HAMILTONIAN STABILITY OF CERTAIN MINIMAL LAGRANGIAN SUBMANIFOLDS IN COMPLEX PROJECTIVE SPACES

Dedicated to Professor Koichi Ogiue on his sixtieth birthday

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Abstract. A compact minimal Lagrangian submanifold immersed in a Kähler manifold is called *Hamiltonian stable* if the second variation of its volume is nonnegative under all Hamilitonian deformations. We study compact Hamiltonian stable minimal Lagrangian submanifolds with parallel second fundamental form embedded in complex projective spaces. Moreover, we completely determine Hamiltonian stability of all real forms in compact irreducible Hermitian symmetric spaces, which were classified previously by M. Takeuchi.

Introduction. Let (M, ω) be a $2n$-dimensional symplectic manifold with a symplectic form ω and L be an n-dimensional smooth manifold. An immersion $\varphi : L \to M$ is called a *Lagrangian immersion* if the 2-form $\varphi^*\omega$ on L pulled back by the immersion $\varphi : L \to M$ vanishes identically. Then L is called a *Lagrangian submanifold* immersed in a symplectic manifold M.

We say that a compact Lagrangian submanifold immersed in a Kähler manifold M is an *H-minimal* Lagrangian submanifold if it has extremal volume under all Hamilitonian variations of the Lagrangian immersion. If a compact Lagrangian submanifold L immersed in a Kähler manifold M is *minimal*, in the sense that it has extremal volume under all smooth variations of the immersion, then it is always H-minimal. A compact H-minimal Lagrangian submanifold in a Kähler manifold M is called *Hamiltonian stable* if the second variation for the volume is nonnegative for all Hamiltonian deformations of the Lagrangian immersion. Any compact *stable*, in the sense that the second variations are nonnegative under all smooth variations of the immersion, minimal submanifold L immersed in a Kähler manifold M is always H-stable.

In [18], [19], [20], [21], Oh developed the fundamental theory for Hamiltonian stability of H-minimal Lagrangian submanifolds of Kähler manifolds. It is known that every compact minimal Lagrangian submanifold L in an Einstein-Kähler manifold M with nonpositive Ricci curvature is stable ([5], [18]).

It is an interesting problem to classify all compact Hamiltonian stable minimal Lagrangian submanifolds L in an Einstein-Kähler manifold M with positive Ricci curvature. In this paper we study this problem in the case when M is either a complex projective space or a compact Hermitian symmetric space. Compact stable minimal submanifolds in compact rank one symmetric spaces have already been classified by Lawson-Simons in [9] and the second named author in [22].

Let $CP^n = CP^n(c)$ be an n-dimensional complex projective space equipped with the Fubini-Study metric of constant holomorphic sectional curvature c. Let $S^m(c)$ denote an m-dimensional standard sphere of constant sectional curvature c or radius $1/\sqrt{c}$. The real projective subspace $\boldsymbol{R}P^n \subset CP^n$ is the first example of a compact minimal Lagrangian submanifold embedded in CP^n. Consider an $(n+1)$-dimensional torus $T^{n+1} = S^1((n+1)c/4) \times \cdots \times S^1((n+1)c/4) \subset S^{2n+1}(c/4)$ naturally embedded in C^{n+1}. By the Hopf fibration $\pi : S^{2n+1}(c/4) \to CP^n$, we have the so-called Clifford torus $T^n = \pi(T^{n+1}) \subset CP^n$ of CP^n, which is the second example of a compact minimal Lagrangian submanifold embedded in CP^n. Oh showed that the real projective subspaces $\boldsymbol{R}P^n$ and the Clifford tori T^n are Hamiltonian stable minimal Lagrangian submanifolds embedded in CP^n ([18]). Then the following is a natural and interesting problem (cf. [4]).

PROBLEM.   Determine all compact Hamiltonian stable minimal Lagrangian submanifolds in CP^n.

The purpose of this paper is to study this problem for a class of compact Hamiltonian stable minimal Lagrangian submanifolds embedded in CP^n, including real projective subspaces $\boldsymbol{R}P^n$ and the Clifford tori T^n. Our main result is

THEOREM.   *Let L be an n-dimensional compact totally real minimal submanifold embedded in CP^n with parallel second fundamental form in the following list*:

(1) $SU(p)/\boldsymbol{Z}_p, n = p^2 - 1$.
(2) $SU(p)/SO(p)\boldsymbol{Z}_p, n = (p-1)(p+2)/2$.
(3) $SU(2p)/Sp(p)\boldsymbol{Z}_{2p}, n = (p-1)(2p+1)$.
(4) $E_6/F_4\boldsymbol{Z}_3, n = 26$.

Then L is a Hamiltonian stable minimal Lagrangian submanifold in CP^n.

In Section 1, we recall fundamental results on Hamiltonian stability of minimal Lagrangian submanifolds in Kähler manifolds. Section 2 is devoted to reviewing the classification theory of totally real submanifolds in CP^n with parallel second fundamental form. The proof of our main theorem is given in Sections 3 and 4. Finally, we study Hamiltonian stability of totally geodesic Lagrangian submanifolds in compact Hermitian symmetric spaces of rank greater than 1 in Section 5.

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1.   Hamiltonian stability of minimal Lagrangian submanifolds in Kähler manifolds.   Let M be a $2n$-dimensional symplectic manifold with symplectic form ω, and let

L be an n-dimensional smooth submanifold of M. Let $\varphi : L \to M$ be a Lagrangian immersion. Set $NL := \varphi^{-1}TM/\varphi_*TL$, the quotient vector bundle of $\varphi^{-1}TM$ by the subbundle φ_*TL. For each $v \in (\varphi^{-1}TM)_x$, define $\alpha_v \in T_x^*L$ by $\alpha_v(X) := \omega_{\varphi(x)}(v, X)$ for $X \in T_xL$. By the nondegeneracy of ω it induces a linear isomorphism $\varpi : N_xL \to T_x^*L$ by $v \mapsto \alpha_v$ and thus a bundle isomorphism $NL \to T^*L$.

By the definition a *Lagrangian deformation* of φ is a smooth family $\{\varphi_t \mid |t| < \varepsilon\}$ of Lagrangian immersions of L into M with $\varphi_0 = \varphi$. Such a deformation is characterized by a property that $\alpha_{V_t} \in \Omega^1(L)$ is closed for each t, where $V_t = \partial\varphi_t/\partial t$ is an infinitesimal deformation of $\{\varphi_t\}$ at a time t. An infinitesimal deformation $V \in C^\infty(\varphi^{-1}TM)$ is called *Lagrangian* if $\alpha_V \in \Omega^1(L)$ is closed and *Hamiltonian* if $\alpha_V \in \Omega^1(L)$ is exact. A smooth family $\{\varphi_t\}$ of Lagrangian immersions of L into M is called a *Hamiltonian deformation* of $\varphi = \varphi_0$ if its infinitesimal deformation $\partial\varphi_t/\partial t$ is Hamiltonian for each t .

Assume that M is an n-dimensional Kähler manifold with complex structure J and Kähler metric g. The Kähler form ω of M, which is defined by $\omega(X, Y) := g(JX, Y)$, defines a symplectic structure of M. If an immersion $\varphi : L \to M$ satisfies $J_x(\varphi_*T_xL) \subset T_x^\perp L$ for each $x \in L$, then L is called a *totally real* submanifold immersed in a Kähler manifold M ([6]). Here for $x \in L$ we define an orthogonal decomposition $T_{\varphi(x)}M = \varphi_*T_xL \oplus T_x^\perp L$ along φ with respect to the metric g. We can identify the bundle NL with the bundle $T^\perp L$. Then the complex structure J induces a bundle isomorphism $NL \to \varphi_*TL$ preserving metrics and connections. Since we have $\alpha_v(X) = g_{\varphi(x)}(Jv, \varphi_*X)$ for $X \in T_xL$, α_v corresponds to Jv through the linear isomorphism $T_x^*L \cong T_xL \cong \varphi_*T_xL$ with respect to g. Then we have a linear isomorphism

(1.1) $$\varpi : C^\infty(T^\perp L) \ni V \mapsto \alpha_V \in \Omega^1(L) .$$

DEFINITION 1.1. A compact Lagrangian submanifold L immersed in a Kähler manifold M is called *Hamiltonian minimal* or H-*minimal* if the first variation for the volume of L vanishes under all Hamiltonian deformations of L in M.

Let H denote the mean curvature vector field of a Lagrangian immersion $\varphi : L \to M$. If M is an Einstein-Kähler manifold, then it satisfies $d\alpha_H = 0$, that is, α_H is a closed 1-form on L ([7]). In [20] it was shown that a Lagrangian submanifold L immersed in a Kähler manifold M is H-minimal if and only if $\delta\alpha_H = 0$, where δ is the codifferential operator of d with respect to the induced metric on L. If a Lagrangian immersion $\varphi : L \to M$ has parallel mean curvature vector field H with respect to the normal connection, then it is H-minimal. Note that if L is a compact H-minimal Lagrangian submanifold in an Einstein-Kähler manifold M with $H^1(L, \boldsymbol{R}) = \{0\}$ or more strongly with positive Ricci curvature, then the mean curvature vector field H of L vanishes, that is, L is minimal.

DEFINITION 1.2. A compact H-minimal Lagrangian submanifold M immersed in a Kähler manifold is called *Hamiltonian stable* if the second variation for the volume of M is nonnegative under all Hamiltonian deformations of M.

Suppose that L is a compact minimal Lagrangian submanifold immersed in an Einstein-Kähler manifold M with Einstein constant κ. Under the correspondence between $C^\infty(NL)$ and $\Omega^1(L) = d(\Omega^0(L)) \oplus \mathrm{Ker}(d^*|\Omega^1(L))$, the Jacobi operator $\mathcal{J}$ of L as a minimal submanifold corresponds to the linear operator $\tilde{\mathcal{J}} = \Delta^1 - \kappa\,\mathrm{Id}$, where $\Delta^1 = d\delta + \delta d$ denotes the Laplacian of L acting on 1-forms and Id denotes the identity operator. The second variation of the volume for a compact minimal Lagrangian submanifold under Hamiltonian deformations is described by the restriction of $\tilde{\mathcal{J}}$ to $d(\Omega^0(L))$. The null space of $\mathcal{J}$ on Hamiltonian deformations corresponds to the null space of $\tilde{\mathcal{J}}$ on $d(\Omega^0(L))$, and it is linearly isomorphic to the eigenspace of the Laplacian on $C^\infty(L)$ with eigenvalue κ. We denote by $n(L)$ the nullity of L, that is, the dimension of the null space.

