Tohoku Math. J.
56 (2004), 423–443

EXAMPLES OF LORENTZIAN GEODESIBLE FOLIATIONS OF CLOSED THREE-MANIFOLDS HAVING HEEGAARD SPLITTINGS OF GENUS ONE

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(Received February 5, 2003, revised August 27, 2003)

Abstract. We construct codimension-one, Lorentzian geodesible foliations of closed three-manifolds having Heegaard splittings of genus one. We prove that all the inner leaves of a Reeb component of a codimension-one, totally geodesic foliation of a Lorentzian three-manifold are spacelike, and the boundary leaf of a Reeb component is lightlike.

1. Introduction. Geodesibility of a given codimension-1 foliation $\mathcal{F}$ has been studied by several authors. Here geodesibility of $\mathcal{F}$ means that $\mathcal{F}$ is totally geodesic for some complete Riemannian metric, and in this case $\mathcal{F}$ is called complete Riemannian geodesible. Let us first recall a couple of results about a codimension-1, totally geodesic foliation $\mathcal{F}$ of a complete Riemannian manifold (M, g). Blumenthal and Hebda ([1]) showed that the universal covering of M is a product $L \times \boldsymbol{R}$ and the lift of $\mathcal{F}$ is the product foliation $\{L \times \{*\}\}$, where L is the universal covering of the leaves of $\mathcal{F}$. Oshikiri ([9], [10]) proved that any Killing field with bounded length preserves $\mathcal{F}$. Regarding codimension-1, complete Riemannian geodesible foliations, we remark the following results. Carrière and Ghys ([3]) classified a codimension-1, complete Riemannian geodesible foliation of a closed 3-manifold. Ghys ([5]) classified a codimension-1, complete Riemannian geodesible foliation of a closed manifold. Thus a codimension-1, complete Riemannian geodesible foliation of a closed manifold is well understood.

Now we consider codimension-1, Lorentzian geodesible foliations. Zeghib ([14]) constructed codimension-1, lightlike totally geodesic foliations. He constructed a lightlike totally geodesic foliation $\mathcal{F}$ in the following cases:

(1) $\mathcal{F}$ is defined by a locally free action with codimension-1 orbits of a Lie group with a 1-dimensional normal subgroup.

(2) $\mathcal{F}$ is the suspension of a foliation $\mathcal{L}$ of a Riemannian manifold (M, g) by a diffeomorphism of M preserving $\mathcal{L}$ and $g|_{T\mathcal{L}}$.

Although this construction gives an example of a codimension-1, Lorentzian geodesible foliation, we do not know whether it is complete Lorentzian geodesible or not (for example, consider a non-singular flow on a 2-torus with Reeb components and the following). Carrière and Rozoy ([4]) proved that the canonical lightlike totally geodesic foliations of a lightlike

2000 *Mathematics Subject Classification*. Primary 57R30; Secondary 53C50.
Key words and phrases. Lorentzian manifolds, totally geodesic foliations.

complete Lorentzian 2-torus are C^0-linearizable. So there is a difference between complete Lorentzian geodesible foliations and Lorentzian geodesible ones.

By the theorem of Blumenthal and Hebda ([1]), there exist no codimension-1, Riemannian geodesible foliations of a closed manifold with finite fundamental group. We have the following question.

QUESTION 1.1. Does there exist a codimension-1, Lorentzian geodesible foliation of a closed manifold with finite fundamental group?

Although a Riemannian metric on a closed manifold is always complete, a Lorentzian metric on a closed manifold is *not* always complete. Hence it is a serious matter whether to suppose completeness of Lorentzian metrics when we consider totally geodesic foliations of closed manifolds. Recall that a geodesic of an incomplete Lorentzian metric on a closed manifold has an infinite Riemannian length whenever it is not closed. Thus if we consider, for example, an Ehresmann connection for a foliation ([2]) of a *closed* manifold, then it seems that the assumption of completeness of the Lorentzian metric is unnecessary. Therefore it seems meaningful to consider totally geodesic foliations of closed Lorentzian manifolds without assuming completeness. Hence, first of all, we consider codimension-1, Lorentzian geodesible foliations. However, completeness must be important. Actually, the present author proved the following (see [12]): There exists no totally geodesic foliation of a lightlike complete Lorentzian 2-torus which contains at least two kinds of leaves among spacelike, timelike, and lightlike ones. There is an example of a totally geodesic foliation of a lightlike incomplete Lorentzian 2-torus which contains at least two kinds of leaves. We have a partial answer to Question 1.1 as follows.

THEOREM 3.1. *If a closed* 3*-manifold has a Heegaard splitting of genus one, then it has a codimension-*1 *Lorentzian geodesible foliation.*

We prove this theorem by constructing examples in Section 3.2. Each example has spacelike and lightlike leaves. So the following question arises.

QUESTION 1.2. Does a codimension-1, totally geodesic foliation of a closed Lorentzian manifold with finite fundamental group have lightlike leaves?

In Section 4, we consider totally geodesic foliations consisting of spacelike leaves of compact Lorentzian manifold, and codimension-1, totally geodesic foliations consisting of timelike leaves of Lorentzian 3-manifold. We do not assume completeness of Lorentzian metrics. We then have the following.

PROPOSITION 4.2. *Let* (M, g) *be a compact Lorentzian manifold and* $\mathcal{F}$ *a totally geodesic foliation of* M *consisting of spacelike leaves. Assume that* $\mathcal{F}$ *is tangent to* ∂M *and* $(T\mathcal{F})^{\perp}$ *is completely integrable. Then the distribution* $(T\mathcal{F})^{\perp}$ *is an Ehresmann connection for* $\mathcal{F}$.

PROPOSITION 4.4. *Let M be an orientable 3-manifold, g a time-orientable Lorentzian metric on M, and $\mathcal{F}$ an orientable, codimension-1, totally geodesic foliation of M consisting of timelike leaves. Denote the foliation determined by $(T\mathcal{F})^{\perp}$ by $\mathcal{H}$. Then the following hold.*

(1) *The lightlike vectors on $T\mathcal{F}$ determine two subfoliations $\mathcal{L}_0, \mathcal{L}_1$ of $\mathcal{F}$.*

(2) *The distribution $T\mathcal{H} \oplus T\mathcal{L}_i$ is completely integrable for $i = 0, 1$. Therefore, if $\mathcal{H}_i$ denotes the foliation determined by $T\mathcal{H} \oplus T\mathcal{L}_i$, $i = 0, 1$, then $(\mathcal{F}, \mathcal{H}_0, \mathcal{H}_1)$ is a total foliation of M (We call it the total foliation associated with $\mathcal{F}$).*

In Section 5, we consider a Reeb component of a codimension-1, totally geodesic foliation of a Lorentzian 3-manifold. Without assuming the completeness of metrics, we prove the following

THEOREM 5.1. *Let (M, g) be a Lorentzian 3-manifold and $(D^2 \times S^1, \mathcal{F}_R)$ a Reeb component of a codimension-1, totally geodesic foliation $\mathcal{F}$ of M. Then all inner leaves of $\mathcal{F}_R$ are spacelike, and the boundary leaf $\partial(D^2 \times S^1)$ is lightlike.*

The following corollary is an answer to Question 1.2 when the manifold is of dimension three.

COROLLARY 5.9. *Let (M, g) be a closed Lorentzian 3-manifold with finite fundamental group and $\mathcal{F}$ a codimension-1, totally geodesic foliation of M. Then $\mathcal{F}$ consists of at least two kinds of leaves among spacelike, timelike, and lightlike ones.*

Theorem 3.1 is a part of the author's doctoral thesis ([13]), which is not published elsewhere. Throughout this paper, we assume that manifolds, foliations and metrics under consideration are smooth.

2. Preliminaries. We recall several basic definitions and results about totally geodesic foliations of Lorentzian manifolds. Section 2.1 is devoted to basic definitions. Section 2.2 contains several results in [12] and a corollary. In Section 2.3 we recall the definition of Heegaard splittings of genus one, which will be used in Section 3.

2.1. Definitions.

DEFINITION 2.1. Let M be a smooth orientable manifold. A *Lorentzian metric* g on M is a nondegenerate, symmetric, covariant 2-tensor of signature $(+, \cdots, +, -)$. We call (M, g) a *Lorentzian manifold.*

DEFINITION 2.2. Let g be a Lorentzian metric. A subspace $E \subset T_xM$ is called *spacelike* (resp. *timelike*, *lightlike*) if the signature of the induced metric $g|_E$ is $(+, \cdots, +)$ (resp. $(+, \cdots, +, -)$, $(+, \cdots, +, 0)$). A vector $v \in T_xM$ is called *spacelike* (resp. *timelike*, *lightlike*) if $g(v, v) > 0$ (resp. $g(v, v) < 0$, $g(v, v) = 0$).

For a Lorentzian manifold, it is well-known that there exists the Levi-Civita connection, that is, a connection which is torsion-free and compatible with the metric (see [8]).

