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MONODROMY GROUPS OF HYPERGEOMETRIC FUNCTIONS SATISFYING ALGEBRAIC EQUATIONS

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Abstract. The solutions of the algebraic equation $y^{mn} + xy^{mp} - 1 = 0$ with $n > p$ and $m \geq 2$ satisfy a generalized hypergeometric differential equation with imprimitive finite irreducible monodromy group. Thanks to this fact, we can determine the monodromy group and the Schwarz map of the differential equation.

1. Introduction. A generalized hypergeometric function

$$ {}_nF_{n-1}(a_0, a_1, a_2, \dots, a_{n-1}; b_1, b_2, \dots, b_{n-1}; z) = \sum_{k=0}^{\infty} \frac{\prod_{j=0}^{n-1}(a_j, k)}{\prod_{j=1}^{n-1}(b_j, k)k!} z^k , $$

where $(a, k) = \Gamma(a + k)/\Gamma(a)$ satisfies a Fuchsian differential equation

$$ {}_nE_{n-1}(a_0, a_1, a_2, \dots, a_{n-1}; b_1, b_2, \dots, b_{n-1}) $$

of rank n with singularities at $z = 0,\ 1$ and ∞. Beukers and Heckman [B-H] determined ${}_nE_{n-1}$ with finite irreducible monodromy groups. In [Kt], for ${}_3E_2$ with finite irreducible primitive monodromy groups, Schwarz maps of $\boldsymbol{P}^1 - \{0, 1, \infty\}$ to $\boldsymbol{P}^2$ defined by linearly independent three solutions are studied. The images of Schwarz maps and their single-valued inverse maps are determined.

1.1. As stated in Theorem 5.8 in [B-H], under some condition, ${}_nE_{n-1}$ with irreducible imprimitive monodromy group is essentially given by

(1.1)
$$ {}_nE_{n-1}\left(\frac{-\alpha}{p}, \frac{-\alpha+1}{p}, \dots, \frac{-\alpha+p-1}{p},\ \frac{\alpha}{q}, \frac{\alpha+1}{q}, \dots, \frac{\alpha+q-1}{q}; \frac{1}{n}, \dots, \frac{n-1}{n}\right), $$

where $(p, q) = 1$ and $n = p + q$.

If we put $z = (-p)^p q^q n^{-n} x^n$, the generalized binomial function (see Section 2)

$$ \psi(\alpha, -p/n, x) \tag{1.2} $$

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is (as a multi-valued function of z) a solution of (1.1). We remark that (1.2) is a typical example of quasi-hypergeometric function studied in [A-I]. If $\alpha = -1/(mn)$ with $m \geq 1$, then (1.2) is also a solution of the algebraic equation

$$y^{mn} + xy^{mp} - 1 = 0\,. \tag{1.3}$$

These facts were found by Lambert (see [Brn, p. 307]), Mellin (see [Blr]) and others.

Let $\alpha = -1/(mn)$ with $m \geq 2$. Then a set of linearly independent n solutions of (1.3) form a fundamental system of solutions of (1.1). As a consequence, we have the following results. The projective monodromy group of (1.1) is imprimitive and irreducible of order $m^{n-1}n!$ (Corollary 4.6). The closure of the image of the Schwarz map of (1.1) defined by the ratio of linearly independent n solutions is an irreducible algebraic curve projectively isomorphic to

$$\{[y_0 : y_1 : \cdots : y_{n-1}] \in \boldsymbol{P}^{n-1} \mid \sigma_k(y_0^m, y_1^m, \ldots, y_{n-1}^m) = 0,\ 1 \leq k \leq n-1,\ k \neq n-p\}\,,$$

where σ_k is the elementary symmetric function of degree k (Theorem 4.5).

1.2. As applications, we state several topics for $n = 3$ case in Section 5. We give a proof of Cardano's formula for a cubic equation, using properties of generalized binomial functions. We also give a uniformization of ${}_3E_2$ by theta functions, that is, if we put $z = J(\tau)$, the elliptic modular function, then the solutions of (1.1) with $\alpha = -1/12$, $p = 1$, $q = 2$ turn out to be single-valued functions of τ and are expressed by the zero values of theta functions.