Let $\mathcal{K}$ denote the space of Killing vector fields on a compact Einstein-Kähler manifold M with positive Einstein constant κ. Assume that $\mathcal{K} \neq \{0\}$, or equivalently by a theorem of Lichnerowicz [10] and Matsushima [12], the first eigenvalue of the Laplacian acting on $C^\infty(M)$ is equal to 2κ. We denote by $\mathrm{V}_1(M)$ the corresponding eigenspace. By a theorem of Matsushima [12], we know that

$$\mathcal{K} = \{J\,\mathrm{grad}\,f \in C^\infty(TM) \mid f \in \mathrm{V}_1(M)\}\,.$$

For each $W \in \mathcal{K}$, we have an orthogonal decomposition $W = W^T + W^\perp$, where W^T and $W^\perp$ denote the tangential and the normal components of the restriction of W to L in M. Set

$$\mathcal{K}^\perp = \{W^\perp \in C^\infty(NL) \mid W \in \mathcal{K}\}\,.$$

Then we have a linear isomorphism

$$\mathcal{K}^\perp \cong \mathcal{K}/\{W \in \mathcal{K} \mid W^\perp = 0\}\,.$$

If $W = -J\,\mathrm{grad}\,f \in \mathcal{K}$ for the first eigenfunction f of the Laplacian acting on $C^\infty(M)$, then it is easy to check that the formula

$$d(f|_L) = \alpha_{W^\perp}$$

holds on L, which means that each $W^\perp \in \mathcal{K}^\perp$ is an infinitesimal Hamiltonian deformation. Hence, for a suitable constant α, $f|_L + \alpha$ is an eigenfunction of the Laplacian acting on $C^\infty(L)$ with eigenvalue κ. Set $n_{\mathcal{K}}(L) = \dim \mathcal{K}^\perp$. Since each $W \in \mathcal{K}$ with $W^\perp = 0$ induces a Killing vector field on L, we obtain inequalities

$$n(L) \geq n_{\mathcal{K}}(L) \geq \dim \mathcal{K} - \dim I_0(L)\,,$$

where $I_0(L)$ denotes the identity component of the isometry group of L. In the case when $M = CP^n$, since $\dim \mathcal{K} = \dim SU(n+1) = (n+1)^2 - 1$ and $\dim I_0(L) \leq n(n+1)/2$, compact minimal Lagrangian submanifolds L in CP^n satisfy

(1.2) $n(L) \geq n_{\mathcal{K}}(L) \geq \dim \mathcal{K} - \dim I_0(L) \geq n(n+3)/2\,.$

The Hamiltonian stability problem of compact minimal Lagrangian submanifolds in an Einstein-Kähler manifold is reduced to the first positive eigenvalue problem of the Laplacian acting on functions.

THEOREM 1.1 ([18]). *Let M be an Einstein-Kähler manifold with Einstein constant* κ. *A compact minimal Lagrangian submanifold L in M is Hamiltonian stable if and only if* $\lambda_1 \geq \kappa$, *where* $\lambda_1 > 0$ *is the first eigenvalue of the Laplacian acting on* $C^\infty(L)$.

It is an important property of compact minimal Lagrangian submanifolds in $CP^n(c)$ that if f is the first eigenfunction of the Laplacian on $CP^n(c)$, then the restriction $f|_L$ of f to L is the eigenfunction of the Laplacian on L with eigenvalue $c(n+1)/2$. This was used by Urbano [28] and a similar result was studied by Ono [23] for a class of generalized flag manifolds including Hermitian symmetric spaces.

THEOREM 1.2 ([28], [23]). *Assume that M is a compact Hermitian symmetric space with standard Einstein-Kähler metric and Einstein constant* κ. *Then a compact minimal Lagrangian submanifold L in M is Hamiltonian stable if and only if* $\lambda_1 = \kappa$. *Here* λ_1 *is the first eigenvalue of the Laplacian acting on* $C^\infty(L)$. *In particular, when M is a complex projective space* $CP^n(c)$ *with constant holomorphic sectional curvature c, a compact minimal Lagrangian submanifold L in* $CP^n(c)$ *is Hamiltonian stable if and only if* $\lambda_1 = c(n+1)/2$.

There is some topological restriction for compact minimal Lagrangian submanifolds in CP^n to be Hamiltonian stable.

THEOREM 1.3. *Let L be a compact minimal Lagrangian submanifold immersed in* CP^n. *If L is Hamiltonian stable, then* $H_1(L; \mathbf{Z}) \neq \{0\}$. *In particular, L cannot be simply connected.*

PROOF. By $\varphi : L \to CP^n$ we denote the Lagrangian immersion and by $\pi : S^{2n+1}(c/4) \to CP^n(c) = CP^n$ the Hopf fibration. On the S^1-bundle $\varphi^{-1}S^{2n+1} \to L$ pulled-back by φ from the Hopf fibration, the induced connection is flat, by virtue of the Lagrangian property of φ. Let $\rho : \pi_1(L) \to S^1$ be the holonomy homomorphism of the flat connection. Since $\pi_1(L)/\mathrm{Ker}\rho \cong \mathrm{Im}\rho \subset S^1$ is abelian, the natural homomorphism $H_1(L; \mathbf{Z}) \cong \pi_1(L)/[\pi_1(L), \pi_1(L)] \to \pi_1(L)/\mathrm{Ker}\rho$ is surjective. Assume that $H_1(L; \mathbf{Z}) = \{0\}$. Then ρ is trivial. Hence there is a minimal isometric immersion $\tilde{\varphi} : L \to S^{2n+1}(c/4)$ which is horizontal with respect to $\pi : S^{2n+1}(c/4) \to CP^n$ such that $\varphi = \tilde{\varphi} \circ \pi$. By Takahashi's theorem [24], component functions of $\tilde{\varphi} : L \to S^{2n+1}(c/4)$ are eigenfunctions of the Laplaician of L with eigenvalue $cn/4$. Hence the first eigenvalue λ_1 of the Laplacian satisfies $\lambda_1 \leq cn/4 < c(n+1)/2$. Thus the minimal Lagrangian immersion $\varphi : L \to CP^n$ is not Hamiltonian stable. $\square$

REMARK 1. This result does not hold in the case of compact Hermitian symmetric space of rank greater than 1 (see Section 4).

2. Minimal Lagrangian submanifolds in CP^n with parallel second fundamental form.
Totally real submanifolds in a complex projective space have been much investigated after the publication of the paper [6] (cf. [5] and [17]). Also totally real submanifolds in complex space forms (that is, a complex Euclidean space $\mathbf{C}^n$, a complex projective space CP^n

and a complex hyperbolic space CH^n) with parallel second fundamental form were elaborately classified by Naitoh and Naitoh-Takeuchi in [14], [15], [13], [16]. Since the parallelism of the second fundamental form implies the parallelism of the mean curvature vector field, such submanifolds provide a nice class of H-minimal Lagrangian submanifolds in complex projective spaces.

By virtue of their theory, all totally real submanifolds in CP^n with parallel second fundamental form are constructed in the following way. Let (U, G) be a Hermitian symmetric pair of compact type with the canonical decomposition $\mathfrak{u} = \mathfrak{g} + \mathfrak{p}$. Set $\dim(U/G) = 2(n+1)$. Let $\langle \ , \ \rangle_{\mathfrak{u}}$ denote the $\mathrm{Ad}(U)$-invariant inner product of $\mathfrak{u}$ defined by (-1)-times Killing-Cartan form of $\mathfrak{u}$. Relative to the complex structure the subspace $\mathfrak{p}$ can be identified with a complex Euclidean space C^{n+1}. We take the decomposition of (U, G) into irreducible Hermitian symmetric pairs of compact type:

$$(2.1) \qquad (U, G) = (U_1, G_1) \oplus \cdots \oplus (U_s, G_s).$$

Set $\dim(U_i/G_i) = 2(n_i + 1)$ for $i = 1, \ldots, s$. Let $\mathfrak{u}_i = \mathfrak{g}_i + \mathfrak{p}_i$ be the canonical decomposition of (U_i, G_i) for each $i = 1, 2, \ldots, s$. Assume that there is an element $\eta_i \in \mathfrak{p}_i$ satisfying the condition $(\mathrm{ad}\eta_i)^3 + 4(\mathrm{ad}\eta_i) = 0$. Choose positive numbers $c_1 > 0, \ldots, c_s > 0$ with $\sum_{i=1}^{s} 1/c_i = 1/c$. Put $a_i = 1/\sqrt{2c_i(n_i + 1)}$ for each $i = 1, \ldots, s$. Set $\hat{L}_i = \mathrm{Ad}(G_i)(a_i\eta_i) \subset S^{2n_i+1}(c_i/4) \subset \mathfrak{p}_i$. It should be remarked that $\hat{L}_i$ is an irreducible symmetric R-space standardly embedded in a complex Euclidean space $\mathfrak{p}_i$.

Let $\eta = a_1\eta_1 + \cdots + a_s\eta_s \in \mathfrak{p}$. Let $\hat{L} = \mathrm{Ad}(G)(\eta) \subset S^{2n+1}(c/4) \subset \mathfrak{p}$, which is a symmetric R-space standardly embedded in a complex Euclidean space $\mathfrak{p}$. Then we have the inclusions:

$$(2.2) \qquad \hat{L} = \hat{L}_1 \times \cdots \times \hat{L}_s \subset S^{2n_1+1}(c_1/4) \times \cdots \times S^{2n_s+1}(c_s/4) \subset S^{2n+1}(c/4).$$

THEOREM 2.1 ([16]). *Let $\pi : S^{2n+1}(c/4) \to CP^n(c)$ be the Hopf fibration and put $L = \pi(\hat{L})$. Then*

(1)   *L is a compact totally real submanifold embedded in $CP^n(c)$ with parallel second fundamental form, and thus L is a symmetric space.*

(2)   *L is a minimal submanifold in $CP^n(c)$ if and only if $c_i(n_i + 1) = c(n + 1)$ for each $i = 1, \ldots, s$.*

(3)   *The dimension of the Euclidean factor of L is equal to $s - 1$.*

(4)   *L is flat if and only if $s = n + 1$.*

(5)   *L has no Euclidean factor if and only if $s = 1$. In this case L is an irreducible symmetric space and a minimal submanifold in CP^n.*

(6)   *Conversely every totally real submanifold immersed in CP^n with parallel second fundamental form covers an open submanfold of L constructed in this way.*

Note that L is always a compact H-minimal Lagrangian submanifold embedded in $CP^n(c)$. Note that $\hat{L}$ is also a compact H-minimal Lagrangian submanifold in C^{n+1}. Their Hamiltonian stability will be discussed in a forthcoming paper [2].

In the case when L is flat, L is the Clifford torus of $\mathbf{C}P^n$. In the case when L has no Euclidean factor, L is locally isometric to one of the following symmetric spaces: $S^n(c/4)$, $SU(p)$, $SU(p)/SO(p)$, $SU(2p)/Sp(p)$, E_6/F_4. Their explicit descriptions are given in Section 5 of [13]. From now on we only discuss this case.