DEFINITION 2.3. A Lorentzian metric g on a manifold M is called *time-orientable* if there exists a non-singular, timelike vector field defined on entire M. When we fix a non-singular, timelike vector field T on M, a Lorentzian metric g is called *time-oriented*, and a lightlike or timelike vector $v \in T_xM$ satisfying $g(v, T) < 0$ is called *positive* (in a usual term, *future-directed*).

Positive lightlike or timelike vectors have the following good property (for a proof, see [8]).

PROPOSITION 2.4 ([8]). *Let (M, g) and T be as the above definition. Define*

$$C(T|_x) = \{v \in T_xM \mid v \text{ is timelike and } g(v, T|_x) < 0\},$$
$$\bar{C}(T|_x) = \{v \in T_xM \mid v \text{ is timelike or lightlike, and } g(v, T|_x) < 0\}.$$

Then we have $av + bw \in C(T|_x)$ (resp. $\bar{C}(T|_x)$) for any $v, w \in C(T|_x)$ (resp. $\bar{C}(T|_x)$), $a \geq 0$ and $b > 0$.

DEFINITION 2.5. A Lorentzian metric g is called (*geodesically*) *complete* if an affine parameter of each geodesic can be defined on the entire $\boldsymbol{R}$. Otherwise g is called (*geodesically*) *incomplete*. A Lorentzian metric g is called *lightlike* (*geodesically*) *complete* if an affine parameter of each geodesic with a lightlike initial vector can be defined on the entire $\boldsymbol{R}$.

REMARK 2.6. Even if a manifold is closed, a Lorentzian metric is not always complete.

DEFINITION 2.7. A foliation $\mathcal{F}$ of a Riemannian or Lorentzian manifold (M, g) is called *totally geodesic* if each leaf L of $\mathcal{F}$ is a totally geodesic submanifold, that is, a submanifold with the property that any geodesic with any initial vector in TL is contained in L.

DEFINITION 2.8. A foliation $\mathcal{F}$ of a manifold M is called *Lorentzian geodesible* if there is a Lorentzian metric g on M for which $\mathcal{F}$ is totally geodesic. We call $\mathcal{F}$ *complete Lorentzian geodesible* if we can choose g to be a complete Lorentzian metric. We define a *lightlike complete Lorentzian geodesible* foliation, in a natural fashion.

DEFINITION 2.9. Let L be a submanifold in a Lorentzian manifold (M, g). We call L *spacelike* (resp. *timelike, lightlike*) if the tangent space T_xL of L at x is a spacelike (resp. timelike, lightlike) subspace of T_xM for each $x \in L$.

We can easily prove the following proposition.

PROPOSITION 2.10. *Every leaf L of a totally geodesic foliation of a Lorentzian manifold is a spacelike, timelike, or lightlike submanifold.*

Based on this proposition, we can call a leaf L a spacelike leaf, a timelike leaf, or a lightlike leaf when L is spacelike, timelike, or lightlike, respectively.

We introduce the concept of "an element of isometric holonomy", which was called "an element of holonomy" in [12], and is a generalization of "an element of holonomy" stated in [1].

DEFINITION 2.11. Let $\mathcal{H}$ be a distribution. A piecewise smooth curve $\sigma : [0, t_0] \to M$ is called an *$\mathcal{H}$-curve* if its tangent vectors lie in $\mathcal{H}$. An *element of isometric holonomy* along the $\mathcal{H}$-curve σ is a family of maps $\{\psi_t : V_{\sigma(0)} \to V_{\sigma(t)}\}_{t\in[0,t_0]}$ which satisfies the following:

(1) The set $V_{\sigma(t)}$ is a plaque of the leaf containing the point $\sigma(t)$ for each $t \in [0, t_0]$.

(2) The map ψ_t is an isometry from $(V_{\sigma(0)}, g|_{V_{\sigma(0)}})$ to $(V_{\sigma(t)}, g|_{V_{\sigma(t)}})$ for each $t \in [0, t_0]$.

(3) The curve $\psi_t(x)$ with parameter $t \in [0, t_0]$ is an $\mathcal{H}$-curve for each $x \in V_{\sigma(0)}$ and $\psi_t(\sigma(0)) = \sigma(t)$.

(4) The map ψ_0 is the identity map of $V_{\sigma(0)}$.

DEFINITION 2.12. Let $\mathcal{F}$ be a foliation of M. A distribution $\mathcal{D}$ is called an *Ehresmann connection* for $\mathcal{F}$ if $\mathcal{D}$ satisfies the following:

(1) $TM = T\mathcal{F} \oplus \mathcal{D}$.

(2) For every $\mathcal{F}$-curve $\tau : I \to M$ and every $\mathcal{D}$-curve $\sigma : I \to M$ with the same initial point $\sigma(0) = \tau(0)$, there exists a map $\delta : I \times I \to M$ such that for every fixed s the curve $\delta(\cdot, s)$ is a $\mathcal{D}$-curve and $\delta(\cdot, 0) = \sigma(\cdot)$, and for every fixed t the curve $\delta(t, \cdot)$ is an $\mathcal{F}$-curve and $\delta(0, \cdot) = \tau(\cdot)$.

2.2. Relevant results about totally geodesic foliations. We state several relevant results about totally geodesic foliations. We first recall an equation discriminating whether a foliation is totally geodesic or not.

PROPOSITION 2.13 ([12]). *Let (M, g) be a pseudo-Riemannian manifold and $\mathcal{F}$ a codimension k foliation of M. Then $\mathcal{F}$ is totally geodesic if and only if $(\mathcal{L}_X g)(Y, Z) = 0$ for all $X \in \Gamma((T\mathcal{F})^\perp)$ and $Y, Z \in \Gamma(T\mathcal{F})$, where $(T\mathcal{F})^\perp$ is the distribution consisting of all vectors perpendicular to $T\mathcal{F}$.*

Now we review the concept of the STL-decomposition.

DEFINITION 2.14. Let (M, g) be a Lorentzian manifold and $\mathcal{F}$ a codimension-k, totally geodesic foliation of M. Denote the union of all spacelike leaves, timelike ones, and lightlike ones of $\mathcal{F}$ by $\boldsymbol{S}$, $\boldsymbol{T}$ and $\boldsymbol{L}$, respectively. The decomposition $M = \boldsymbol{S} \sqcup \boldsymbol{T} \sqcup \boldsymbol{L}$ (disjoint union) is called the *STL-decomposition* of M by $\mathcal{F}$.

The STL-decomposition satisfies the following

PROPOSITION 2.15 ([12]). *The sets $\boldsymbol{S}$ and $\boldsymbol{T}$ are open in M, and $\boldsymbol{L}$ is closed in M.*

Totally geodesic foliations of a lightlike complete Lorentzian 2-torus have the following property.

THEOREM 2.16 ([12]). *Let (T^2, g) be a lightlike complete Lorentzian 2-torus. There exists no totally geodesic foliation containing at least two kinds of leaves among spacelike, timelike, and lightlike ones.*

We have an easy corollary of this theorem.

COROLLARY 2.17. *Let (M, g) be a Lorentzian manifold. Assume that there is a timelike totally geodesic submanifold N diffeomorphic to the 2-torus such that there is a totally geodesic foliation of N consisting of at least two kinds of leaves among spacelike, timelike, and lightlike ones. Then g is lightlike incomplete.*

Finally, we review a result about an element of isometric holonomy. Let $\mathcal{F}$ be a totally geodesic foliation of a Lorentzian manifold (M, g) and $\mathcal{H}$ the distribution perpendicular to $T\mathcal{F}$.

PROPOSITION 2.18 ([12]). *If an $\mathcal{H}$-curve $\sigma : [0, t_0] \to M$ intersects only spacelike or timelike leaves, then there exists an element of isometric holonomy along σ.*

2.3. Heegaard splittings of genus one. We recall Heegaard splittings of genus one of closed 3-manifolds. See [6] for further detail about Heegaard splittings.

Let V_1 and V_2 denote two copies of oriented $D^2 \times S^1 \subset \boldsymbol{C} \times \boldsymbol{C}$, where D^2 denotes a 2-disk. Let $f : \partial V_2 \to \partial V_1$ be an orientation reversing diffeomorphism. Consider a topological space $V_1 \cup_f V_2$ and give it a differentiable structure in a certain way. We call the resulting manifold M. The couple (V_1, V_2) is called a *Heegaard splitting of genus one* of M. Define curves l, m by $l : t \mapsto (1, e^{2\pi it}) \in D^2 \times S^1$ and $m : t \mapsto (e^{2\pi it}, 1)$. A simple closed curve in $\partial(D^2 \times S^1)$ is called a *meridian* (resp. *longitude*) if it is homotopic to m (resp. l) on $\partial(D^2 \times S^1)$. The fundamental group $\pi_1(\partial(D^2 \times S^1))$ is isomorphic to $\langle l \mid - \rangle \oplus \langle m \mid - \rangle$, where $\langle l \mid - \rangle$ denotes the free group generated by l.

