xi)\Psi(\xi))] \exp(-2\pi i \langle k, \xi \rangle) d\xi .$$

Let ψ be as in Section 2. Then by applying the Hausdorff-Young inequality we have, for $l \geq 0$,

$$\begin{aligned} (4.1) \quad & \int_{|k| \approx |k_m|, 2^l \leq |k| \leq 2^{l+1}} |A_j(x, k)|^{q'} (1 + |k|^q)^{sq'/q} dk \\ & \leq C 2^{q'sl} \int_{|k| \approx |k_m|} |\psi(2^{-l}k)A_j(x, k)|^{q'} dk \\ & \leq C 2^{q'dl} \left(\int_{\mathbf{R}^n} |((\hat{\psi})_{2^{-l}} * F(x, \cdot))(\xi)|^q d\xi \right)^{q'/q} , \end{aligned}$$

where $F(x, \xi) = (\partial/\partial \xi_m)^{[n/q]}(\sigma(x, 2^j \xi)\Psi(\xi))$. Then, by (1.4) and (1.5) with $L = [n/q]$ we have, for all $x, \xi, \eta \in \mathbf{R}^n$,

$$(4.2) \quad |F(x, \xi)| \leq C \quad \text{and} \quad |F(x, \xi + \eta) - F(x, \xi)| \leq C|\eta|^a .$$

By (4.2) we see that

$$\begin{aligned} & |((\hat{\psi})_{2^{-l}} * F(x, \cdot))(\xi)| = \left| \int [F(x, \xi + \eta) - F(x, \xi)](\hat{\psi})_{2^{-l}}(\eta) d\eta \right| \\ & \leq \left| \int_{|\eta| < |\xi|/2} [F(x, \xi + \eta) - F(x, \xi)](\hat{\psi})_{2^{-l}}(\eta) d\eta \right| \\ & \quad + \left| \int_{|\eta| \geq |\xi|/2} [F(x, \xi + \eta) - F(x, \xi)](\hat{\psi})_{2^{-l}}(\eta) d\eta \right| \\ & \leq C \chi_0(\xi) \int |\eta|^a |(\hat{\psi})_{2^{-l}}(\eta)| d\eta + C \min(2^{-al}, (2^l |\xi|)^{-2n}) \\ & \leq C 2^{-al} (1 + |\xi|)^{-2n} , \end{aligned}$$

where χ_0 is the characteristic function of the ball $\{|\xi| \leq 5\}$. Using this in (4.1), we have

$$(4.3) \quad \int_{|k| \approx |k_m|, |k| \geq 1} |A_j(x, k)|^{q'} (1 + |k|^q)^{sq'/q} dk \leq \sum_{l \geq 0} C 2^{q'dl} 2^{-q'al} \leq C .$$

It is easier to get the estimate

$$\int_{|k| \leq 1} |A_j(x, k)|^{q'} (1 + |k|^q)^{sq'/q} dk \leq C .$$

Using this and (4.3) for $m = 1, \dots, n$, we see that the condition (1.8) holds.

Next we show that the condition (1.9) holds. Let $|\beta| = [r]$. Put

$$A(x, y, k, j, \beta) = \int_{\mathbf{R}^n} [\sigma_{(\beta)}(x + y, 2^j \xi) - \sigma_{(\beta)}(x, 2^j \xi)] \Psi(\xi) \exp(-2\pi i \langle k, \xi \rangle) d\xi .$$

Then, by integration by parts

$$A(x, y, k, j, \beta) = (2\pi i k_m)^{-[n/q]} \int_{\mathbf{R}^n} G(x, y, \xi) \exp(-2\pi i \langle k, \xi \rangle) d\xi ,$$

where $G(x, y, \xi) = (\partial/\partial \xi_m)^{[n/q]} [(\sigma_{(\beta)}(x + y, 2^j \xi) - \sigma_{(\beta)}(x, 2^j \xi)) \Psi(\xi)]$. By the Hausdorff-Young inequality we have, as above, for $l \geq 0$

$$(4.4) \quad \int_{|k| \approx |k_m|, 2^l \leq |k| \leq 2^{l+1}} |A(x, y, k, j, \beta)|^{q'} (1 + |k|^q)^{sq'/q} dk \leq C 2^{q'l} \left(\int_{\mathbf{R}^n} |((\hat{\psi})_{2^{-l}} * G(x, y, \cdot))(\xi)|^q d\xi \right)^{q'/q} .$$

By (1.6), (1.7) with $L = [n/q]$ and $a = b$, we have, for all $x, y, \xi, \eta \in \mathbf{R}^n$,

$$(4.5) \quad |G(x, y, \xi)| \leq C 2^{j\delta[r]} \omega(2^j, |y|) \quad \text{and} \\ |G(x, y, \xi + \eta) - G(x, y, \xi)| \leq C |\eta|^a 2^{j\delta[r]} \omega(2^j, |y|) .$$

Using (4.5) and arguing as in the proof for (1.8) above, we can see that

$$|(\hat{\psi}_{2^{-l}} * G(x, y, \cdot))(\xi)| \leq C 2^{-al} 2^{j\delta[r]} \omega(2^j, |y|) (1 + |\xi|)^{-2n} .$$

Using this in (4.4) and summing up in $l \geq 0$, we have

$$(4.6) \quad \int_{|k| \approx |k_m|, |k| \geq 1} |A(x, y, k, j, \beta)|^{q'} (1 + |k|^q)^{sq'/q} dk \leq C 2^{q'j\delta[r]} \omega(2^j, |y|)^{q'} .$$

We also have

$$\int_{|k| \leq 1} |A(x, y, k, j, \beta)|^{q'} (1 + |k|^q)^{sq'/q} dk \leq C 2^{q'j\delta[r]} \omega(2^j, |y|)^{q'} .$$

Using this and (4.6) for $m = 1, \dots, n$, we can get (1.9).