Suppose that (U, G) is irreducible ($s = 1$). In order to determine the Hamiltonian stability of the Lagrangian submanifold L embedded in $\mathbf{C}P^n$, we have to calculate the first eigenvalue λ_1 of the Laplacian on $C^\infty(L)$ with respect to the induced metric on L from the Fubini-Study metric of $\mathbf{C}P^n$.

Let $\widetilde{L}$ denote the universal covering space of L. Let $\tilde{\lambda}_1$ denote the first eigenvalue of the Laplacian acting on $C^\infty(\widetilde{L})$. The complete list of all irreducible Hermitian symmetric spaces U/G of compact type and corresponding L, $\widetilde{L}$ is given as follows:

Class	U/G	$n + 1$	L	$\widetilde{L}$
AIII	$\dfrac{SU(2p)}{S(U(p) \times U(p))}$	p^2	$SU(p)\mathbf{Z}_p$	$SU(p)$
DIII	$\dfrac{SO(4p)}{U(2p)}$	$p(2p-1)$	$\dfrac{SU(2p)}{Sp(p)\mathbf{Z}_{2p}}$	$\dfrac{SU(2p)}{Sp(p)}$
BDI	$\dfrac{SO(p+2)}{SO(p) \times SO(2)}$	p	$\mathbf{R}P^n$	S^n
CI	$\dfrac{Sp(p)}{U(p)}$	$\dfrac{p(p+1)}{2}$	$\dfrac{SU(p)}{SO(p)\mathbf{Z}_p}$	$\dfrac{SU(p)}{SO(p)}$
EVII	$\dfrac{E_7}{E_6 \cdot T}$	27	$\dfrac{E_6}{F_4\mathbf{Z}_3}$	$\dfrac{E_6}{F_4}$

Here note that the compact symmetric space L is a bottom space.

We take the decomposition

$$(2.3) \qquad \mathfrak{g} = \mathfrak{c}(\mathfrak{g}) \oplus \mathfrak{g}'$$

into the center and the semisimple part of $\mathfrak{g}$.

Let us denote by $B_{\mathfrak{h}}(\ ,\ )$ the Killing-Cartan form of a Lie algebra $\mathfrak{h}$. Define the invariant inner products on $\mathfrak{u}$ and $\mathfrak{g}'$ by

$$(2.4) \qquad \langle\ ,\ \rangle_{\mathfrak{u}} := -B_{\mathfrak{u}}(\ ,\ )$$

and

$$(2.5) \qquad \langle\ ,\ \rangle_{\mathfrak{g}'} := -B_{\mathfrak{g}'}(\ ,\ ),$$

respectively. For $\eta \in \mathfrak{p}$ with $\langle\eta, \eta\rangle_{\mathfrak{u}} = 4/c$, put

$$(2.6) \qquad \mathfrak{k} := \{X \in \mathfrak{g}' \mid [X, \eta] = 0\}.$$

Then we have an orthogonal decomposition

(2.7) $\mathfrak{g}' = \mathfrak{k} + \mathfrak{m}$.

Define an inner product on $\mathfrak{m}$ by

(2.8) $\langle X, Y \rangle := \langle [X, \eta], [Y, \eta] \rangle_{\mathfrak{u}}$

for $X, Y \in \mathfrak{m}$, which corresponds to the induced metric on L from the Fubini-Study metric of CP^n.

Now we assume that there exists a constant $C > 0$ such that

(2.9) $\langle X, Y \rangle = C \langle X, Y \rangle_{\mathfrak{g}'}$

for $X, Y \in \mathfrak{m}$. Then the constant $C > 0$ can be given explicitly as follows.

LEMMA 2.1.

(2.10) $$C = \frac{2}{c} \left(\frac{1}{n+1} - \frac{2}{\dim \mathfrak{g}'} a(\mathfrak{p}) \right) ,$$

where $a(\mathfrak{p})$ denotes the eigenvalue of the Casimir operator on the representation space $\mathfrak{p}$ of $\mathfrak{g}'$ with respect to $\langle \, , \, \rangle_{\mathfrak{g}'}$.

This lemma follows immediately from Propositions 2.1 and 2.2.

Assume that there is a constant $l > 0$ such that

(2.11) $\langle X, Y \rangle_{\mathfrak{u}} = l \langle X, Y \rangle_{\mathfrak{g}'}$

for $X, Y \in \mathfrak{g}'$.

Set $m = \dim \mathfrak{m}$. Then we obtain the following formula for C in terms of l.

PROPOSITION 2.1.   *Let $\langle \, , \, \rangle$ be the inner product of $\mathfrak{m}$ defined in (2.8). Assume that*

$$\langle X, Y \rangle = C \langle X, Y \rangle_{\mathfrak{g}'}$$

for $X, Y \in \mathfrak{m}$. Then we have

(2.12) $$C = \frac{l}{m} \frac{2n}{c(n+1)} .$$

PROOF.   Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{m}$ with respect to $\langle \, , \, \rangle_{\mathfrak{u}}$. By using (2.9) and (2.11), we have

(2.13)
$$\sum_{i=1}^{m} \langle e_i, e_i \rangle = C \sum_{i=1}^{m} \langle e_i, e_i \rangle_{\mathfrak{g}'} = \frac{C}{l} \sum_{i=1}^{m} \langle e_i, e_i \rangle_{\mathfrak{u}}$$
$$= \frac{C}{l} \dim \mathfrak{m} = \frac{C}{l} m .$$

On the other hand, we obtain

$$
\begin{aligned}
\sum_{i=1}^{m} \langle e_i, e_i \rangle &= \sum_{i=1}^{m} \langle [e_i, \eta], [e_i, \eta] \rangle_{\mathfrak{u}} \\
&= -\sum_{i=1}^{m} \langle (\mathrm{ad}\eta)^2 e_i, e_i \rangle_{\mathfrak{u}} = -\mathrm{tr}_{\mathfrak{g}'}(\mathrm{ad}\eta)^2 \\
&= -\left(\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\eta)^2 - \mathrm{tr}_{\mathfrak{c}_{\mathfrak{g}}}(\mathrm{ad}\eta)^2 \right) \\
&= -\frac{1}{2}\mathrm{tr}_{\mathfrak{u}}(\mathrm{ad}\eta)^2 + \mathrm{tr}_{\mathfrak{c}_{\mathfrak{g}}}(\mathrm{ad}\eta)^2 \\
&= \frac{1}{2}\langle \eta, \eta \rangle_{\mathfrak{u}} - \frac{1}{2(n+1)}\langle \eta, \eta \rangle_{\mathfrak{u}} \\
&= \frac{1}{2}\frac{4}{c} - \frac{1}{2(n+1)}\frac{4}{c} = \frac{2n}{c(n+1)} \,.
\end{aligned}
$$
(2.14)

$\square$

Next we are going to prove a formula determining the positive constant l. Here we can show a similar formula in a slightly general setting. Let $(\mathfrak{u}, \mathfrak{g})$ be an irreducible symmetric Lie algebra of compact type with canonical decomposition $\mathfrak{u} = \mathfrak{g} + \mathfrak{p}$. Assume that $\mathfrak{g} = \mathfrak{g}'_0 \oplus \mathfrak{g}'_1 \oplus \cdots \oplus \mathfrak{g}'_t$ is a decomposition into the center and simple ideals, where $\mathfrak{g}'_0 = \mathfrak{c}(\mathfrak{g})$.

PROPOSITION 2.2. *If there is a constant $l_1 > 0$ such that*

$$\langle X, Y \rangle_{\mathfrak{u}} = l_1 \langle X, Y \rangle_{\mathfrak{g}'_1}$$
(2.15)

for $X, Y \in \mathfrak{g}'_1$, where

$$\langle \,,\, \rangle_{\mathfrak{g}'_1} = -B_{\mathfrak{g}'_1}(\,,\,).$$

Then the constant $l_1 > 0$ is given by

$$l_1 = \frac{1}{k_1}\left(k_1 - (\dim \mathfrak{p})a'(\mathfrak{p}) \right),$$
(2.16)

where $k_1 = \dim \mathfrak{g}'_1$ and $a'(\mathfrak{p})$ denotes the eigenvalue of the Casimir operator on the representation space $\mathfrak{p}$ of $\mathfrak{g}'_1$ with respect to $\langle \,,\, \rangle_{\mathfrak{g}'_1}$.

PROOF. Let $\{X_1, \ldots, X_k\}$ be an orthonormal basis of $\mathfrak{g}'_1$ with respect to $\langle \,,\, \rangle_{\mathfrak{g}'_1}$. Then we have

$$\sum_{j=1}^{k_1} \langle X_j, X_j \rangle_{\mathfrak{u}} = l_1 \sum_{j=1}^{k_1} \langle X_j, X_j \rangle_{\mathfrak{g}'_1} = l_1 k_1 \,.$$
(2.17)

On the other hand, we obtain

$$\sum_{j=1}^{k_1} \langle X_j, X_j \rangle_{\mathfrak{u}} = -\sum_{j=1}^{k_1} \mathrm{tr}_{\mathfrak{u}}(\mathrm{ad}X_j)^2$$

$$= -\sum_{j=1}^{k_1} \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}X_j)^2 - \sum_{j=1}^{k_1} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}X_j)^2$$

(2.18)

$$= -\sum_{j=1}^{k_1} \mathrm{tr}_{\mathfrak{g}'_1}(\mathrm{ad}X_j)^2 - \sum_{j=1}^{k_1} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}X_j)^2$$

$$= \sum_{j=1}^{k_1} \langle X_j, X_j \rangle_{\mathfrak{g}'_1} - \mathrm{tr}_{\mathfrak{p}}(\mathcal{C})$$

$$= k_1 - (\dim \mathfrak{p}) a'(\mathfrak{p}),$$

where $\mathcal{C} = \mathcal{C}_{(\mathfrak{g}'_1, \langle\ ,\ \rangle_{\mathfrak{g}'_1})}$ denotes the Casimir operator of $\mathfrak{g}'_1$ with respect to $\langle\ ,\ \rangle_{\mathfrak{g}'_1}$. $\square$

3.   Calculation of the first eigenvalues of minimal Lagrangian submanifolds in CP^n with parallel second fundamental form.