Let (V_1, V_2) be a Heegaard splitting of M and $f : \partial V_2 \to \partial V_1$ a gluing map defining M. Let l_i and m_i be a longitude and a meridian in V_i, respectively. It is known that if $f_*(m_2) = pl_1 + qm_1 \in \pi_1(\partial V_1)$, then $(p, q) = 1$. When $f_*(m_2) = pl_1 + qm_1$, we denote M by $L(p, q)$. It is known that $L(0, 1) \cong S^2 \times S^1$ and $L(1, 0) \cong S^3$. When $L(p, q)$ is diffeomorphic to neither $S^2 \times S^1$ nor S^3, it is called a *lens space* of type (p, q).

3. Constructions of Lorentzian geodesible foliations. The goal of this section is the following

THEOREM 3.1. *If a closed 3-manifold has a Heegaard splitting of genus one, then it has a codimension-1 Lorentzian geodesible foliation.*

3.1. Examples and propositions. First, we construct an example of a codimension-1, Lorentzian geodesible foliation of $S^{n-1} \times S^1$.

EXAMPLE 3.2. Let $\tilde{M} = \boldsymbol{R}^n \setminus (0, \ldots, 0)$. Define the Lorentzian metric $\tilde{g}$ on $\tilde{M}$ by

$$\tilde{g} = \frac{1}{\sum_{i=1}^{n} x_i^2} \left(\sum_{i=1}^{n-1} dx_i \otimes dx_i - dx_n \otimes dx_n \right),$$

where $(x_1, \ldots, x_n)$ is the canonical coordinates of $\boldsymbol{R}^n$. Define the vector field $\tilde{X}$ by

$$\tilde{X} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i},$$

which is a Killing field on $\tilde{M}$. The distribution $T\tilde{\mathcal{F}}$ defined by $\ker \tilde{g}(\tilde{X}, \cdot)$ is completely integrable, and hence defines a totally geodesic foliation $\tilde{\mathcal{F}}$ by Proposition 2.13. The map $\psi : (x_1, \ldots, x_n) \mapsto (2x_1, \ldots, 2x_n)$ is an isometry preserving $\tilde{\mathcal{F}}$. Thus we have a manifold $M = \tilde{M}/\sim$, a Lorentzian metric g on M, and a codimension-1, totally geodesic foliation $\mathcal{F}$ of M. Clearly, M is diffeomorphic to $S^{n-1} \times S^1$. Hence $\mathcal{F}$ is a Lorentzian geodesible foliation of $S^{n-1} \times S^1$. □

REMARK 3.3. The Lorentzian manifold (M, g) obtained above is lightlike incomplete, because $\{x_1 = \cdots = x_{n-2} = 0\}/\sim$ satisfies the assumption of Corollary 2.17.

Second, we construct a totally geodesic foliation of $D^2 \times S^1$.

EXAMPLE 3.4. Recall Example 3.2. Consider the case when $n = 3$, that is,

$$\left(\boldsymbol{R}^3 \setminus \{0\},\ \tilde{g} = \frac{1}{\sum_{i=1}^{3} x_i^2} \left(\sum_{i=1}^{2} dx_i \otimes dx_i - dx_3 \otimes dx_3 \right) \right).$$

Put

$$M_+ = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 - x_3^2 \leq 0,\ x_3 > 0\}.$$

Define the diffeomorphism $\Phi : M_+ \to D^2(1) \times \boldsymbol{R}$ by

$$(x_1, x_2, x_3) \mapsto \left(\frac{\sqrt{2}x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{\sqrt{2}x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \log\sqrt{x_1^2 + x_2^2 + x_3^2} \right).$$

The Lorentzian metric $(\Phi^{-1})^* \tilde{g}$ on $D^2(1) \times \boldsymbol{R}$ is invariant by the additive $\boldsymbol{R}$-action. Hence we have the Lorentzian metric g_1' on $D^2(1) \times \boldsymbol{R}/2\pi\boldsymbol{Z}$.

Let (x, y, t) denote coordinates of $D^2(1) \times S^1$, where (x, y) and (t) are the canonical coordinates of $\boldsymbol{R}^2$ and $\boldsymbol{R}$, respectively. We then have

$$g_1' = \begin{pmatrix} 1/2 - x^2/(4 - 2x^2 - 2y^2) & -xy/(4 - 2x^2 - 2y^2) & x \\ -xy/(4 - 2x^2 - 2y^2) & 1/2 - y^2/(4 - 2x^2 - 2y^2) & y \\ x & y & x^2 + y^2 - 1 \end{pmatrix},$$

where the right hand side is the matrix of components of g_1' with respect to (x, y, t). The foliation defined by $\ker g_1'(\partial/\partial t, \cdot)$ is totally geodesic by Proposition 2.13. □

Third, we change g_1' with a Lorentzian metric g_1 as follows. We change coordinates from (x, y, t) to (r, θ, t), where $x = r\cos\theta$ and $y = r\sin\theta$. Then we have

$$g_1' = \begin{pmatrix} (r^2-1)/(r^2-2) & 0 & r \\ 0 & r^2/2 & 0 \\ r & 0 & r^2-1 \end{pmatrix}$$

with respect to (r, θ, t). Consider a C^∞ monotone increasing function $a : [0, 1] \to [0, 1]$ satisfying the following conditions:

(1) $a(0) = 0$ and $a(1) = 1$.
(2) There exists an $\varepsilon > 0$ such that $a(r) = r$ for all $r \in [0, \varepsilon)$.
(3) $(d^n a/dr^n)(1) = 0$ for all integer $n > 0$.

We change r with $a(r)$. The resulting metric is just g_1 in the next example.

EXAMPLE 3.5. Consider $D^2(1) \times S^1$, where $D^2(1)$ is the unit 2-disk in $\boldsymbol{R}^2$ and $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$. We denote the coordinates of $D^2(1) \times S^1$ by (x, y, t), where (x, y) and (t) are the canonical coordinates of $\boldsymbol{R}^2$ and $\boldsymbol{R}$, respectively. Define the Lorentzian metric g_1 by

$$g_1 = \begin{pmatrix} \dfrac{G_{11}}{2(a^2-2)(x^2+y^2)^2} & \dfrac{G_{12}}{2(a^2-2)(x^2+y^2)^2} & \dfrac{ax}{\sqrt{x^2+y^2}} \\ \dfrac{G_{12}}{2(a^2-2)(x^2+y^2)^2} & \dfrac{G_{22}}{2(a^2-2)(x^2+y^2)^2} & \dfrac{ay}{\sqrt{x^2+y^2}} \\ \dfrac{ax}{\sqrt{x^2+y^2}} & \dfrac{ay}{\sqrt{x^2+y^2}} & a^2-1 \end{pmatrix},$$

where

$$\begin{aligned} G_{11} &= 2(a^2-1)x^4 + (a^4 + 2(x^2-1)a^2 - 2x^2)y^2\,, \\ G_{12} &= xy(-a^4 + 2(x^2+y^2+1)a^2 - 2(x^2+y^2))\,, \\ G_{22} &= a^2(a^2-2)x^2 + 2(a^2-1)y^2(x^2+y^2)\,. \end{aligned}$$

Here, a is the function $a(\sqrt{x^2+y^2})$ defined above, the right hand side denotes the matrix of components of the metric with respect to coordinates (x, y, t), and we assume that the numerator is divided by the denominator in $0 \le \sqrt{x^2+y^2} < \varepsilon$ in each component of the matrix.

The vector field $\partial/\partial t$ is a Killing field, because all the components of the matrix are independent of t. Since

$$g_1(\partial/\partial t, \cdot) = \frac{ax}{\sqrt{x^2+y^2}}dx + \frac{ay}{\sqrt{x^2+y^2}}dy + (a^2-1)dt\,,$$

the foliation defined by $\ker(g_1(\partial/\partial t, \cdot))$ is totally geodesic by Proposition 2.13. This foliation is a Reeb foliation. Clearly, $\partial(D^2 \times S^1)$ is a lightlike leaf, and the other leaves are spacelike. □

Fourth, we want to glue two copies of Example 3.5 together. Since it is not possible in general, we adopt the following tricks to accomplish it.

PROPOSITION 3.6. *Let (M_i, g_i) be a Lorentzian manifold with boundary, N_i a connected component of ∂M_i, and $h_i : N_i \times [0,1] \to M_i$ a collar neighborhood such that $h_i(N_i \times 0) = N_i$ $(i = 1, 2)$. Let $f : N_1 \to N_2$ be a diffeomorphism. Denote $M_1 \bigcup_f M_2$ by M. Define a differentiable structure on M such that*

$$\begin{array}{ccc} h : N_1 \times [-1,1] & \to & M = M_1 \bigcup_f M_2 \\ (x,t) & \mapsto & \begin{cases} h_1(x,t) & \text{for } t \geq 0 \\ h_2(f(x),-t) & \text{for } t \leq 0 \end{cases} \end{array}$$

is a diffeomorphism into M in a certain way. Assume that for all $x \in N_1$ there exists a local coordinate system $(x_1, \ldots, x_{n-1})$ around x such that the local coordinate system $(x_1, \ldots, x_{n-1}, t)$ in $h(N_1 \times [-1,1]) \subset M$ satisfies the following:

(1) *The (i, j)-component of g_1 equals the (i, j)-component of g_2 on $h(N_1 \times 0)$ for all i and j.*

(2) *There exists an $\varepsilon_1 > 0$ such that all the components of g_1 are functions depend on only t on $h(N_1 \times [0, \varepsilon_1))$, and their differentials of order ≥ 1 vanish on $h(N_1 \times 0)$.*

(3) *There exists an $\varepsilon_2 > 0$ such that all the components of g_2 are functions depending only on t on $h(N_1 \times (-\varepsilon_2, 0])$, and their differentials of order ≥ 1 vanish on $h(N_1 \times 0)$.*

Then

$$g = \begin{cases} g_1 \text{ on } M_1 \subset M \\ g_2 \text{ on } M_2 \subset M \end{cases}$$

is a C^∞ Lorentzian metric on M.