2. Generalized binomial function. In this section, we summarize several known results which can be found in [Brn], [Blr], etc.

For any complex numbers α and s, put

$$\begin{aligned} &c_0(\alpha, s) = 1\,, \\ &c_k(\alpha, s) = \alpha(\alpha + ks + 1, k-1)/k! \quad (k \geq 1)\,, \end{aligned} \tag{2.1}$$

and define

$$\psi(\alpha, s, x) = \sum_{k=0}^{\infty} c_k(\alpha, s)x^k\,. \tag{2.2}$$

We call $\psi(\alpha, s, x)$ a generalized binomial function because $\psi(\alpha, 0, x) = (1-x)^{-\alpha}$.

We will prove some properties of $\psi(\alpha, s, x)$.

LEMMA 2.1.

$$\psi(\alpha, s, x) = \psi(-\alpha, -s-1, -x)\,. \tag{2.3}$$

PROOF.

$$\begin{aligned} &(-1)^k c_k(-\alpha, -s-1) \\ &\quad = (-1)^k(-\alpha)(-\alpha - (s+1)k + 1, k-1)/k! \\ &\quad = \alpha(\alpha + sk + k - 1)(\alpha + sk + k - 2)\cdots(\alpha + sk + 1) \\ &\quad = c_k(\alpha, s)\,. \end{aligned}$$

□

We note that $\psi(\alpha, -1, x) = (1+x)^\alpha$ and $\psi(0, s, x) = 1$.

PROPOSITION 2.2. *If none of* $\alpha, s, s+1$ *is zero, then the radius of convergence of* $\psi(\alpha, s, x)$ *is* $|s^s/(s+1)^{s+1}|$, *where* z^z *denotes the principal value.*

PROOF. Put

$$\tilde{c}_k(\alpha, s) = (\alpha + sk + 1, k-1)/k! = \frac{\Gamma(\alpha + (s+1)k)}{\Gamma(1+k)\Gamma(\alpha + 1 + sk)} .$$

Then the radius of convergence of $\psi(\alpha, s, x)$ is the reciprocal of the upper limit of $|\tilde{c}_k|^{1/k}$.

First assume that s is not a negative real number. Then, from the Stirling's formula:

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z} \quad \text{as} \quad z \to \infty \quad \text{and} \quad |\arg z| < \pi - \delta , \quad \delta > 0 ,$$

we have

$$\begin{aligned} |\tilde{c}_k(\alpha, s)|^{1/k} &\sim \frac{|(\alpha + (s+1)k)^{s+1}|}{(1+k)|(\alpha + 1 + sk)^s|} \sim \left| \frac{\alpha + (s+1)k}{1+k} \left(\frac{\alpha + (s+1)k}{\alpha + 1 + sk} \right)^s \right| \\ &\sim |(s+1)^{s+1}/s^s| . \end{aligned}$$

This proves the proposition for s which is not a negative real number.

Assume $-1 < s < 0$. For large $k \in \boldsymbol{N}$, choose $n_k \in \boldsymbol{N}$ and δ_k with $0 \leq \delta_k < 1$ such that

$$\mathrm{Re}(\alpha) + sk = -n_k - \delta_k .$$

Then

$$\begin{aligned} |\tilde{c}_k(\alpha, s)| &= |(\alpha + 1 + sk, k-1)|/k! \\ &= |(\alpha + 1 + sk) \cdots (\alpha + 1 + sk + n_k - 1)| \\ &\quad \times |(\alpha + 1 + sk + n_k) \cdots (\alpha + (s+1)k - 1)|/k! \\ &= |(-\alpha - sk - n_k, n_k)| \cdot |(\alpha + sk + n_k + 1, k - 1 - n_k)|/k! \\ &= \frac{|\Gamma(-\alpha - sk)| \cdot |\Gamma(\alpha + (s+1)k)|}{|\Gamma(1+k)\Gamma(-\alpha - sk - n_k)\Gamma(\alpha + sk + n_k + 1)|} . \end{aligned}$$