To calculate the eigenvalues of the Laplacian acting on functions we review relevant results from the theory of spherical functions on compact symmetric spaces ([26]). Let G/K be a compact symmetric space with symmetric pair (G, K, σ), where G is a compact connected Lie group. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be its canonical decomposition and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$. We fix an $\mathrm{Ad}G$-invariant inner product $(\ ,\ )$ of $\mathfrak{g}$. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then we have $\mathfrak{t} = \mathfrak{b} + \mathfrak{a}$, where $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$. We fix a σ-linear order $<$ on $\mathfrak{t}$. The maximal torus T of G is generated by $\mathfrak{t}$. For each $\alpha \in \mathfrak{t}$, we put

(3.1) $\tilde{\mathfrak{g}}_\alpha = \{X \in \mathfrak{g}^C \mid (\mathrm{ad}H)X = 2\pi\sqrt{-1}(\alpha, H)X \text{ for each } H \in \mathfrak{t}\}.$

An element $\alpha \in \mathfrak{t}$ is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{t}$ if $\tilde{\mathfrak{g}}_\alpha$ is nonzero. We denote by $\Sigma(G)$ and $\Sigma^+(G)$ the set of all roots and all positive roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, respectively, and put $\Sigma_0(G) = \Sigma(G) \cap \mathfrak{b}$. Then we have the root decomposition

$$\mathfrak{g}^C = \mathfrak{t}^C + \sum_{\alpha \in \Sigma(G)} \tilde{\mathfrak{g}}_\alpha.$$

We always denote by $\Pi(\mathfrak{g})$ the fundamental system of roots and set $\Pi_0(\mathfrak{g}) = \Pi(\mathfrak{g}) \cap \Sigma_0(G)$. Set

(3.2)
$$\Gamma(G) := \{H \in \mathfrak{t} \mid \exp H = e\},$$
$$Z(G) := \{\Lambda \in \mathfrak{t} \mid (\Lambda, H) \in \mathbf{Z} \text{ for each } H \in \Gamma(G)\},$$
$$D(G) := \{\Lambda \in Z(G) \mid (\Lambda, \alpha) \geq 0 \text{ for each } \alpha \in \Sigma^+(G)\}.$$

Let $\mathcal{D}(G)$ be the complete set of inequivalent irreducible unitary representations of G. Then it is known that for each $(V, \rho) \in \mathcal{D}(G)$ the highest weight Λ_ρ of (V, ρ) belongs to $D(G)$, and the mapping $\mathcal{D}(G) \to D(G)$ is bijective.

Let A be the torus of G generated by $\mathfrak{a}$ and $\hat{A} = Ao$ be a maximal torus of G/K, where o denotes the origin eK of G/K. For each $\gamma \in \mathfrak{a}$, we put

$$\mathfrak{g}_\gamma = \{X \in \mathfrak{g}^C \mid (\mathrm{ad}H)X = 2\pi\sqrt{-1}(\alpha, H)X \text{ for each } H \in \mathfrak{a}\}.$$

An element $\gamma \in \mathfrak{a}$ is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{a}$ if $\mathfrak{g}_\gamma$ is nonzero. We denote by $\Sigma(G, K)$ and $\Sigma^+(G, K)$ the set of all roots and all positive roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, respectively. Then we have the decomposition

$$\mathfrak{g}^C = \mathfrak{g}_0 + \sum_{\gamma \in \Sigma(G,K)} \mathfrak{g}_\gamma.$$

Put

$$\Gamma(G, K) := \{H \in \mathfrak{a} \mid (\exp H)o = o\},$$

$$Z(G, K) := \{\Lambda \in \mathfrak{a} \mid (\Lambda, H) \in \mathbf{Z} \text{ for each } H \in \Gamma(G, K)\},$$

$$D(G, K) := \{\Lambda \in Z(G, K) \mid (\Lambda, \gamma) \geq 0 \text{ for each } \gamma \in \Sigma^+(G, K)\}.$$

Then we have $Z(G, K) \subset Z(G)$ and $D(G, K) \subset D(G)$. Let $\mathcal{D}(G, K)$ be the complete set of inequivalent class-one unitary representations of pair (G, K). Then for each $(V_\rho, \rho) \in \mathcal{D}(G, K)$ the subspace $(V_\rho)_K = \{v \in V_\rho \mid \rho(k)v = v \text{ for each } k \in K\}$ is of complex dimension 1, and the bijection $\mathcal{D}(G) \to D(G)$ induces the bijection $\mathcal{D}(G, K) \to D(G, K)$.

Let $\Pi(G, K) = \{\gamma_1, \ldots, \gamma_l\}$ be the fundamental root system for $\Sigma(G, K)$. We define

$$\beta_i = \begin{cases} \gamma_i & \text{if } 2\gamma_i \notin \Sigma(G, K) \\ 2\gamma_i & \text{if } 2\gamma_i \in \Sigma(G, K), \end{cases}$$

and set $\beta_i^* = 2\beta_i/(\beta_i, \beta_i)$. It is known (cf. [11]) that if G/K is simply connected, then

$$\Gamma(G, K) = \sum_{i=1}^{l} \mathbf{Z}\left(\frac{1}{2}\beta_i^*\right),$$

and if G/K is a bottom space, then

$$\Gamma(G, K) = \left\{H \in \mathfrak{a} \,\middle|\, \left(H, \frac{1}{2\pi}\gamma_i\right) \in \frac{1}{2}\mathbf{Z}, \; i = 1, 2, \ldots, p - 1\right\}.$$

Let g be a G-invariant Riemannian metric on G/K induced by $(\ ,\ )$. If G/K is irreducible and g is a G-invariant Riemannian metric induced from the (-1)-times Killing-Cartan form, then $(G/K, g)$ is an Einstein manifold with Einstein constant $1/2$ (cf. [27]). Let Δ be the Laplacian of $(G/K, g)$ acting on functions, and let $\Lambda = \Lambda_\rho$ be the highest weight of the representation $\rho \in \mathcal{D}(G, K)$. Then by the Freudenthal formula the complete set of eigenvalues of Δ is given by

$$(3.3) \qquad \{-a_\Lambda = 4\pi^2(\Lambda_\rho + 2\delta(G), \Lambda_\rho) \mid \rho \in \mathcal{D}(G, K)\}.$$

Here we set

$$\delta(G) = \frac{1}{2} \sum_{\alpha \in \Sigma^+(G)} \alpha \,.$$

The multiplicity of the k-th eigenvalue μ_k of Δ is given by $\sum_\rho d_\rho$, where the summation runs over all $\rho \in \mathcal{D}(G, K)$ such that $\mu_k = -a_\Lambda$. By Weyl's dimension formula d_ρ is given by

$$(3.4) \qquad d_\rho = \dim(V_\rho, \rho) = \prod_{\alpha \in \Sigma^+(G)} \frac{(\alpha, \Lambda_\rho + \delta(G))}{(\alpha, \delta(G))} \,.$$

Let $\{\Lambda_1, \ldots, \Lambda_{l'}\}$ be the fundamental weight system of the semisimple part $\mathfrak{g}'$ of $\mathfrak{g}$, where $l' = \operatorname{rank} \mathfrak{g}'$. Then the highest weight $\Lambda(\mathfrak{p})$ of the isotropy representation of irreducible Hermitian symmetric pair (U, G) restricted to $\mathfrak{g}'$ and the corresponding eigenvalue $a(\mathfrak{p})$ of their Casimir operator with respect to the (-1)-times Killing-Cartan form of $\mathfrak{g}'$ is given in the following list (cf. [8, Chapter 8]). Moreover, the constant C in (2.9) is also given in each case.

Class	$\mathfrak{g}'$	$\Lambda(\mathfrak{p})$	$n+1$	$\dim \mathfrak{g}'$	$-a(\mathfrak{p})$	C
AIII	$\mathfrak{su}(p) \oplus \mathfrak{su}(p)$	$\Lambda_1 \otimes \Lambda_{p-1}$	p^2	$2(p^2 - 1)$	$\dfrac{p^2 - 1}{p^2}$	$\dfrac{4}{p^2 c}$
DIII	$\mathfrak{su}(2p)$	Λ_2	$2p^2 - p$	$4p^2 - 1$	$\dfrac{2p^2 - p - 1}{p^2}$	$\dfrac{2}{p^2 c}$
BI	$\mathfrak{so}(2p - 1)$	Λ_1	$2p - 1$	$2p^2 - 3p + 1$	$\dfrac{p - 1}{2p - 3}$	$\dfrac{1}{(n - 1)c}$
DI	$\mathfrak{so}(2p)$	Λ_1	$2p$	$p(2p - 1)$	$\dfrac{2p - 1}{4(p - 1)}$	$\dfrac{1}{(n - 1)c}$
CI	$\mathfrak{su}(p)$	$2\Lambda_1$	$\dfrac{p(p + 1)}{2}$	$p^2 - 1$	$\dfrac{p^2 + p - 2}{p^2}$	$\dfrac{8}{p^2 c}$
EVII	$\mathfrak{e}_6$	Λ_1	27	78	$\dfrac{13}{18}$	$\dfrac{1}{9c}$

Here we give a computation of C only in the case of EVII. In this case $(U, G) = (E_7, T \cdot E_6)$ and $(G, K) = (T \cdot E_6, F_4)$. From (2.12) and (2.16) we have

$$(3.5) \qquad C = \frac{2}{c} \left(\frac{1}{n + 1} - \frac{2}{\dim \mathfrak{g}'} a(\mathfrak{p}) \right) = \frac{2}{c} \left(\frac{1}{27} + \frac{2}{78} \frac{13}{18} \right) = \frac{1}{9c} \,.$$

LEMMA 3.1. *Assume that G/K is one of compact irreducible symmetric spaces $\boldsymbol{R}P^n = SO(n + 1)/S(O(n) \times O(1))$, $SU(p)/\boldsymbol{Z}_p$, $SU(p)/SO(p)\boldsymbol{Z}_p$, $SU(2p)/Sp(p)\boldsymbol{Z}_{2p}$ and $E_6/F_4\boldsymbol{Z}_3$. Then the first eigenvalue μ_1 of the Laplacian for functions on G/K with respect to the metric defined by the (-1)-times Killing-Cartan form of $\mathfrak{g}$ is given in the following list*:

G/K	$\widetilde{G/K}$	$n = \dim G/K$	μ_1	$\tilde{\mu}_1$
$\boldsymbol{R}P^n$	S^n	n	$\dfrac{n+1}{2(n-1)}$	$\dfrac{n}{4(n-1)}$
$\dfrac{SU(p)}{\boldsymbol{Z}_p}$	$SU(p)$	$p^2 - 1$	2	$\dfrac{2(p^2-1)}{p^2}$
$\dfrac{SU(p)}{SO(p)\boldsymbol{Z}_p}$	$\dfrac{SU(p)}{SO(p)}$	$\dfrac{(p-1)(p+2)}{2}$	$\dfrac{2(p+1)}{p}$	$\dfrac{(p-1)(p+2)}{p^2}$
$\dfrac{SU(2p)}{Sp(p)\boldsymbol{Z}_{2p}}$	$\dfrac{SU(2p)}{Sp(p)}$	$(p-1)(2p+1)$	$\dfrac{2p-1}{p}$	$\dfrac{(p-1)(2p+1)}{2p^2}$
$\dfrac{E_6}{F_4\boldsymbol{Z}_3}$	$\dfrac{E_6}{F_4}$	26	$\dfrac{3}{2}$	$\dfrac{13}{18}$

Here $\widetilde{G/K}$ is the universal covering space of G/K, $\tilde{\mu}_1$ denotes the first eigenvalue of the Laplacian acting on functions for $\widetilde{G/K}$ and $p \geq 2$ is an integer.