PROPOSITION 3.7. *Let V_1 and V_2 be oriented manifolds with compact connected boundary. Let $f : \partial V_2 \to \partial V_1$ be an orientation reversing diffeomorphism. Put $M = V_1 \bigcup_f V_2$. Consider $\partial V_1 \times [0,1]$, and regard $\mathrm{id}_{\partial V_1}$ and f as*

$$\mathrm{id}_{\partial V_1} : \partial V_1 \to \partial V_1 \times 0 \quad \text{and} \quad f : \partial V_2 \to \partial V_1 \times 1 \,.$$

If we take the orientation of $V_1 \times [0,1]$ such that $f : \partial V_2 \to \partial V_1 \times 1$ is orientation reversing, then the map id $: \partial V_1 \to \partial V_1 \times 0$ *is orientation reversing, and M is diffeomorphic to $V_1 \bigcup_{\mathrm{id}} (\partial V_1 \times [0,1]) \bigcup_f V_2$.*

3.2. The proof of Theorem 3.1. We will construct a totally geodesic foliation of $L(p,q)$. Let V_1 and V_2 be two copies of an oriented $D^2(1) \times S^1$. Consider (V_i, g_1), where g_1 is the Lorentzian metric defined in Example 3.5. We change coordinates from $(x_i, y_i, t_i) \in V_i$ to (r_i, θ_i, t_i), where $x_i = r_i \cos\theta_i$ and $y_i = r_i \sin\theta_i$. We regard the r_i-direction as a "collar direction," that is, the collar neighborhood is defined by

$$h_i : \partial V_i \times [0, \varepsilon] \to V_i \,, \quad (\theta_i, t_i, s_i) \mapsto (1 - s_i, \theta_i, t_i) \,.$$

Consider the gluing map $f : \partial V_2 \cong \boldsymbol{R}^2/2\pi\boldsymbol{Z}^2 \to \partial V_1 \cong \boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$ defined by

$$f : \begin{pmatrix} \theta_2 \\ t_2 \end{pmatrix} \mapsto \begin{pmatrix} q & r' \\ p & s' \end{pmatrix} \begin{pmatrix} \theta_2 \\ t_2 \end{pmatrix} \quad \text{for some } r', s' \in \boldsymbol{Z} \,,$$

where $qs' - pr' = -1$. Note that f determines $L(p, q)$. In general, we cannot glue (V_1, g_1) and (V_2, g_1) by using only Proposition 3.6. So we use Proposition 3.7 to get a Lorentzian metric and a totally geodesic foliation of $L(p, q)$. We will carry out this by several steps.

Step 1. Denote coordinates of $\partial V_1 \times [0, 1]$ by (θ, t, s), where $\theta = \theta_1$ and $t = t_1$. Consider the glued manifold $V_1 \bigcup_{\mathrm{id}} (\partial V_1 \times [0, 1]) \bigcup_f V_2$.

LEMMA 3.8. *We can join the metric restricted on $\partial V_1 \times 0$ to the metric on $\partial V_1 \times 1$ so that the constructed metric g' on $\partial V_1 \times [0, 1]$ satisfies the following conditions*:

(1) *All the components of g' with respect to (θ, t, s) depend only on $s \in [0, 1]$.*

(2) *The manifold $\partial V_1 \times \{s\}$ is lightlike for all $s \in [0, 1]$.*

PROOF. Note that

$$g_1 = \begin{pmatrix} (a^2-1)/(a^2-2) & 0 & a \\ 0 & a^2/2 & 0 \\ a & 0 & a^2-1 \end{pmatrix},$$

where the right hand side is the matrix of components of g_1 with respect to $(r_i, \theta_i, t_i) \in V_i$. Hence we have

$$g_1 = \begin{pmatrix} a^2/2 & 0 & 0 \\ 0 & a^2-1 & -a \\ 0 & -a & (a^2-1)/(a^2-2) \end{pmatrix}$$

with respect to the collar coordinates $(\theta_i, t_i, s_i) \in \partial V_i \times [0, \varepsilon]$. Since

$$g_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

on $\partial V_i \times 0 \subset \partial V_i \times [0, \varepsilon]$, the metric on $\partial V_1 \times 0 \subset \partial V_1 \times [0, 1]$ is represented by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the coordinates $(\theta, t, s) \in \partial V_1 \times [0, 1]$. Note that the inverse map $f^{-1} : \partial V_1 \times 1 \to \partial V_2$ is represented by

$$\begin{pmatrix} -s' & r' \\ p & -q \end{pmatrix}.$$

Hence we have

$$\begin{pmatrix} -s' & p & 0 \\ r' & -q & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -s' & r' & 0 \\ p & -q & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s'^2/2 & -r's'/2 & -p \\ -r's'/2 & r'^2/2 & q \\ -p & q & 0 \end{pmatrix}$$

on $\partial V_1 \times 1 \subset \partial V_1 \times [0, 1]$ with respect to $(\theta, t, s) \in \partial V_1 \times [0, 1]$. (See Figure 1.)

Consider a C^∞ $GL(2, \boldsymbol{R})$-valued function

$$\begin{pmatrix} b_s & c_s \\ d_s & k_s \end{pmatrix}, \quad s \in [0, 1]$$

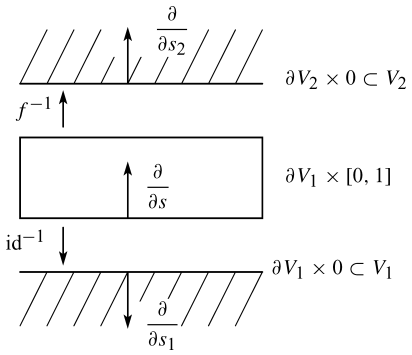

FIGURE 1. f and id.

satisfying

$$\begin{pmatrix} b_0 & c_0 \\ d_0 & k_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} GL_+(2, \boldsymbol{R})$$

and

$$\begin{pmatrix} b_1 & c_1 \\ d_1 & k_1 \end{pmatrix} = \begin{pmatrix} -s' & r' \\ p & -q \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} GL_+(2, \boldsymbol{R}) .$$

Note that such a function exists. Put

$$\begin{aligned} g' &= \begin{pmatrix} b_s & d_s & 0 \\ c_s & k_s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} b_s & c_s & 0 \\ d_s & k_s & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} b_s^2/2 & b_s c_s/2 & -d_s \\ b_s c_s/2 & c_s^2/2 & -k_s \\ -d_s & -k_s & 0 \end{pmatrix} . \end{aligned}$$

Note that signature of this matrix is $(+, +, -)$. Thus we regard g' as the matrix of components of the Lorentzian metric with respect to $(\theta, t, s) \in \partial V_1 \times [0, 1]$.

Now we show that g' satisfies the condition (2). We have

$$g'|_{\partial V_1 \times \{s\}} = \begin{pmatrix} b_s^2/2 & b_s c_s/2 \\ b_s c_s/2 & c_s^2/2 \end{pmatrix}$$

with respect to $(\theta, t) \in \partial V_1 \times \{s\}$. Thus $\partial V_1 \times \{s\}$ is lightlike. This proves Lemma 3.8. □

REMARK 3.9. The foliation $\{\partial V_1 \times \{s\}\}_{s \in [0,1]}$ is a totally geodesic foliation of $(\partial V_1 \times [0, 1], g')$ by (1), (2) and Proposition 2.13.

Step 2. We change the parameter s of each component of g' to $u(s)$, where u is a function satisfying the following:

(1) The function $u : [0, 1] \to [0, 1]$ is a C^∞ monotone increasing function.

(2) $(d^n u/ds^n)(0) = (d^n u/ds^n)(1) = 0$ for all integer $n > 0$.

We denote a new metric by the same symbol g'.

Step 3. Put

$$g = \begin{cases} g_1 & \text{on} \quad V_1\,, \\ g' & \text{on} \quad \partial V_1 \times [0,1]\,, \\ g_1 & \text{on} \quad V_2\,. \end{cases}$$

Note that g is a C^∞ Lorentzian metric on $V_1 \bigcup_{\mathrm{id}} (\partial V_1 \times [0,1]) \bigcup_f V_2$ by Proposition 3.6. We define the desired foliation $\mathcal{F}$ by

$$\mathcal{F} = \begin{cases} \ker(g_1(\partial/\partial t_1, \cdot)) & \text{on} \quad V_1\,, \\ \{\partial V_1 \times \{s\}\}_{s\in[0,1]} & \text{on} \quad \partial V_1 \times [0,1]\,, \\ \ker(g_1(\partial/\partial t_2, \cdot)) & \text{on} \quad V_2\,. \end{cases}$$

Note that $\mathcal{F}$ is a C^∞ totally geodesic foliation. Hence we have a Lorentzian geodesible foliation of $L(p,q)$. This completes the proof.