If s is a rational number, then the set $\delta := \{\delta_k \mid k \in \boldsymbol{N}\}$ is finite, otherwise δ is dense in the open interval $(0, 1)$. In any case we have

$$\begin{aligned} \limsup_{k \to \infty} |\tilde{c}_k(\alpha, s)|^{1/k} &= \lim_{k \to \infty} \left| \frac{(-\alpha - sk)^{-s} (\alpha + (s+1)k)^{s+1}}{1+k} \right| \\ &= \lim_{k \to \infty} \left| \left(\frac{-\alpha - sk}{1+k} \right)^{-s} \left(\frac{\alpha + (s+1)k}{1+k} \right)^{s+1} \right| \\ &= |(-s)^{-s} (s+1)^{s+1}| = |(s+1)^{s+1}/s^s| . \end{aligned}$$

This proves the proposition for s with $-1 < s < 0$. From Lemma 2.1, the proposition holds for any negative real number s which is not -1. This completes the proof. □

LEMMA 2.3.

(2.4) $$c_k(\alpha, s) - c_k(\alpha - 1, s) = c_{k-1}(\alpha + s, s) , \quad k \geq 1 .$$

PROOF.

$$\begin{aligned} &c_k(\alpha, s) - c_k(\alpha - 1, s) \\ &\quad = \frac{\alpha(\alpha + ks + 1, k - 1) - (\alpha - 1)(\alpha + ks, k - 1)}{k!} \\ &\quad = \frac{(\alpha + s)(\alpha + s + (k - 1)s + 1, k - 2)}{(k - 1)!} = c_{k-1}(\alpha + s, s) . \end{aligned}$$

□

PROPOSITION 2.4. *We have the following two equalities.*

$$\psi(\alpha, s, x) - \psi(\alpha - 1, s, x) = x\psi(\alpha + s, s, x) , \tag{2.5}$$

$$\psi(\alpha + \beta, s, x) = \psi(\alpha, s, x)\psi(\beta, s, x) . \tag{2.6}$$

PROOF. (2.5) follows immediately from (2.4).

Proof of (2.6). It is sufficient to prove

$$c_k(\alpha + \beta, s) = \sum_{i+j=k} c_i(\alpha, s)c_j(\beta, s) , \tag{2.7}$$

which is proved by induction for k. Consider

$$d_k(\beta) = c_k(\alpha + \beta, s) - \sum_{i+j=k} c_i(\alpha, s)c_j(\beta, s)$$

as a polynomial of β (α being a parameter) of degree at most k. From (2.4), we have

$$d_k(\beta) - d_k(\beta - 1) = d_{k-1}(\beta + s) ,$$

which vanishes by induction. Hence $d_k(\beta)$ must be constant C. Since $c_j(0, s) = 0$ for $j > 0$, we hace $C = d_k(0) = 0$. This completes the proof of (2.7) whence of (2.6). □

COROLLARY 2.5. *Let* $\psi'(s, x) = \partial\psi/\partial\alpha(0, s, x)$. *Then we have the following*:

(1) $\psi'(s, x)$ *is holomorphic in* $\{x \mid |x| < |s^s/(s + 1)^{s+1}|\}$ *with* $\psi'(s, 0) = 0$.

(2) $\psi(\alpha, s, x) = \exp(\alpha\psi'(s, x))$.

PROOF. (1) holds because $\psi'(s, x) = \sum_{k\geq 1} \tilde{c}_k(\alpha, s)x^k$, where $\tilde{c}_k(\alpha, s) = c_k(\alpha, s)/\alpha$ as in the proof of Proposition 2.2. (2) follows from (2.6). □

PROPOSITION 2.6. *Let* $\varepsilon_k = e^{2\pi i/k}$. *For positive integers* p, q *with* $n = p + q$, *the equation* (1.3) *with* $m = 1$

$$y^n + xy^p - 1 = 0 \tag{2.8}$$

has solutions

$$f_j(x) := \varepsilon_n^j \psi(-1/n, -p/n, \varepsilon_n^{pj} x) , \quad 0 \leq j \leq n - 1 , \tag{2.9}$$

in a neighborhood of $x = 0$,

$$\varepsilon_p^{-j} x^{-1/p} \psi \left(1/p, q/p, -(\varepsilon_p^{-j} x^{-1/p})^n\right) , \quad 0 \leq j \leq p - 1 , \tag{2.10}$$

$$\varepsilon_q^j (-x)^{1/q} \psi(-1/q, p/q, -(\varepsilon_q^j (-x)^{1/q})^{-n}) , \quad 0 \leq j \leq q - 1 , \tag{2.11}$$

in a neighborhood of $x = \infty$.