PROOF. We use the table of root systems for complex simple Lie algebras in [3].

The case $G/K = \big(SU(p)/\boldsymbol{Z}_p \times SU(p)/\boldsymbol{Z}_p\big)/\big(SU(p)/\boldsymbol{Z}_p\big) \cong SU(p)/\boldsymbol{Z}_p$: Suppose that $\mathfrak{k} = \mathfrak{su}(p)$ is a compact simple Lie algebra of type A_{p-1}. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. If $\{\varepsilon_1, \ldots, \varepsilon_p\}$ denotes the standard orthonormal basis of $\boldsymbol{R}^p$, then $\mathfrak{t}$ can be identified with the subspace of $\boldsymbol{R}^p$ as follows:

$$\mathfrak{t} = \left\{ \sum_{i=1}^p x_i \varepsilon_i \ \middle| \ \sum_{i=1}^p x_i = 0 \right\}.$$

The fundamental root system of $(\mathfrak{k}, \mathfrak{t})$ is

$$\Pi(\mathfrak{k}) = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \ldots, p-1\}.$$

Set $\gamma_i^* = 2\gamma_i/(\gamma_i, \gamma_i) = \gamma_i$, $i = 1, 2, \ldots, p-1$.
The sets $\Gamma(SU(p))$, $Z(SU(p))$ and $D(SU(p))$ are given explicitly as follows:

$$\Gamma(SU(p)) = \left\{ 2\pi \sum_{i=1}^p x_i \varepsilon_i \ \middle| \ \sum_{i=1}^p x_i = 0, \ x_i \in \boldsymbol{Z}, \ i = 1, 2, \ldots, p \right\},$$

$$Z(SU(p)) = \left\{ \frac{1}{2\pi} \sum_{i=1}^p m_i \varepsilon_i \ \middle| \ \sum_{i=1}^p m_i = 0, \right.$$

$$\left. m_i - m_{i+1} \in \boldsymbol{Z}, \ i = 1, 2, \ldots, p-1 \right\},$$

$$D(SU(p)) = \left\{ \frac{1}{2\pi} \sum_{i=1}^{p} m_i \varepsilon_i \ \middle| \ \sum_{i=1}^{p} m_i = 0 \,, \right.$$

$$\left. 0 \le m_i - m_{i+1} \in \mathbf{Z}, \ i = 1, 2, \ldots, p-1 \right\}.$$

Furthermore, $\Gamma(SU(p)/\mathbf{Z}_p)$, $Z(SU(p)/\mathbf{Z}_p)$ and $D(SU(p)/\mathbf{Z}_p)$ are given explicitly as follows:

$$\Gamma(SU(p)/\mathbf{Z}_p) = \left\{ 2\pi \sum_{i=1}^{p} x_i \varepsilon_i \ \middle| \ \sum_{i=1}^{p} x_i = 0, \ x_i - x_{i+1} \in \mathbf{Z}, \ i = 1, 2, \ldots, p-1 \right\},$$

$$Z(SU(p)/\mathbf{Z}_p) = \left\{ \frac{1}{2\pi} \sum_{i=1}^{p} m_i \varepsilon_i \ \middle| \ \sum_{i=1}^{p} m_i = 0, \ m_i \in \mathbf{Z}, \ i = 1, 2, \ldots, p \right\},$$

$$D(SU(p)/\mathbf{Z}_p) = \left\{ \frac{1}{2\pi} \sum_{i=1}^{p} m_i \varepsilon_i \ \middle| \ \sum_{i=1}^{p} m_i = 0, m_i \in \mathbf{Z}, \ i = 1, 2, \ldots, p, \right.$$

$$\left. m_i - m_{i+1} \ge 0, \ i = 1, 2, \ldots, p-1 \right\}.$$

Here $\delta(G)$ is given by

$$\delta(G) = \frac{1}{2\pi} \frac{1}{2} \sum_{i=1}^{p} (p - 2i + 1)\varepsilon_i \,.$$

By the Freudenthal formula we determine all eigenvalues of the Casimir opetrator of $SU(p)/\mathbf{Z}_p$. For each $\Lambda \in D(SU(p)/\mathbf{Z}_p)$, the corresponding eigenvalue $-a_\Lambda$ is given by

$$-a_\Lambda = 4\pi^2 \left(\frac{1}{2\pi} \sum_{i=1}^{p} m_i \varepsilon_i + \frac{1}{2\pi} \sum_{i=1}^{p} (p - 2i + 1)\varepsilon_i, \frac{1}{2\pi} \sum_{i=1}^{p} m_i \varepsilon_i \right)$$

$$= \sum_{i=1}^{p} m_i^2 - 2 \sum_{i=1}^{p} i m_i \,.$$

For $(m_1, m_2, \ldots, m_p) = (1, 0, \ldots, 0, -1)$, we have $-a_\Lambda = 2 - 2(1 - p) = 2p$.

By using Lemma 3.2, we see that $-a_\Lambda = 2p$ is the first eigenvalue of $SU(p)/\mathbf{Z}_p$. By using Weyl's dimension formula (3.4), we can show that its multiplicity is equal to $p^2(p^2 - 1)$. Hence the first eigenvalue of $SU(p)/\mathbf{Z}_p$ relative to the metric $(-1)B_{\mathfrak{k}}(\ ,\ )$ is equal to $2p/2p = 1$. Thus the first eigenvalue of $\left(SU(p)/\mathbf{Z}_p \times SU(p)/\mathbf{Z}_p \right)/\left(SU(p)/\mathbf{Z}_p \right)$ relative to the metric $(-1)B_{\mathfrak{g}}(\ ,\ )$ is equal to $1 \times 2 = 2$.

The case $G/K = SU(p)/SO(p)\mathbf{Z}_p$: Suppose that $\mathfrak{g} = \mathfrak{su}(p)$ and $\mathfrak{k} = \mathfrak{so}(p)$. Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p\}$ be the standard orthonormal basis of $\mathbf{R}^p$.

Recall the Satake diagram of AI:
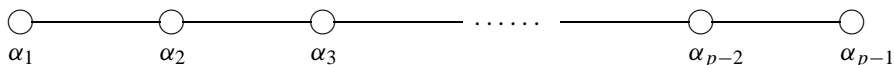
The fundamental root system of $(\mathfrak{g}, \mathfrak{t})$ can be found as follows:

$$\Pi(\mathfrak{g}) = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \ldots, p - 1\}.$$
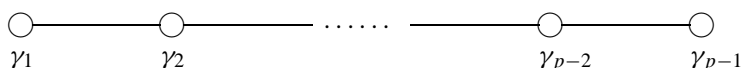
In this case $\Pi_0(\mathfrak{g}) = \emptyset$, $\mathfrak{b} = \{0\}$ and

$$\Pi(\mathfrak{g}, \mathfrak{a}) = \{\gamma_i = \bar{\alpha}_i = \alpha_i \mid i = 1, 2, \ldots, p - 1\} = \Pi(\mathfrak{g}).$$

Then the subspace $\mathfrak{a}$ is given by

$$\mathfrak{a} = \sum_{i=1}^{p-1} \mathbf{R} \cdot (\varepsilon_i - \varepsilon_{i+1}).$$

The Dynkin diagram of $\Pi(\mathfrak{g}, \mathfrak{a})$ is of type A_{p-1} and given by

Set $\beta_i := \gamma_i$, $i = 1, 2, \ldots, p - 1$, and $\beta_i^* := \gamma_i^* = 2\gamma_i/(\gamma_i, \gamma_i) = \gamma_i$, $i = 1, 2, \ldots, p - 1$.

The sets $\Gamma(SU(p), SO(p))$, $Z(SU(p), SO(p))$ and $D(SU(p), SO(p))$ are described as follows:

$$\Gamma(SU(p), SO(p)) = \left\{ \pi \sum_{i=1}^{p} x_i \varepsilon_i \,\bigg|\, \sum_{i=1}^{p} x_i = 0,\ x_i \in \mathbf{Z},\ i = 1, 2, \ldots, p \right\},$$

$$Z(SU(p), SO(p)) = \left\{ \frac{1}{\pi} \sum_{i=1}^{p} m_i \varepsilon_i \,\bigg|\, \sum_{i=1}^{p} m_i = 0, \right.$$
$$\left. m_i - m_{i+1} \in \mathbf{Z},\ i = 1, 2, \ldots, p - 1 \right\},$$

$$D(SU(p), SO(p)) = \left\{ \frac{1}{\pi} \sum_{i=1}^{p} m_i \varepsilon_i \,\bigg|\, \sum_{i=1}^{p} m_i = 0, \right.$$
$$\left. 0 \leq m_i - m_{i+1} \in \mathbf{Z},\ i = 1, 2, \ldots, p - 1 \right\}.$$

Similarly, $\Gamma(SU(p), SO(p)\mathbf{Z}_p)$, $Z(SU(p), SO(p)\mathbf{Z}_p)$ and $D(SU(p), SO(p)\mathbf{Z}_p)$ are described as follows:

$$\Gamma(SU(p), SO(p)\mathbf{Z}_p) = \left\{ \pi \sum_{i=1}^{p} x_i \varepsilon_i \ \middle| \ \sum_{i=1}^{p} x_i = 0, \ x_i - x_{i+1} \in \mathbf{Z}, i = 1, 2, \ldots, p-1 \right\},$$

$$Z(SU(p), SO(p)\mathbf{Z}_p) = \left\{ \frac{1}{\pi} \sum_{i=1}^{p} m_i \varepsilon_i \ \middle| \ \sum_{i=1}^{p} m_i = 0, \ m_i \in \mathbf{Z}, i = 1, 2, \ldots, p \right\},$$

$$D(SU(p), SO(p)\mathbf{Z}_p) = \left\{ \frac{1}{\pi} \sum_{i=1}^{p} m_i \varepsilon_i \ \middle| \ \sum_{i=1}^{p} m_i = 0, \ m_i \in \mathbf{Z}, \ i = 1, 2, \ldots, p, \right.$$
$$\left. m_i - m_{i+1} \geq 0, \ i = 1, 2, \ldots, p-1 \right\}.$$

Here

$$\delta(SU(p)) = \frac{1}{2\pi} \frac{1}{2} \sum_{i=1}^{p} (p - 2i + 1)\varepsilon_i .$$

For each $\Lambda \in D(SU(p), SO(p)\mathbf{Z}_p)$, the corresponding eigenvalue $-a_\Lambda$ is

$$-a_\Lambda = 4\pi^2 \left(\frac{1}{\pi} \sum_{i=1}^{p} m_i \varepsilon_i + \frac{1}{2\pi} \sum_{i=1}^{p} (p - 2i + 1)\varepsilon_i, \frac{1}{\pi} \sum_{i=1}^{p} m_i \varepsilon_i \right)$$

$$= 4 \sum_{i=1}^{p} (m_i)^2 - 4 \sum_{i=1}^{p} i m_i .$$

For $(m_1, m_2, \ldots, m_p) = (1, 0, \ldots, 0, -1)$, we have $-a_\Lambda = 4(p+1)$. This is the first eigenvalue of $SU(p)/SO(p)\mathbf{Z}_p$ by Lemma 3.2. The first eigenvalue of $SU(p)/SO(p)\mathbf{Z}_p$ relative to the metric $(-1)B_{\mathfrak{g}}(\ ,\ )$ is equal to $2(p+1)/p$. The first eigenvalue of $SU(p)/SO(p)\mathbf{Z}_p$ relative to the metric $(-2)B_{\mathfrak{g}}(\ ,\ )/p^2$ is equal to $p(p+1)/4$. By using Weyl's dimension formula (3.4), we can show that its multiplicity is equal to $p^2(p-1)(p+3)/4$.