4. Some properties of totally geodesic foliations. In this section, we study totally geodesic foliations consisting of *spacelike* leaves, and codimension-1, totally geodesic foliations consisting of *timelike* leaves of Lorentzian 3-manifolds. We do not assume the completeness of Lorentzian metrics.

First, we consider totally geodesic foliations consisting of spacelike leaves.

PROPOSITION 4.1. *Let (M, g) be a Lorentzian manifold with or without boundary and $\mathcal{F}$ a codimension-k, totally geodesic foliation consisting of spacelike leaves. Assume that $\mathcal{F}$ is tangent to ∂M and the distribution $(T\mathcal{F})^\perp$ is integrable. Then for an arbitrary metric h on $(T\mathcal{F})^\perp$, the foliation $\mathcal{F}$ is totally geodesic with respect to the Riemannian metric $g|_{T\mathcal{F}} + h$.*

PROOF. By the assumption, we have $TM = T\mathcal{F} \oplus (T\mathcal{F})^\perp$. Put $g_R = g|_{T\mathcal{F}} + h$. Then the distribution $T\mathcal{F}$ is perpendicular to $(T\mathcal{F})^\perp$ with respect to g_R. Let $\mathcal{H}$ be the foliation determined by $(T\mathcal{F})^\perp$. Since $\mathcal{F}$ is totally geodesic, we have

$$(\mathcal{L}_X g)(Y, Z) = 0 \quad \text{for all } \ X \in \Gamma(T\mathcal{H}) \ \text{ and } \ Y, Z \in \Gamma(T\mathcal{F})\,.$$

There exists a product neighborhood $U \times V$ for each $x \in M$, where U is a plaque of $\mathcal{F}$ and V is a plaque of $\mathcal{H}$. Hence there exist linear independent, $\mathcal{F}$-preserving, non-singular local vector fields $X_1, \ldots, X_k \in \Gamma(T\mathcal{H})$. If we describe $X \in \Gamma(T\mathcal{H})$ as $X = \sum a_i X_i$, then we have

$$\begin{aligned} (\mathcal{L}_X g_R)(Y, Z) &= (\mathcal{L}_{(\sum a_i X_i)}\, g_R)(Y, Z) \\ &= \sum a_i \ \{X_i(g_R(Y, Z)) - g_R([X_i, Y], Z) - g_R(Y, [X_i, Z])\} \\ &= \sum a_i \{X_i((g|_{T\mathcal{F}})(Y, Z)) - (g|_{T\mathcal{F}})([X_i, Y], Z) - (g|_{T\mathcal{F}})(Y, [X_i, Z])\} \\ &= \sum a_i \{X_i(g(Y, Z)) - g([X_i, Y], Z) - g(Y, [X_i, Z])\} \\ &= \sum a_i (\mathcal{L}_{X_i} g)(Y, Z) = 0\,. \end{aligned}$$

Therefore $\mathcal{F}$ is totally geodesic with respect to g_R. □

If M is closed, by Proposition 4.1 and [1] we can prove the following

PROPOSITION 4.2. *Let (M, g) be a compact Lorentzian manifold and $\mathcal{F}$ be a totally geodesic foliation consisting of spacelike leaves. Assume that $\mathcal{F}$ is tangent to ∂M and $(T\mathcal{F})^{\perp}$ is completely integrable. Then the distribution $(T\mathcal{F})^{\perp}$ is an Ehresmann connection for $\mathcal{F}$.*

PROOF. Let $\mathcal{H}$ be the foliation determined by $(T\mathcal{F})^{\perp}$. Cover M by a product foliation charts $\{U_i \times V_i\}_{i=1}^m$, where U_i is an $\mathcal{F}$-plaque and V_i is an $\mathcal{H}$-plaque. Let h be a metric on $(T\mathcal{F})^{\perp}$, and define the Riemannian metric g_R by $g|_{T\mathcal{F}} + h$. We have $T\mathcal{F} \perp (T\mathcal{F})^{\perp}$ with respect to g_R. Let d be the distance determined by g_R. Consider a Lebesgue number ρ with respect to $\{U_i \times V_i\}_{i=1}^m$, that is, any set with the diameter $< \rho$ is contained in $U_i \times V_i$ for some i. Hence any $\mathcal{F}$-plaque P with the diameter $< \rho$ with respect to $g_R|_P$ is contained in $U_i \times V_i$ for some i.

Let $\sigma : [0, 1] \to M$ be an arbitrary $\mathcal{H}$-curve and $\tau : [0, 1] \to M$ be an $\mathcal{F}$-curve with the length $< \rho/2$. We will construct a map $\delta : [0, 1] \times [0, 1] \to M$ satisfying Definition 2.12 (2). Decompose $[0, 1] = [t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{l-1}, t_l]$ so that $t_0 = 0$, $t_l = 1$, and $\sigma([t_{i-1}, t_i])$ has the length $< \rho/2$ for all i. The curve $\sigma([t_0, t_1]) \cup \tau([0, 1])$ has the length $< \rho$. Hence there exists an i_0 such that $\sigma([t_0, t_1]) \cup \tau([0, 1]) \subset U_{i_0} \times V_{i_0}$.

LEMMA 4.3. *We can construct an element of isometric holonomy along $\sigma|_{[t_0,t_1]}$*

$$\{\psi_t : V_{\sigma(0)} \to V_{\sigma(t)}\}_{t\in[t_0,t_1]}$$

so that $V_{\sigma(0)} = U_{i_0} \times \sigma(0)$ and $V_{\sigma(t)} = U_{i_0} \times \sigma(t)$.

PROOF. Let $p : U_{i_0} \times V_{i_0} \to V_{i_0}$ be the projection. Define the vector field Y on $p \circ \sigma([t_0, t_1])$ tangent to V_{i_0} by

$$Y_{p\circ\sigma(t)} = \frac{\mathrm{d}}{\mathrm{d}t}(p \circ \sigma)\bigg|_t .$$

Define the vector field X on $U_{i_0} \times \sigma([t_0, t_1])$ tangent to $\mathcal{H}$ by $p_*X = Y$. The vector field X is $\mathcal{F}$-preserving and tangent to $\mathcal{H}$. Hence a local one-parameter group generated by X determines an element of isometric holonomy along $\sigma|_{[t_0,t_1]}$

$$\{\psi_t : U_{i_0} \times \sigma(0) \to U_{i_0} \times \sigma(t)\}_{t\in[t_0,t_1]} . \qquad \square$$

Define the map $\delta_1 : [t_0, t_1] \times [0, 1] \to M$ by

$$\delta_1(t, s) = \psi_t(\tau(s)) .$$

Put $\tau_1(s) = \delta_1(t_1, s)$. By the definition of elements of isometric holonomy, the curve τ_1 has the length $< \rho/2$. Construct $\delta_2 : [t_1, t_2] \times [0, 1] \to M$ by applying the same argument as above to $\sigma([t_1, t_2]) \cup \tau_1([0, 1])$. Repeat this process and define $\delta : [0, 1] \times [0, 1] \to M$ by

$$\delta(t, s) = \delta_i(t, s) \quad \text{when } t \in [t_{i-1}, t_i] .$$

This δ satisfies Definition 2.12 (2). Therefore $(T\mathcal{F})^{\perp}$ is an Ehresmann connection for $\mathcal{F}$. This proves the proposition. $\square$

Now we consider codimension-1, totally geodesic foliations consisting of timelike leaves of Lorentzian 3-manifolds.

PROPOSITION 4.4. *Let M be an orientable 3-manifold, g a time-orientable Lorentzian metric on M, and $\mathcal{F}$ an orientable, codimension-1, totally geodesic foliation consisting of timelike leaves. Denote the foliation determined by $(T\mathcal{F})^{\perp}$ by $\mathcal{H}$. Then the following hold.*

(1) *The lightlike vectors on $T\mathcal{F}$ determine two subfoliations $\mathcal{L}_0, \mathcal{L}_1$ of $\mathcal{F}$.*

(2) *The distribution $T\mathcal{H} \oplus T\mathcal{L}_i$ is completely integrable for $i = 0, 1$. Therefore, if $\mathcal{H}_i$ denotes the foliation determined by $T\mathcal{H} \oplus T\mathcal{L}_i$, $i = 0, 1$, then $(\mathcal{F}, \mathcal{H}_0, \mathcal{H}_1)$ is a total foliation of M (We call it the total foliation associated with $\mathcal{F}$).*

PROOF. Fix orientations of M and $\mathcal{F}$. By the assumption of g, there exists a non-singular, timelike vector field T on M. Since all the leaves of $\mathcal{F}$ are timelike, we have a splitting $TM = T\mathcal{F} \oplus T\mathcal{H}$. Let $\pi : TM \to T\mathcal{F}$ denote the projection defined by the splitting $TM = T\mathcal{F} \oplus T\mathcal{H}$. By straight computation, $\pi(T)$ is non-singular and timelike. Hence for all $L \in \mathcal{F}$ we can regard $(L, g|_L)$ as an oriented, time-oriented, Lorentzian 2-manifold by $\pi(T)|_L$. Fix a leaf L of $\mathcal{F}$ and a point $x \in L$. Define two lightlike subspaces $T_x\mathcal{L}_0, T_x\mathcal{L}_1$ of T_xL by the following: Take linear independent, positive, lightlike vectors V_0, V_1 on T_xL so that $\{V_0, V_1\}$ equals the orientation of T_xL, and put $T_x\mathcal{L}_0 = \mathrm{Span}\{V_0\}$ and $T_x\mathcal{L}_1 = \mathrm{Span}\{V_1\}$. Define distributions $T\mathcal{L}_0, T\mathcal{L}_1$ by

$$T\mathcal{L}_0 = \bigcup_{x \in M} T_x\mathcal{L}_0, \quad T\mathcal{L}_1 = \bigcup_{x \in M} T_x\mathcal{L}_1 ,$$

which are C^∞ subdistributions of $T\mathcal{F}$, proving (1).