The validity of the conditions (1.10) and (1.11) can be proved similarly. Since $\rho(x) = (1 + |x|^q)^s$ satisfies (1.12), now Theorem 2 follows from Theorem 3.

5. Boundedness on the weighted Besov spaces. As applications of Theorems 1–3, we have the following:

THEOREM 4. *Let $1 < t \leq 2$. Suppose that $\sigma(x, \xi)$ satisfies (1.2) and (1.3) with $L = [n/t] + 1$. Then*

$$\|T_\sigma(f)\|_{B_p^{s,q}(w)} \leq C \|f\|_{B_p^{s,q}(w)}, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

where $t \leq p < \infty, s \in ((\delta - 1)r, r), 0 < q \leq \infty$ and $w \in A_{p/t}$.

THEOREM 5. Let $1 < t \leq 2$, $0 < a \leq 1$ and $[n/t] + a > n/t$. Suppose that $\sigma(x, \xi)$ satisfies (1.4)–(1.7) with $L = [n/t]$, $a = b$. Let $t \leq p < \infty$, $s \in ((\delta - 1)r, r)$, $0 < q \leq \infty$ and $w \in A_{p/t}$. Then, we have

$$\|T_\sigma(f)\|_{B_p^{s,q}(w)} \leq C\|f\|_{B_p^{s,q}(w)}, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

THEOREM 6. Let $1 < t \leq 2$. Suppose that (1.8)–(1.11) hold with t in place of q . We further assume (1.12). Then

$$\|T_\sigma(f)\|_{B_p^{s,q}(w)} \leq C\|f\|_{B_p^{s,q}(w)}, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

where $t \leq p < \infty$, $s \in ((\delta - 1)r, r)$, $0 < q \leq \infty$ and $w \in A_{p/t}$.

Also, we have remarks similar to those in Theorems 1–3 for the dependence of the bounds on σ . We can derive Theorems 4–6 from Theorems 1–3, respectively, by applying interpolation arguments (see [2, Theorem 3.5]).

REFERENCES

- [1] G. BOURDAUD, L^p -estimates for certain non-regular pseudo-differential operators, *Comm. Partial Differential Equations* 7 (1982), 1023–1033.
- [2] H.-Q. BUI, Weighted Besov and Triebel spaces: Interpolation by the real method, *Hiroshima Math. J.* 12 (1982), 581–605.
- [3] R. R. COIFMAN AND Y. MEYER, Au delà des opérateurs pseudo-différentiels, *Astérisque* 57, Soc. Math. France, Paris, 1978.
- [4] M. FRAZIER, B. JAWERTH AND G. WEISS, Littlewood-Paley theory and study of function spaces, *CBMS Regional Conf. Ser. in Math.* 79, Conf. Board Math. Sci., American Mathematical Society, Providence, R.I., 1991.
- [5] J. MARSCHALL, Pseudo-differential operators with non-regular symbols of the class $S_{\rho,\delta}^m$, *Comm. Partial Differential Equations* 12 (1987), 921–965.
- [6] J. MARSCHALL, Pseudo-differential operators with coefficients in Sobolev spaces, *Trans. Amer. Math. Soc.* 307 (1988), 335–361.
- [7] A. MIYACHI AND K. YABUTA, L^p -boundedness of pseudo-differential operators with non-regular symbols, *Bull. Fac. Sci. Ibaraki Univ. Ser. A* 17 (1985), 1–20.
- [8] M. NAGASE, The L^p -boundedness of pseudo-differential operators with non-regular symbols, *Comm. Partial Differential Equations* 2 (1977), 1045–1061.
- [9] J. L. RUBIO DE FRANCIA, F. J. RUIZ AND J. L. TORREA, Calderón-Zygmund theory for operator-valued kernels, *Adv. in Math.* 62 (1986), 7–48.
- [10] S. SATO, A note on weighted estimates for certain classes of pseudo-differential operators, *Rocky Mountain J. Math.* 35 (2005), 267–284.
- [11] M. SUGIMOTO, L^p -boundedness of pseudo-differential operators satisfying Besov estimate II, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 35 (1988), 149–162.
- [12] M. SUGIMOTO, Pseudo-differential operators on Besov spaces, *Tsukuba J. Math.* 12 (1988), 43–63.
- [13] H. TRIEBEL, *Theory of function spaces II*, *Monogr. Math.* 84, Birkhäuser Verlag, Basel, 1992.
- [14] K. YABUTA, Calderón-Zygmund operators and pseudo-differential operators, *Comm. Partial Differential Equations* 10 (1985), 1005–1022.
- [15] K. YABUTA, Weighted norm inequalities for pseudo differential operators, *Osaka J. Math.* 23 (1986), 703–723.

DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
KANAZAWA UNIVERSITY
KANAZAWA 920-1192
JAPAN

E-mail address: shuichi@kenroku.kanazawa-u.ac.jp