The case $G/K = SU(2p)/Sp(p)\mathbf{Z}_{2p}$: Suppose that $\mathfrak{g} = \mathfrak{su}(2p)$ and $\mathfrak{k} = \mathfrak{sp}(p)$. Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2p}\}$ be the standard orthonormal basis of $\mathbf{R}^{2p}$. The subspace $\mathfrak{t}$ can be identified with

$$\mathfrak{t} = \left\{ \sum_{i=1}^{2p} \xi^i \varepsilon_i \ \middle| \ \sum_{i=1}^{2p} \xi^i = 0 \right\} .$$

Recall the Satake diagram of AII:
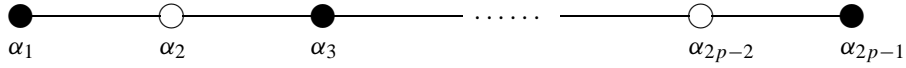
The fundamental root system of $(\mathfrak{g}, \mathfrak{k})$ is given by

$$\Pi(\mathfrak{g}) = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \ldots, 2p - 1\}.$$
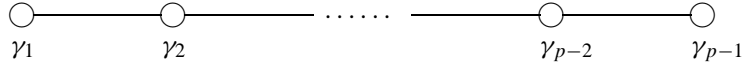
In this case

$$\Pi_0(\mathfrak{g}) = \{\alpha_{2i-1} \mid i = 1, 2, \ldots, p\},$$

$$\mathfrak{b} = \sum_{i=1}^{p} \boldsymbol{R} \cdot \alpha_{2i-1} = \sum_{i=1}^{p} \boldsymbol{R} \cdot (\varepsilon_{2i-1} - \varepsilon_{2i})$$

and

$$\Pi(\mathfrak{g}, \mathfrak{a}) = \{\gamma_i = \bar{\alpha}_{2i} \mid i = 1, 2, \ldots, p - 1\}.$$

The Dynkin diagram of $\Pi(\mathfrak{g}, \mathfrak{a})$ is of type A_{p-1}:

Define p vectors f_i of $\boldsymbol{R}^{2p}$ by

$$f_i := \frac{1}{\sqrt{2}}(\varepsilon_{2i-1} + \varepsilon_{2i}), \quad i = 1, 2, \ldots, p - 1, p.$$

Then, for $i = 1, 2, \ldots, p - 1$ we have

$$\alpha_{2i} = \varepsilon_{2i} - \varepsilon_{2i+1}, \quad \gamma_i = \frac{1}{\sqrt{2}}(f_i - f_{i+1}).$$

The subspace $\mathfrak{a}$ can be identified with

$$\mathfrak{a} = \sum_{i=1}^{p-1} \boldsymbol{R} \cdot (f_i - f_{i+1}).$$

Set $\beta_i := \gamma_i$, and $\beta_i^* := \gamma_i^* = 2\gamma_i/(\gamma_i, \gamma_i) = 2\gamma_i$, $i = 1, 2, \ldots, p - 1$. Then $\Gamma(SU(2p), Sp(p))$, $Z(SU(2p), Sp(p))$ and $D(SU(2p), Sp(p))$ are described as follows:

$$\Gamma(SU(2p), Sp(p)) = \left\{ \sqrt{2}\pi \sum_{i=1}^{p} x_i f_i \ \Big| \ \sum_{i=1}^{p} x_i = 0, \ x_i \in \mathbf{Z}, \ i = 1, 2, \ldots, p \right\},$$

$$Z(SU(2p), Sp(p)) = \left\{ \frac{1}{\sqrt{2}\pi} \sum_{i=1}^{p} m_i f_i \ \Big| \ \sum_{i=1}^{p} m_i = 0, \right.$$
$$\left. m_i - m_{i+1} \in \mathbf{Z}, \ i = 1, 2, \ldots, p - 1 \right\},$$

$$D(SU(2p), Sp(p)) = \left\{ \frac{1}{\sqrt{2}\pi} \sum_{i=1}^{p} m_i f_i \ \Big| \ \sum_{i=1}^{p} m_i = 0, \right.$$
$$\left. 0 \le m_i - m_{i+1} \in \mathbf{Z}, \ i = 1, 2, \ldots, p - 1 \right\}.$$

Moreover, $\Gamma(SU(2p), Sp(p)\mathbf{Z}_{2p})$, $Z(SU(2p), Sp(p)\mathbf{Z}_{2p})$ and $D(SU(2p), Sp(p)\mathbf{Z}_{2p})$ are described as follows:

$$\Gamma(SU(2p), Sp(p)\mathbf{Z}_{2p}) = \left\{ \sqrt{2}\pi \sum_{i=1}^{p} x_i f_i \ \Big| \ \sum_{i=1}^{p} x_i = 0, \right.$$
$$\left. x_i - x_{i+1} \in \mathbf{Z}, \ i = 1, 2, \ldots, p - 1 \right\},$$

$$Z(SU(2p), Sp(p)\mathbf{Z}_{2p}) = \left\{ \frac{1}{\sqrt{2}\pi} \sum_{i=1}^{p} m_i f_i \ \Big| \ \sum_{i=1}^{p} m_i = 0, \ m_i \in \mathbf{Z}, \ i = 1, 2, \ldots, p \right\},$$

$$D(SU(2p), Sp(p)\mathbf{Z}_{2p}) = \left\{ \frac{1}{\sqrt{2}\pi} \sum_{i=1}^{p} m_i f_i \ \Big| \ \sum_{i=1}^{p} m_i = 0, \ m_i \in \mathbf{Z}, \ i = 1, 2, \ldots, p, \right.$$
$$\left. m_i - m_{i+1} \ge 0, \ i = 1, 2, \ldots, p - 1 \right\}.$$

Here

$$\delta(SU(2p)) = \frac{1}{2\pi} \frac{1}{2} \sum_{i=1}^{2p} (2p - 2i + 1)\varepsilon_i$$

$$= \frac{1}{2\pi} \frac{1}{2} \sum_{i=1}^{p} ((2p - 4i + 3)\varepsilon_{2i-1} + (2p - 4i + 1)\varepsilon_{2i}).$$

For each $\Lambda \in D(SU(2p), Sp(p)\mathbf{Z}_{2p})$, the corresponding eigenvalue $-a_\Lambda$ is

$$-a_\Lambda = 4\pi^2 \left(\frac{1}{\sqrt{2\pi}} \sum_{i=1}^{p} m_i f_i + \frac{1}{2\pi} \sum_{i=1}^{2p} (2p - 2i + 1)\varepsilon_i, \ \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{p} m_i f_i \right)$$

$$= 2\sum_{i=1}^{p} (m_i)^2 - 8\sum_{i=1}^{p} i \, m_i \,.$$

All eigenvalues of the Laplacian on functions for $SU(2p)/Sp(p)\mathbf{Z}_{2p}$ relative to the metric $(-1)B_\mathfrak{g}(\ ,\ )$ are given by the formula

$$-\frac{1}{4p}a_\Lambda = \frac{1}{2p} \left(\sum_{i=1}^{p} (m_i)^2 - 4\sum_{i=1}^{p} i m_i \right) = \frac{1}{2p} \sum_{i=1}^{p} m_i (m_i - 4i) \,.$$

For $(m_1, m_2, \ldots, m_p) = (1, 0, \ldots, 0, -1)$,

$$-\frac{1}{4p}a_\Lambda = \frac{1}{2p} \sum_{i=1}^{p} m_i (m_i - 4i) = \frac{2p - 1}{p} \,.$$

By using Lemma 3.2, we see that $-a_\Lambda/4p = (2p - 1)/p$ is the first eigenvalue of $SU(2p)/Sp(p)\mathbf{Z}_{2p}$. By Weyl's dimension formula (3.4), we can show that its multiplicity is equal to $p^2(2p + 1)(2p - 3)$.

The case $G/K = E_6/F_4\mathbf{Z}_3$: Suppose that $\mathfrak{g} = \mathfrak{e}_6$ and $\mathfrak{k} = \mathfrak{f}_4$. Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_8\}$ be the standard orthonormal basis of $\mathbf{R}^8$. Let

$$\mathfrak{t} = \sum_{i=1}^{5} \mathbf{R} \cdot \varepsilon_i + \mathbf{R} \cdot (\varepsilon_6 + \varepsilon_7 - \varepsilon_8) \subset \mathbf{R}^8 \,.$$
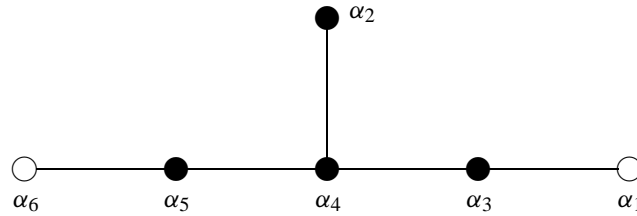
The fundamental root system of $(\mathfrak{g}, \mathfrak{t})$ is given as

$$\Pi(\mathfrak{g}) = \{\alpha_1, \alpha_2, \ldots, \alpha_6\} \,,$$

where

$$\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7) \,,$$
$$\alpha_2 = \varepsilon_1 + \varepsilon_2 \,,$$
$$\alpha_3 = \varepsilon_2 - \varepsilon_1 \,,$$
$$\alpha_4 = \varepsilon_3 - \varepsilon_2 \,,$$
$$\alpha_5 = \varepsilon_4 - \varepsilon_3 \,,$$
$$\alpha_6 = \varepsilon_5 - \varepsilon_4 \,.$$