PROOF. Put $s = -p/n$ and $\alpha = 0$ in (2.5). Then we have

$$1 - \psi(-1, s, x) = x\psi(-p/n, s, x)\,,$$

which is equivalent to

$$\psi(-1/n, s, x)^n + x\psi(-1/n, s, x)^p - 1 = 0\,. \tag{2.12}$$

If we replace x by $\varepsilon_n^{pj}x$, we know that (2.9) are solutions of (2.8).

Put $s = q/p$ and $\alpha = 1$ in (2.5). Then we have

$$\psi(1/p, s, x)^p - 1 = x\psi(1/p, s, x)^n\,,$$

which is equivalent to

$$[(-x)^{1/n}\psi(1/p, s, x)]^n + (-x)^{-p/n}[(-x)^{1/n}\psi(1/p, s, x)]^p - 1 = 0\,.$$

Put $x_1 = (-x)^{-p/n}$, and write x instead of x_1. Then we know that functions in (2.10) are solutions of (2.8).

Now, put $s = p/q$ and $\alpha = -s$ in (2.5). Then we have

$$\psi(-1/q, s, x)^n - \psi(-1/q, s, x)^p + x = 0\,.$$

Then, by the same way as above, we know that functions in (2.11) are solutions of (2.8). This completes the proof. □

COROLLARY 2.7. *If* $\sigma_k(y_0, y_1, \dots, y_{n-1})$ *denotes the elementary symmetric function of degree* k*, then we have*

$$\sigma_k(f_0(x), f_1(x), \dots, f_{n-1}(x)) = 0\,, \quad 1 \le k \le n-2,\ k \ne n-p\,, \tag{2.13}$$

$$\sigma_{n-p}(f_0(x), f_1(x), \dots, f_{n-1}(x)) = (-1)^{n-p}x\,, \tag{2.14}$$

$$\sigma_n(f_0(x), f_1(x), \dots, f_{n-1}(x)) = (-1)^{n-1}\,. \tag{2.15}$$

For any positive integer n, put

$$\varphi_j(\alpha, s, x) = x^j \sum_{l=0}^{\infty} c_{j+ln}(\alpha, s)x^{ln}\,. \tag{2.16}$$

Then we have

$$\psi(\alpha, s, x) = \sum_{j=0}^{n-1} \varphi_j(\alpha, s, x)\,. \tag{2.17}$$

PROPOSITION 2.8. *Let $s=-p/n$ and $n=p+q$. Then we have*

$$\begin{aligned}\varphi_j(\alpha,s,x)&=c_j(\alpha,s)x^j\\ &\times {}_nF_{n-1}\biggl(\frac{-\alpha+\mu}{p}+\frac{j}{n},\ 0\le\mu\le p-1,\ \frac{\alpha+\nu}{q}+\frac{j}{n},\ 0\le\nu\le q-1;\\ &\qquad\frac{j+1}{n},\dots,\frac{n-1}{n},\frac{n+1}{n},\dots,\frac{n+j}{n};\ \frac{(-1)^p p^p q^q}{n^n}x^n\biggr).\end{aligned}\tag{2.18}$$

PROOF. If $k=nl$ $(l\ge 1)$, then we have

$$\begin{aligned}c_k(\alpha,s)&=\frac{1}{k!}\alpha(\alpha-pl+1,nl-1)=\frac{1}{k!}\alpha(\alpha-pl+1,pl-1)(\alpha,ql)\\ &=(-1)^{pl}\frac{(-\alpha,pl)(\alpha,ql)}{(1,nl)}\\ &=(-1)^{pl}\frac{p^{pl}q^{ql}\prod_{\mu=0}^{p-1}(-\alpha/p+\mu/p,l)\prod_{\nu=0}^{q-1}(\alpha/q+\nu/q,l)}{n^{nl}\prod_{\lambda=0}^{n-1}(1/n+\lambda/n,l)}.\end{aligned}$$