We will prove that $T\mathcal{H} \oplus T\mathcal{L}_i$ is completely integrable for $i = 0, 1$. Let $\sigma : [0, 1] \to M$ be an arbitrary $\mathcal{H}$-curve. Let $\{\psi_t : V_{\sigma(0)} \to V_{\sigma(t)}\}_{t \in [0,1]}$ be an element of isometric holonomy along σ. Since ψ_t is an isometry, the map ψ_t preserves $\mathcal{L}_0$ and $\mathcal{L}_1$. Define the vector field H defined on $\bigcup_t V_{\sigma(t)}$ by

$$H_{\psi_t(p)} = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(p) .$$

Let $L_i \in \Gamma(T\mathcal{L}_i|_{\cup_t V_{\sigma(t)}})$ be a frame. We have $[H, L_i] = l_i L_i$ for some function l_i. Therefore $T\mathcal{H} \oplus T\mathcal{L}_i$ is integrable. This proves the proposition. □

5. Reeb components of totally geodesic foliations of Lorentzian 3-manifolds. In this section, we study Reeb components of codimension-1, totally geodesic foliations of Lorentzian 3-manifolds. We do not assume the completeness of metrics under consideration. We have the following

THEOREM 5.1. *Let (M, g) be a Lorentzian 3-manifold and $(D^2 \times S^1, \mathcal{F}_R)$ a Reeb component of a codimension-1, totally geodesic foliation $\mathcal{F}$ of M. Then all inner leaves of $\mathcal{F}_R$ are spacelike, and the boundary leaf $\partial(D^2 \times S^1)$ is lightlike.*

PROOF. By taking a finite covering of M, we can assume that M is orientable, g is time-orientable, and $\mathcal{F}$ is orientable. By the property of the STL-decomposition of M by $\mathcal{F}$ (Proposition 2.15), we have the following three cases:

(i) The set $\boldsymbol{L} \cap (D^2 \times S^1)$ is empty.

(ii) The set $\boldsymbol{L} \cap (D^2 \times S^1)$ contains $\partial(D^2 \times S^1)$ and does not equal $\partial(D^2 \times S^1)$.

(iii) The set $\boldsymbol{L} \cap (D^2 \times S^1)$ equals $\partial(D^2 \times S^1)$.

Consequently, we have the following five cases:

(1) All the leaves of $\mathcal{F}_R$ are spacelike.

(2) All the leaves of $\mathcal{F}_R$ are timelike.

(3) The boundary leaf $\partial(D^2 \times S^1)$ and at least one of the inner leaves are lightlike.

(4) All the inner leaves of $\mathcal{F}_R$ are timelike and the boundary is lightlike.

(5) All the inner leaves of $\mathcal{F}_R$ are spacelike and the boundary is lightlike.

We will prove that the case (1) through the case (4) do not occur in the following propositions. Therefore only the case (5) occurs. □

PROPOSITION 5.2. *The case* (1) *does not occur, that is, there exists no Reeb component* $(D^2 \times S^1, \mathcal{F}_R)$ *of a codimension*-1*, totally geodesic foliation such that all the leaves of* $\mathcal{F}_R$ *are spacelike.*

PROOF. Assume that there is a Reeb component $(D^2 \times S^1, \mathcal{F}_R)$ consisting of spacelike leaves. By Proposition 4.2, the distribution $(T\mathcal{F}_R)^{\perp}$ is an Ehresmann connection for $\mathcal{F}_R$. But an inner leaf never cover the boundary leaf for any normal distribution of $\mathcal{F}_R$, which is contradiction. □

PROPOSITION 5.3. *The case* (2) *does not occur, that is, there exists no Reeb component* $(D^2 \times S^1, \mathcal{F}_R)$ *such that all the leaves of* $\mathcal{F}_R$ *are timelike.*

We need two lemmas to prove Proposition 5.3.

LEMMA 5.4. *Let* (M_i, g_i) *be a pseudo-Riemannian manifold,* $i = 0, 1$*, and* $\varphi : (M_0, g_0) \to (M_1, g_1)$ *be an isometry. Then the following hold*:

(a) $\mathrm{d}\varphi(\nabla_X Y) = \nabla_{\mathrm{d}\varphi(X)}\mathrm{d}\varphi(Y)$ *for all* $X, Y \in \Gamma(TM_0)$*, where* ∇ *in the left* (*resp. right*) *hand side is the Levi-Civita connection of* g_0 (*resp.* g_1).

(b) *If* $\exp_x X$ *is defined, then* $\varphi(\exp_x X) = \exp_{\varphi(x)} \mathrm{d}\varphi(X)$.

PROOF. (a) Define $\hat{\nabla}_X Y = \mathrm{d}\varphi^{-1}(\nabla_{\mathrm{d}\varphi(X)}\mathrm{d}\varphi(Y))$ for $X, Y \in \Gamma(TM_0)$. By a straight computation, we see that $\hat{\nabla}$ is the Levi-Civita connection. Hence $\hat{\nabla} = \nabla$.

(b) We can easily prove this by (a). □

LEMMA 5.5. *The identity component of* $SO(1, 1)$ *is*

$$\left\{ \begin{pmatrix} \sqrt{c^2+1} & c \\ c & \sqrt{c^2+1} \end{pmatrix} \,\middle|\, c \in \boldsymbol{R} \right\},$$

the eigenvalues of $\begin{pmatrix} \sqrt{c^2+1} & c \\ c & \sqrt{c^2+1} \end{pmatrix}$ *are* $\sqrt{c^2+1}+c$ *and* $\sqrt{c^2+1}-c$*, and the eigenvectors are* ${}^t(1, 1)$ *and* ${}^t(1, -1)$ (*these are lightlike*)*. Moreover,*

$$\sqrt{c^2+1}+c > 1 > \sqrt{c^2+1}-c \ (c > 0),$$
$$\sqrt{c^2+1}-c > 1 > \sqrt{c^2+1}+c \ (c < 0).$$

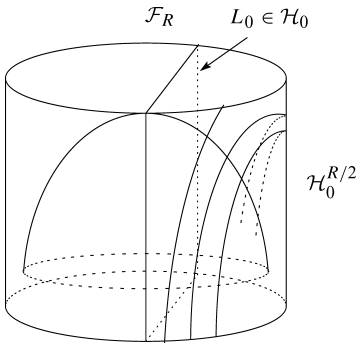

FIGURE 2. A half Reeb component in $\mathcal{H}_0$.

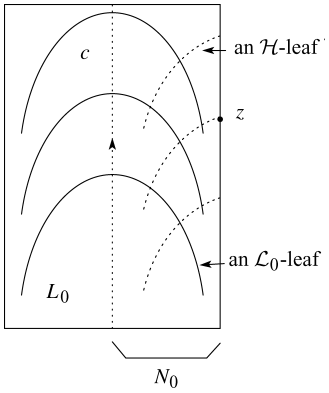

FIGURE 3. $\mathcal{L}_0$ and $\mathcal{H}$ on L_0.

PROOF OF PROPOSITION 5.3. Let $(\mathcal{F}_R, \mathcal{H}_0, \mathcal{H}_1)$ be the total foliation associated with $\mathcal{F}_R$. We use the same notation as in Proposition 4.4. By [11], there exists a half Reeb component $\mathcal{H}_0^{R/2}$ in $\mathcal{H}_0$ and an annular leaf $L_0 \in \mathcal{H}_0^{R/2}$ (see Figure 2). The foliation $\mathcal{F}_R \cap L_0 = \mathcal{L}_0|_{L_0}$ is a Reeb foliation. Since $T\mathcal{H}_0 = T\mathcal{H} \oplus T\mathcal{L}_0$, there is the foliation $\mathcal{H}|_{L_0}$ on L_0. Take an arbitrary point $z \in \partial L_0$. Start from z along $\mathcal{H}$ and consider the limit $\mathcal{H}$-leaf c. The leaf c is closed. Put

$$N_0 = c \bigcup (\text{the connected component of } L_0 \setminus c \text{ containing } z)\,.$$

All the $\mathcal{H}$-leaves in $N_0 \setminus c$ approach c (see Figure 3). Fix an arbitrary point $x \in c$. By parametrizing c, we define an $\mathcal{H}$-curve

$$\sigma : [0,1] \to D^2 \times S^1, \quad \sigma(0) = \sigma(1) = x\,,$$

assuming that σ proceeds in the direction where all the $\mathcal{H}$-leaves in $N_0 \setminus c$ leave from c. Let $L_1 \in \mathcal{H}_1$ be the leaf passing through c. Consider a tubular neighborhood T of c in L_1, and let N_1 be the subset of T which intersects $\mathcal{H}_0^{R/2}$. All the $\mathcal{H}$-leaves near c in N_1 leave from c along σ (see Figure 4).