The Satake diagram of EIV is given by

Moreover,

$$\Pi_0(\mathfrak{g}) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

and

$$\mathfrak{b} = \boldsymbol{R} \cdot \alpha_2 + \boldsymbol{R} \cdot \alpha_3 + \boldsymbol{R} \cdot \alpha_4 + \boldsymbol{R} \cdot \alpha_5$$
$$= \boldsymbol{R} \cdot \varepsilon_1 + \boldsymbol{R} \cdot \varepsilon_2 + \boldsymbol{R} \cdot \varepsilon_3 + \boldsymbol{R} \cdot \varepsilon_4 \,,$$
$$\mathfrak{a} = \boldsymbol{R} \cdot \varepsilon_5 + \boldsymbol{R} \cdot (\varepsilon_6 + \varepsilon_7 - \varepsilon_8) \,.$$
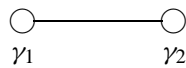
The fundamental root system of $(\mathfrak{g}, \mathfrak{a})$ is given as

$$\Pi(\mathfrak{g}, \mathfrak{a}) = \{\bar{\alpha}_1, \bar{\alpha}_6\} \,.$$

Set

$$\gamma_1 := \bar{\alpha}_1 = -\frac{1}{2}(\varepsilon_5 + \varepsilon_6 + \varepsilon_7 - \varepsilon_8) \,, \quad \gamma_2 := \bar{\alpha}_6 = \varepsilon_5 \,.$$

The Dynkin diagram of $\Pi(\mathfrak{g}, \mathfrak{a})$ is of type A_2 and is given by

Define

$$\gamma_1^* := \frac{2}{(\gamma_1, \gamma_1)}\gamma_1 = 2\gamma_1 \,, \quad \gamma_2^* := \frac{2}{(\gamma_2, \gamma_2)}\gamma_2 = 2\gamma_2 \,.$$

Let $\{f_1, f_2\}$ be an orthonormal basis of $\mathfrak{a}$ defined by

(3.6) $$f_1 = \varepsilon_5 \,, \quad f_2 = -\frac{1}{\sqrt{3}}(\varepsilon_6 + \varepsilon_7 - \varepsilon_8) \,.$$

Then we have

$$\gamma_1 = -\frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2 \,, \quad \gamma_2 = f_1 \,.$$

Moreover, we have

$$\Gamma(E_6, F_4) = \{\pi(k_1 f_1 + \sqrt{3}k_2 f_2) \mid k_1, k_2 \in 2\mathbf{Z} \text{ or } k_1, k_2 \in 2\mathbf{Z} + 1\},$$

$$Z(E_6, F_4) = \left\{ x_1 f_1 + x_2 f_2 \;\middle|\; x_1 \in \frac{1}{2\pi}\mathbf{Z}, x_2 \in \frac{1}{2\sqrt{3}\pi}\mathbf{Z}, \; x_1 + \sqrt{3}x_2 \in \frac{1}{\pi}\mathbf{Z} \right\}$$

$$= \left\{ \frac{1}{2\pi}\left(k_1 f_1 + \frac{1}{\sqrt{3}}k_2 f_2 \right) \;\middle|\; k_1, k_2 \in \mathbf{Z}, k_1 + k_2 \in 2\mathbf{Z} \right\},$$

$$D(E_6, F_4) = \left\{ \frac{1}{2\pi}\left(k_1 f_1 + \frac{1}{\sqrt{3}}k_2 f_2 \right) \;\middle|\; k_1, k_2 \in \mathbf{Z}, k_1 + k_2 \in 2\mathbf{Z}, k_2 \geq k_1 \geq 0 \right\}.$$

Similarly $\Gamma(E_6, F_4\mathbf{Z}_3)$, $Z(E_6, F_4\mathbf{Z}_3)$ and $D(E_6, F_4\mathbf{Z}_3)$ are described as follows:

$$\Gamma(E_6, F_4\mathbf{Z}_3) = \left\{ \pi\left(k_1 f_1 + \frac{k_2}{\sqrt{3}} f_2 \right) \;\middle|\; k_1, k_2 \in 2\mathbf{Z} \text{ or } k_1, k_2 \in 2\mathbf{Z} + 1 \right\}$$

$$Z(E_6, F_4\mathbf{Z}_3) = \left\{ x_1 f_1 + x_2 f_2 \;\middle|\; x_1 \in \frac{1}{2\pi}\mathbf{Z}, x_2 \in \frac{\sqrt{3}}{2\pi}\mathbf{Z}, x_1 + \frac{1}{\sqrt{3}}x_2 \in \frac{1}{\pi}\mathbf{Z} \right\}$$

$$= \left\{ \frac{1}{2\pi}\left(k_1 f_1 + \frac{1}{\sqrt{3}}k_2 f_2 \right) \;\middle|\; k_1 \in \mathbf{Z}, k_2 \in 3\mathbf{Z}, k_1 + k_2 \in 2\mathbf{Z} \right\},$$

$$D(E_6, F_4\mathbf{Z}_3) = \left\{ \frac{1}{2\pi}\left(k_1 f_1 + \frac{1}{\sqrt{3}}k_2 f_2 \right) \;\middle|\; k_1 \in \mathbf{Z}, k_2 \in 3\mathbf{Z}, \; k_1 + k_2 \in 2\mathbf{Z}, \right.$$

$$\left. k_2 \geq k_1 \geq 0 \right\}.$$

Here

$$\delta = \frac{1}{2\pi}(\varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_4 + 4\varepsilon_5 + 4(\varepsilon_8 - \varepsilon_7 - \varepsilon_6))$$

$$= \frac{1}{2\pi}(\varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_4 + 4 f_1 + 4\sqrt{3} f_2).$$

Note that $\varepsilon_2 + 2\varepsilon_3 + 3\varepsilon_4 \in \mathfrak{b}$ and $4 f_1 + 4\sqrt{3} f_2 \in \mathfrak{a}$.

For each $\Lambda = \Lambda(k_1, k_2) \in D(E_6, F_4)$, the corresponding eigenvalue $-a_\Lambda$ of E_6/F_4 is found by the following formula:

$$(3.7) \qquad\qquad -a_\Lambda = k_1(k_1 + 8) + k_2\left(\frac{k_2}{3} + 8 \right).$$

Here we examine eigenvalues corresponding to small values of pairs (k_1, k_2). For $(k_1, k_2) = (1, 1)$ or $(k_1, k_2) = (0, 2)$, we have

$$(3.8) \qquad\qquad -a_\Lambda = \frac{52}{3} \quad \text{and} \quad -\frac{a_\Lambda}{24} = \frac{13}{18}.$$

For $(k_1, k_2) = (2, 2)$, we have

$$(3.9) \qquad\qquad -a_\Lambda = \frac{112}{3} \quad \text{and} \quad -\frac{a_\Lambda}{24} = \frac{14}{9}.$$

For $(k_1, k_2) = (1, 3)$, we have

(3.10) $-a_\Lambda = 36$ and $-\dfrac{a_\Lambda}{24} = \dfrac{3}{2}$.

For $(k_1, k_2) = (3, 3)$, we have

(3.11) $-a_\Lambda = 60$ and $-\dfrac{a_\Lambda}{24} = \dfrac{5}{2}$.

Otherwise, we obtain

(3.12) $-\dfrac{a_\Lambda}{24} > \dfrac{3}{2}$.

The first eigenvalue of E_6/F_4 is equal to 13/18. The second eigenvalue of E_6/F_4 and the first eigenvalue of $E_6/F_4\mathbf{Z}_3$ are equal to 3/2. By Weyl's dimension formula (3.4), we can show that the multiplicity of the first eigenvalue of $E_6/F_4\mathbf{Z}_3$ is equal to 650.   □

LEMMA 3.2. *Let $(m_1, \ldots, m_p)$ be a p-tuple of integers satisfying*

(3.13) $\displaystyle\sum_{i=1}^{p} m_i = 0 \ and \ m_i - m_{i+1} \geq 0, \ i = 1, 2, \ldots, p$.

Suppose that $(m_1, \ldots, m_p) \neq 0$ and t is a positive real number. Then the following inequality holds:

$$\sum_{i=1}^{p}(m_i)^2 - t\sum_{i=1}^{p} i m_i \geq 2 + t(p-1)(m_1 - m_p)/2 .$$

The equality holds if and only if $(m_1, m_2, \ldots, m_{p-1}, m_p) = (1, 0, \ldots, 0, -1)$.

PROOF.   Put $\mathbf{Z} \ni k_i := m_i - m_{i+1} \geq 0, \ i = 1, 2, \ldots, p - 1$. Then we have $m_p = -\sum_{j=1}^{p-1} jk_j/p$. By using the identity

$$\sum_{i=1}^{p} i m_i = \sum_{i=1}^{p} i(m_p + k_{p-1} + k_{p-2} + \cdots + k_i)$$

$$= m_p \frac{p(p+1)}{2} + \sum_{j=1}^{p-1} \frac{j(j+1)}{2} k_j$$

$$= -\frac{p+1}{2}\sum_{j=1}^{p-1} jk_j + \sum_{j=1}^{p-1} \frac{j(j+1)}{2} k_j$$

$$= -\frac{1}{2}\sum_{j=1}^{p-1} j(p-j)k_j ,$$

we obtain

$$\sum_{i=1}^{p} m_i^2 - t \sum_{i=1}^{p} i m_i = \sum_{i=1}^{p} m_i^2 + \frac{t}{2} \sum_{j=1}^{p-1} j(p-j)k_j$$

$$\geq 2 + \frac{t}{2} \sum_{j=1}^{p-1} (p-1)k_j = 2 + t(p-1)(m_1 - m_p)/2 \,.$$

Here we see that

$$\sum_{i=1}^{p} m_i^2 + \frac{t}{2} \sum_{j=1}^{p-1} j(p-j)k_j = 2 + t(p-1)(m_1 - m_p)/2$$

if and only if $(k_1, k_2, \dots, k_{p-2}, k_{p-1}) = (1, 0, \dots, 0, 1)$ $(p \geq 3)$ or $k_1 = 2$ $(p = 2)$, and equivalently $(m_1, m_2, \dots, m_{p-2}, m_{p-1}, m_p) = (1, 0, \dots, 0, 0, -1)$. $\qquad\square$

4. Hamiltonian stability of minimal Lagrangian submanifolds in CP^n with parallel second fundamental form.

First, combining Proposition 2.1 and Lemma 3.1, we obtain the following lemma.