Consider an element of isometric holonomy along σ

$$\{\psi_t : V_{\sigma(0)} \to V_{\sigma(t)}\}_{t \in [0,1]}\,,$$

where $V_{\sigma(t)}$ is an $\mathcal{F}_R$-plaque. Let i denote 0 or 1. Note that the set $N_i \cap V_{\sigma(0)}$ is a subset of an $\mathcal{L}_i$-leaf. Fix a point $y^{(i)} \in N_i \cap V_{\sigma(0)} \setminus \{x\}$. Let $\alpha^{(i)}$ be the geodesic satisfying $\alpha^{(i)}(0) = x$ and $\alpha^{(i)}(1) = y^{(i)}$. The vector $\dot{\alpha}^{(i)}(0)$ is lightlike and $y^{(i)} = \exp_x \dot{\alpha}^{(i)}(0)$. The map ψ_1 is an isometry from $(V_{\sigma(0)}, g|_{V_{\sigma(0)}})$ to $(V_{\sigma(1)}, g|_{V_{\sigma(1)}})$. By Lemma 5.4 and $\psi_1(x) = x$, we have

$$\psi_1(y^{(i)}) = \exp_x \mathrm{d}\psi_1(\dot{\alpha}^{(i)}(0))\,.$$

All the $\mathcal{H}$-leaves near c in N_i leave from c along σ. Hence $\psi_1(y^{(i)})$ is farther than $y^{(i)}$ from x (see Figure 5).

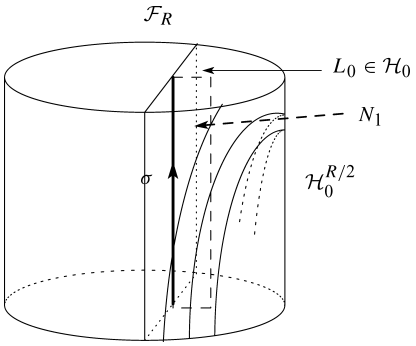

FIGURE 4. The set N_1.

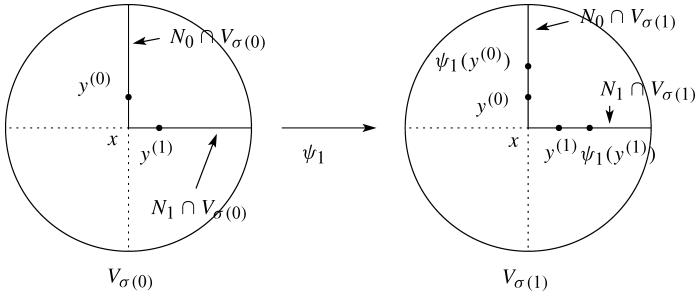

FIGURE 5. The map ψ_1.

Thus there exists a constant $l_i > 1$ such that

$$\mathrm{d}\psi_1(\dot{\alpha}^{(i)}(0)) = l_i\, \dot{\alpha}^{(i)}(0)\,,$$

which is a contradiction by Lemma 5.5. Therefore there exists no Reeb component consisting of timelike leaves. □

PROPOSITION 5.6. *The case* (3) *does not occur, that is, there exists no Reeb component* $(D^2 \times S^1, \mathcal{F}_R)$ *such that the boundary leaf* $\partial(D^2 \times S^1)$ *and at least one of the inner leaves are lightlike.*

PROOF. Assume that there is a Reeb component $(D^2 \times S^1, \mathcal{F}_R)$ such that the boundary leaf $\partial(D^2 \times S^1)$ and one of the inner leaves of $\mathcal{F}_R$ are lightlike. Put $F_1 = \partial(D^2 \times S^1)$. Let F_2 be a lightlike leaf of $\mathcal{F}_R|_{\mathrm{Int}(D^2 \times S^1)}$. Let $\mathcal{N}_i$ be the foliation of F_i determined by the lightlike vectors for $i = 1, 2$. By [14], the one-dimensional foliation determined by the lightlike vectors of a lightlike leaf F is a Riemannian foliation of F. Since the foliation $\mathcal{N}_1$ is a Riemannian foliation of $F_1 \cong T^2$, we have the following two cases:

(c) All the leaves of $\mathcal{N}_1$ are closed.

(d) All the leaves of $\mathcal{N}_1$ are dense.

Let l and m be a longitude and a meridian on F_1, respectively.

Case 1. *There exists a C^∞ closed curve $c : S^1 \to F_1$ which is transverse to $\mathcal{N}_1$ and homotopic to m.* (*This case occurs except the case* (c) *where the homology class of a leaf $L \in \mathcal{N}_1$ is equal to* $[m]$.)

We extend c to a C^∞ embedding $\hat{c} : S^1 \times [0, \varepsilon) \to D^2 \times S^1$ which satisfies the following:

(1) The set $\hat{c}(S^1 \times \{t\})$ is on an $\mathcal{F}_R$-leaf for all $t \in [0, \varepsilon)$.

(2) $\hat{c}(s, 0) = c(s)$ for all $s \in S^1$.

Since the curve c is spacelike, there exists an $\varepsilon_1 \in (0, \varepsilon]$ such that the curve $\hat{c}(\cdot, t_0)$ is spacelike for all $t_0 \in [0, \varepsilon_1)$. Since the leaf F_2 approaches F_1, there exists a $t_1 \in (0, \varepsilon_1)$ such that $\hat{c}(S^1 \times \{t_1\}) \cap F_2 \neq \emptyset$. Hence the curve $\hat{c}(\cdot, t_1)$ on F_2 is transverse to the foliation $\mathcal{N}_2$. One of the connected components of $F_2 \setminus \hat{c}(S^1 \times \{t_1\})$ is a 2-disk, which is a contradiction by the standard Euler class argument.

Case 2. *The case* (c) *where the homology class of a leaf $L \in \mathcal{N}_1$ is equal to* $[m]$.

Fix a Riemannian metric h on $D^2 \times S^1$. Let $\mathcal{N}_i^\perp$ be the foliation of F_i perpendicular to $\mathcal{N}_i$ with respect to h for $i = 1, 2$. Fix a leaf $L \in \mathcal{N}_1$ and parametrize L by a diffeomorphism $c : S^1 \to L \subset F_1$. The curve c is transverse to $\mathcal{N}_1^\perp$. We extend c to a C^∞ embedding $\hat{c} : S^1 \times [0, \varepsilon) \to D^2 \times S^1$ which satisfies the following:

(1) The set $\hat{c}(S^1 \times \{t\})$ is on an $\mathcal{F}_R$-leaf for all $t \in [0, \varepsilon)$.

(2) $\hat{c}(s, 0) = c(s)$ for all $s \in S^1$.

Let $X : \hat{c}(S^1 \times [0, \varepsilon)) \to T\mathcal{F}$ be a C^∞ non-singular vector field satisfying

$$X|_{\hat{c}(s,t)} \perp \hat{c}_* \left(\frac{\partial}{\partial s} \right) \quad \text{for all } (s, t) \in S^1 \times [0, \varepsilon)$$

with respect to h. The vector field $X|_{c(S^1)}$ is spacelike and tangent to $\mathcal{N}_1^\perp$. So there exists an $\varepsilon_1 \in (0, \varepsilon]$ such that the vector field $X|_{\hat{c}(S^1 \times [0, \varepsilon_1))}$ is spacelike. Since the leaf F_2 approaches F_1, there exists a $t_1 \in (0, \varepsilon_1)$ such that $\hat{c}(S^1 \times \{t_1\}) \subset F_2$. Since the vector field $X|_{\hat{c}(S^1 \times \{t_1\})}$ is spacelike, it is transverse to $\mathcal{N}_2$. Hence the orthogonal complement $(\mathrm{Span}\{X\})^\perp|_{\hat{c}(S^1 \times \{t_1\})}$ in TF_2 with respect to h is transverse to $\mathcal{N}_2^\perp$. Therefore the curve $\hat{c}(\cdot, t_1)$ is transverse to $\mathcal{N}_2^\perp$, which is a contradiction. □

PROPOSITION 5.7. *The case* (4) *does not occur, that is, there exists no Reeb component $(D^2 \times S^1, \mathcal{F}_R)$ such that all the inner leaves of $\mathcal{F}_R$ are timelike and the boundary is lightlike.*

PROOF. By the assumption of time-orientability of g, we fix a non-singular, timelike vector field $T \in \Gamma(T(D^2 \times S^1))$. There exist two subfoliations $\mathcal{L}_0, \mathcal{L}_1$ of $\mathcal{F}_R|_{\mathrm{Int}(D^2 \times S^1)}$ by Proposition 4.4. There exists the foliation $\mathcal{N}$ determined by the lightlike vectors on $\partial(D^2 \times S^1) \in \mathcal{F}_R$. Three foliations $\mathcal{L}_0, \mathcal{L}_1, \mathcal{N}$ are orientable.

LEMMA 5.8. *For all point $p \in \partial(D^2 \times S^1)$, there exists a neighborhood V_p around p in $D^2 \times S^1$ and a non-singular, $\mathcal{F}_R$-tangent vector field $X^{(p)}$ defined on V_p which satisfies the following*:

(1) $X^{(p)}|_{V_p \cap \partial(D^2 \times S^1)}$ *is positive and lightlike.*

(2) $X^{(p)}|_{V_p \cap \mathrm{Int}(D^2 \times S^1)}$ *is positive and timelike.*

PROOF. Fix an arbitrary point p in $\partial(D^2 \times S^1)$. Let $\varphi : U_p \to \boldsymbol{R}^2 \times [0, \infty)$ be a foliation chart around p, where we assume that $\varphi(\partial(D^2 \times S^1) \cap U_p) \subset \boldsymbol{R}^2 \times \{0\}$ and that $\varphi^{-1}(\boldsymbol{R}^2 \times \{*\})$ is an $\mathcal{F}_R$-plaque. We regard the standard coordinate $(x_1, x_2, x_3) \in \boldsymbol{R}^2 \times \boldsymbol{R}$ as a local coordinate of $\boldsymbol{R}^2 \times [0, \infty)$. Let (g_{ij}) be the matrix of components of g with respect to (x_1, x_2, x_3). Fix a positive lightlike vector v_p at p. We can assume that v_p equals $\partial/\partial x_1 + b(p)\partial/\partial x_2$, where $b(p) \in \boldsymbol{R}$ is a constant. We then have

$$\begin{aligned} 0 &= g\left(\frac{\partial}{\partial x_1} + b(p)\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_1} + b(p)\frac{\partial}{\partial x_2}\right) \\ &= g_{11}(p) + 2b(p)g_{12}(p) + (b(p))^2 g_{22}(p)\,. \end{aligned}$$

Consider the equation

$$g_{11} + 2\,b\,g_{12} + b^2 g_{22} = 0\,. \tag{1}$$

Since all the leaves of $\mathcal{F}_R$ are timelike or lightlike, we have

$$g_{11}\,g_{22} - g_{12}^2 \leq 0\,.$$

Now we will prove that there exists a neighborhood $V_p \subset U_p$ around p such that the function g_{22} is positive on V_p. Since $\partial/\partial x_1 + b(p)\partial/\partial x_2$ is lightlike and $g_{11}(p)\,g_{22}(p) - (g_{12}(p))^2 = 0$, we have $g_{22}(p) \neq 0$. Since $g|_{T_p(\partial(D^2 \times S^1))}$ is positive indefinite, we have

$$g_p\left(\frac{\partial}{\partial x_2}\,,\ \frac{\partial}{\partial x_2}\right) = g_{22}(p) \geq 0\,.$$

Thus $g_{22}(p) > 0$. Therefore there exists a neighborhood $V_p \subset U_p$ around p such that $g_{22}|_{V_p} > 0$.

Since $g_{22} \neq 0$ on V_p, we can solve the equation (1) as

$$b = (-g_{12} \pm \sqrt{{g_{12}}^2 - g_{11}\,g_{22}})/g_{22}\,.$$

Define the vector field $X^{(p)}$ defined on V_p by

$$X^{(p)} = \frac{\partial}{\partial x_1} + \frac{-g_{12}}{g_{22}}\frac{\partial}{\partial x_2}\,.$$

This is an $\mathcal{F}_R$-tangent, non-singular vector field. We then have

$$g(X^{(p)}, X^{(p)}) = (g_{11}\,g_{22} - {g_{12}}^2)/g_{22}\,.$$

Hence $X^{(p)}|_{V_p \cap \partial(D^2 \times S^1)}$ is positive and lightlike, and $X^{(p)}|_{V_p \cap \mathrm{Int}(D^2 \times S^1)}$ is positive and timelike. □

Fix a collar neighborhood $C : \partial(D^2 \times S^1) \times [0, 1] \to D^2 \times S^1$, where $C(\partial(D^2 \times S^1) \times \{0\}) = \partial(D^2 \times S^1)$. For all $p \in \partial(D^2 \times S^1)$, define an open subset $V'_p \subset \partial(D^2 \times S^1)$ and a constant $\varepsilon_p > 0$ such that

$$p \in V'_p \subset \partial(D^2 \times S^1) \cap V_p\,, \quad C(V'_p \times [0, \varepsilon_p)) \subset V_p\,.$$

Put $\hat{V}_p = C(V'_p \times [0, \varepsilon_p))$. We have $\bigcup_{p\in\partial(D^2\times S^1)} V'_p \supset \partial(D^2 \times S^1)$. Hence there exist points $p_1, \ldots, p_k \in \partial(D^2 \times S^1)$ such that $\bigcup_{i=1}^k V'_{p_i} = \partial(D^2 \times S^1)$. Let $\{\rho_i\}_{i=1}^k$ be a partition of unity subordinate to the covering $\{V'_{p_i}\}_{i=1}^k$. We extend ρ_i to the function $\hat{\rho}_i$ defined on $\hat{V}_{p_i} = C(V'_{p_i} \times [0, \varepsilon_{p_i}))$ by

$$C(V'_{p_i} \times [0, \varepsilon_{p_i})) \xrightarrow{C^{-1}} V'_{p_i} \times [0, \varepsilon_{p_i}) \xrightarrow{\text{prj.}} V'_{p_i} \xrightarrow{\rho_i} [0, 1]\,.$$

Put $\varepsilon = \min_{i\in\{1,\ldots,k\}} \varepsilon_{p_i}$. Then we have $\varepsilon > 0$ and

$$C(\partial(D^2 \times S^1) \times [0, \varepsilon)) \subset \bigcup_{i=1}^k \hat{V}_{p_i}\,.$$

Put $O = C(\partial(D^2 \times S^1) \times [0, \varepsilon))$ and

$$X = \sum_{i=1}^k \hat{\rho}_i (X^{(p_i)}|_{\hat{V}_{p_i}\cap O})\,.$$

By Proposition 2.4, the vector field X is a non-singular, $\mathcal{F}_R$-tangent vector field on O which satisfies the following:

(1) $X|_{\partial(D^2\times S^1)}$ is positive and lightlike.

(2) $X|_{\mathrm{Int}(D^2\times S^1)\cap O}$ is positive and timelike.

Now, let N be a non-singular, $\mathcal{L}_0$-tangent, positive, lightlike vector field on $\mathrm{Int}(D^2 \times S^1)$. Put

$$U_1 = C(\partial(D^2 \times S^1) \times [0, \varepsilon))\,,$$
$$U_2 = D^2 \times S^1 \setminus C(\partial(D^2 \times S^1) \times [0, \varepsilon/2])\,.$$

The family $\{U_1, U_2\}$ is an open covering of $D^2 \times S^1$. Let $\{\lambda_1, \lambda_2\}$ be a partition of unity subordinate to $\{U_1, U_2\}$. Put $\hat{X} = \lambda_1 X + \lambda_2 N$. Then the vector field $\hat{X}$ is a non-singular, $\mathcal{F}_R$-tangent vector field on $D^2 \times S^1$ which satisfies the following:

(1) $\hat{X}|_{\partial(D^2\times S^1)}$ is positive and lightlike.

(2) $\hat{X}|_{\mathrm{Int}(D^2\times S^1)}$ is positive.

Let $\hat{\mathcal{N}}$ be the subfoliation of $\mathcal{F}_R$ determined by $\hat{X}$. So we have $\hat{\mathcal{N}}|_{\partial(D^2\times S^1)} = \mathcal{N}$. By [11], the restriction of any subfoliation of $\mathcal{F}_R$ to $\partial(D^2 \times S^1)$ has Reeb components. So the foliation $\hat{\mathcal{N}}|_{\partial(D^2\times S^1)}$ has Reeb components. Hence $\mathcal{N}$ has Reeb components. However, $\mathcal{N}$ must be a Riemannian foliation on $\partial(D^2 \times S^1)$ by [14], which is a contradiction. Therefore there exists no Reeb component $(D^2 \times S^1, \mathcal{F}_R)$ such that all the inner leaves of $\mathcal{F}_R$ are timelike and the boundary is lightlike. □

We have an easy corollary of Theorem 5.1.

COROLLARY 5.9. *Let (M, g) be a closed Lorentzian 3-manifold with finite fundamental group and $\mathcal{F}$ a codimension-1, totally geodesic foliation of M. Then $\mathcal{F}$ consists of at least two kinds of leaves among spacelike, timelike, and lightlike ones.*

PROOF. By [7], the foliation $\mathcal{F}$ has a Reeb component. Therefore $\mathcal{F}$ consists of at least two kinds of leaves. □

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