LEMMA 4.1. *Suppose that L is an n-dimensional compact totally real submanifold embedded in CP^n constructed in Section 2 with $s = 1$. Then the Einstein constant, the first eigenvalue λ_1 of the Laplacian acting on $C^\infty(L)$ with respect to the induced metric and its multiplicity d_1 are given in the following table:*

L	$\widetilde{L}$	κ_L	λ_1	$\tilde{\lambda}_1$	d_1
$\boldsymbol{R}P^n$	S^n	$\dfrac{(n-1)c}{4}$	$\dfrac{(n+1)c}{2}$	$\dfrac{nc}{4}$	$\dfrac{n(n+3)}{2}$
$\dfrac{SU(p)}{\boldsymbol{Z}_p}$	$SU(p)$	$\dfrac{p^2 c}{8}$	$\dfrac{p^2 c}{2}$	$\dfrac{(p^2-1)c}{2}$	$(p^2-1)^2$
$\dfrac{SU(p)}{SO(p)\boldsymbol{Z}_p}$	$\dfrac{SU(p)}{SO(p)}$	$\dfrac{p^2 c}{16}$	$\dfrac{p(p+1)c}{4}$	$\dfrac{(p+2)(p-1)c}{8}$	$\dfrac{p^2(p-1)(p+3)}{4}$
$\dfrac{SU(2p)}{Sp(p)\boldsymbol{Z}_{2p}}$	$\dfrac{SU(2p)}{Sp(p)}$	$\dfrac{p^2 c}{4}$	$\dfrac{p(2p-1)c}{2}$	$\dfrac{(p-1)(2p+1)c}{4}$	$p^2(2p+1)(2p-3)$
$\dfrac{E_6}{F_4\boldsymbol{Z}_3}$	$\dfrac{E_6}{F_4}$	$\dfrac{9c}{2}$	$\dfrac{27c}{2}$	$\dfrac{13c}{2}$	650

Here κ_L is the Einstein constant of L and $p \geq 2$ is an integer.

Comparing tables of Lemma 3.1 and 4.1, we see that L satisfies the Hamiltonian stability condition of Theorem 1.2, that is, the first eigenvalue λ_1 of the Laplacian of L is equal to $c(n+1)/2$ in each case. We also note that this condition is not satisfied for $\widetilde{L}$.

In Lemma 4.1, the multiplicies d_1 of λ_1 were computed by the Weyl's dimension formula (3.4). In these cases we have

$$n(L) = d_1 \,.$$

In each case $L = G/K$, we can observe the fact that

$$d_1 = \dim(\mathfrak{su}(n+1)) - \dim G$$
$$= \dim\mathcal{K} - \dim I_0(L) \,.$$

By (1.2), we obtain

$$n(L) = n_{\mathcal{K}}(L) \,.$$

Therefore we obtain the following theorem.

THEOREM 4.1. *Let L be an n-dimensional compact totally real minimal submanifold embedded in $\mathbf{C}P^n$ with parallel second fundamental form in the following list*:

(1) $SU(p)/\mathbf{Z}_p$, $n = p^2 - 1$.
(2) $SU(p)/SO(p)\mathbf{Z}_p$, $n = (p-1)(p+2)/2$.
(3) $SU(2p)/Sp(p)\mathbf{Z}_{2p}$, $n = (p-1)(2p+1)$.
(4) $E_6/F_4\mathbf{Z}_3$, $n = 26$.

Then L is a Hamiltonian stable minimal Lagrangian submanifold in $\mathbf{C}P^n$. Moreover the null space of L is exactly the span of the normal projections of Killing vector fields on $\mathbf{C}P^n$.

In our cases we see that all the eigenfunctions of the Laplacian on L with eigenvalue $c(n+1)/2$ are obtained as the restriction of the first eigenfunctions of the Laplacian on $\mathbf{C}P^n$ to L.

The submanifolds from (1) through (4) of Theorem 4.1 and the real projective space $\mathbf{R}P^n$ exhaust all compact irreducible totally real minimal submanifolds embedded in the complex projective space $\mathbf{C}P^n$ with parallel second fundamental form. Hence we state the following

THEOREM 4.2. *All n-dimensional compact totally real minimal submanifolds embedded in $\mathbf{C}P^n$ with parallel second fundamental form and positive Ricci curvature are Hamiltonian stable as minimal Lagrangian submanifolds.*

Based on our results we shall now remark on some related open problems.

PROBLEM 4.1. Is it true that all compact n-dimensional totally real submanifolds embedded in $\mathbf{C}P^n$ with parallel second fundamental form are Hamiltonian stable as H-minimal Lagrangian submanifolds?

PROBLEM 4.2. Is it true that compact Hamiltonian stable H-minimal Lagrangian submanifolds in $\mathbf{C}P^n$ have parallel second fundamental form?

PROBLEM 4.3. Is such a compact Hamiltonian stable H-minimal Lagrangian submanifold L in $\mathbf{C}P^n$ *globally* Hamiltonian stable or not, that is, volume minimizing with respect to every Hamiltonian deformation of L?

Few examples of globally Hamiltonian stable (H-)minimal Lagrangian submanifolds seems to be known in specific Kähler manifolds such as complex projective spaces, compact Hermitian symmetric spaces and so on. It is known that the real projective subspaces $RP^n \subset CP^n$ are *globally* Hamiltonian stable minimal Lagrangian submanifolds in CP^n (cf. [1] and [18]).

5. Hamiltonian stability of symmetric R-spaces canonically embedded in Hermitian symmetric spaces. In this section we discuss Hamiltonian stability of totally geodesic Lagrangian submanifolds in compact Hermitian symmetric spaces of rank greater than 1. Let M be a compact Hermitian symmetric space and L be a symmetric R-space canonically embedded in M. Such pairs represent a nice class of minimal Lagrangian submanifolds embedded in Einstein-Kähler manifolds.

A real form of a compact Hermitian symmetric space M is defined as the fixed point subset by an involutive anti-holomorphic isometry. It is a totally real and totally geodesic submanifold of M with dimension equal to the half of dim M, and hence it is a totally geodesic Lagrangian submanifold of M. In [25], Takeuchi proved that every compact totally geodesic Lagrangian submanifold of a compact Hermitian symmetric space M is obtained as a real form by an involutive anti-holomorphic isometry of M, and it is nothing but a symmetric R-space canonically embedded in a compact Hermitian symmetric space. Moreover in [25], he showed that a symmetric R-space L canonically embedded in a compact Hermitian symmetric space is stable if and only if L is simply connected. The theory of symmetric R-spaces is well-investigated and we refer [25] for a complete list of symmetric R-spaces. In [25], he also proved the following.

THEOREM 5.1 ([25]). *Let L be a symmetric R-space canonically embedded in a compact irreducible Hermitian symmetric space $M = G/K$ equipped with a metric g induced from the (-1)-times Killing-Cartan form of $\mathfrak{g} = \mathrm{Lie}(G)$. If g is an Einstein metric, then the first eigenvalue of the Laplacian of L acting on $C^\infty(L)$ is equal to $1/2$.*

Moreover, we already know that a compact irreducible Hermitian symmetric space M with a metric g induced from the (-1)-times Killing-Cartan form is an Einstein manifold with Einstein constant $1/2$. Hence Theorem 5.1 implies that, if an irreducible symmetric R-space embedded in a compact Hermitian symmetric space as a real form is an Einstein manifold, then it is a Hamiltonian stable minimal Lagrangian submanifold (see [18]). Analyzing root space of symmetric R-spaces, Takeuchi classified all irreducible symmetric R-spaces into five classes: The Hermitian and four other types (cf. [25]). The irreducible symmetric R-spaces of Hermitian type are just irreducible Hermitian symmetric spaces of compact type, which are Einstein. The Hermitian symmetric spaces of compact type canonically embedded in a Hermitian symmetric space M as a real form are always stable and hence Hamiltonian stable. By calculating the first eigenvalues of symmetric R-spaces of other types canonically embedded in a compact irreducible Hermitian symmetric space, we can determine their Hamiltonian

stability. Here we give a complete list of the Hamiltonian stability for all irreducible symmetric R-spaces of non-Hermitian type canonically embedded in compact irreducible Hermitian symmetric spaces.

M	L	Einstein	μ_1	H-stable	stable
$G_{p,q}(\boldsymbol{C}), q \geq p \geq 1$	$G_{p,q}(\boldsymbol{R})$	Yes	$1/2$	Yes	No
$G_{2p,2q}(\boldsymbol{C}), q \geq p \geq 1$	$G_{p,q}(\boldsymbol{H})$	Yes	$1/2$	Yes	Yes
$G_{m,m}(\boldsymbol{C}), m \geq 2$	$U(m)$	No	$1/2$	Yes	No
$\dfrac{SO(2m)}{U(m)}$	$SO(m), m \geq 5$	Yes	$1/2$	Yes	No
$\dfrac{SO(4m)}{U(2m)}, m \geq 3$	$\dfrac{U(2m)}{Sp(m)}$	No	$\dfrac{m}{4m-2}$	No	No
$\dfrac{Sp(2m)}{U(2m)}$	$Sp(m), m \geq 2$	Yes	$1/2$	Yes	Yes
$\dfrac{Sp(m)}{U(m)}, m \geq 3$	$\dfrac{U(m)}{O(m)}$	No	$1/2$	Yes	No
$Q_{p+q-2}(\boldsymbol{C}), q-p \geq 3$	$Q_{p,q}(\boldsymbol{R}), p \geq 2$	No	$\dfrac{p}{p+q-2}$	No	No
$Q_{2p-2}(\boldsymbol{C})$	$Q_{p,p}(\boldsymbol{R}), p \geq 2$	Yes	$1/2$	Yes	No
$Q_{2p+k-2}(\boldsymbol{C}), k = 1, 2$	$Q_{p,p+k}(\boldsymbol{R}), p \geq 2$	No	$1/2$	Yes	No
$Q_q(\boldsymbol{C}), q < 5$	$Q_{2,q}(\boldsymbol{R})$	No	$1/2$	Yes	No
$Q_{q-1}(\boldsymbol{C}), q < 4$	$Q_{1,q}(\boldsymbol{R})$	Yes	$1/2$	Yes	No
$E_6/T \cdot Spin(10)$	$P_2(\boldsymbol{K})$	Yes	$1/2$	Yes	Yes
$E_6/T \cdot Spin(10)$	$G_{2,2}(\boldsymbol{H})/\boldsymbol{Z}_2$	Yes	$1/2$	Yes	No
$E_7/T \cdot E_6$	$SU(8)/Sp(4)\boldsymbol{Z}_2$	Yes	$1/2$	Yes	No
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	No	$1/6$	No	No

Here we use the following notation:

$G_{p,q}(\boldsymbol{F})$: Grassmann manifold of all p-dimensional subspaces of $\boldsymbol{F}^{p+q}$
for $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ and a real quaternion algebra $\boldsymbol{H}$.

$\boldsymbol{P}_2(K)$: Cayley projective plane,

$Q_n(\boldsymbol{C})$: complex quadric of dimension n,

$Q_{p,q}(\boldsymbol{R}) = \{[x] \in \boldsymbol{R}P^{p+q-1} \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 0\}$
: real quadric.

Note that the third column indicates whether L is Einstein or not.

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