If $k=nl+j$ $(1\le j\le n-1)$, then we have

$$\begin{aligned}&c_k(\alpha,s)\\ &=\frac{1}{k!}\alpha\left(\alpha-\frac{p}{n}(nl+j)+1,nl+j-1\right)\\ &=\frac{1}{j!(j+1,nl)}\alpha\left(\alpha-\frac{p}{n}(nl+j)+1,pl\right)\left(\alpha-\frac{pj}{n}+1,j-1\right)\left(\alpha+\frac{qj}{n},ql\right)\\ &=\frac{\alpha(\alpha+qj/n-j+1,j-1)}{j!}(-1)^{pl}\frac{(-\alpha+pj/n,pl)(\alpha+qj/n,ql)}{(j+1,nl)}\\ &=c_j(\alpha,s)(-1)^{pl}\frac{p^{pl}q^{ql}\prod_{\mu=0}^{p-1}(-\alpha/p+j/n+\mu/p,l)\prod_{\nu=0}^{q-1}(\alpha/q+j/n+\nu/q,l)}{n^{nl}\prod_{\lambda=0}^{n-1}((j+1)/n+\lambda/n,l)}.\end{aligned}$$

This implies (2.18). □

COROLLARY 2.9. *Let $s=-p/n, n=p+q$ and $\varepsilon_n=e^{2\pi i/n}$. Then $\psi(\alpha,s,\varepsilon_n^k x)$ is, as a multi-valued function of $z=(-p)^p q^q n^{-n}x^n$, a solution of the differential equation (1.1). If $c_j(\alpha,s)\ne 0$ for $0\le j\le n-1$, then $\psi(\alpha,s,\varepsilon_n^k x)$ $0\le k\le n-1$ are linearly independent.*

PROOF. From (2.18), we know that $\varphi_j(\alpha, s, x)$ is a solution of (1.1) (see the lemma below). From (2.16) and (2.17), we have

$$\psi(\alpha, s, \varepsilon_n^k x) = \sum_{j=0}^{n-1} \varepsilon_n^{jk} \varphi_j(\alpha, s, x)\,, \tag{2.19}$$

which is thus a solution of (1.1). If $c_j(\alpha, s) \neq 0$, then $\varphi_j(\alpha, s, x) \neq 0$ and $\psi(\alpha, s, \varepsilon_n^k x)$, $0 \leq k \leq n-1$, are linearly independent from (2.19). □

The following lemma is well-known.

LEMMA 2.10. *If $b_0 = 1$, then the differential equation*

$${}_nE_{n-1}(a_0, a_1, a_2, \dots, a_{n-1}; b_1, b_2, \dots, b_{n-1})$$

has solutions

$$\begin{aligned} &z^{1-b_j}{}_nF_{n-1}(a_0 + 1 - b_j, \dots, a_{n-1} + 1 - b_j\,; \\ &\qquad b_0 + 1 - b_j, \dots, \widehat{b_j + 1 - b_j}, \dots, b_{n-1} + 1 - b_j; z);\ 0 \leq j \leq n-1 \end{aligned}$$

at $z = 0$ and

$$\begin{aligned} &z^{-a_j}{}_nF_{n-1}(a_j + 1 - b_0, \dots, a_j + 1 - b_{n-1}\,; \\ &\qquad a_j + 1 - a_0, \dots, \widehat{a_j + 1 - a_j}, \dots, a_j + 1 - a_{n-1}; 1/z);\ 0 \leq j \leq n-1 \end{aligned}$$

at $z = \infty$.

PROOF. ${}_nE_{n-1}$ is defined by

$$\left[\prod_{j=0}^{n-1}(\vartheta + b_j - 1) - z \prod_{j=0}^{n-1}(\vartheta + a_j)\right]u = 0\,, \tag{2.20}$$

where $\vartheta = z\partial/\partial z$ (see [Bly]). It is easily verified that functions in Lemma satisfy (2.20). □

REMARK 2.1. If $s = p/q$ with $n = p + q$, then we have, for $0 \leq j \leq q-1$,

$$\begin{aligned} \varphi_j(\alpha, s, x) &= x^j \sum_{l=0}^{\infty} c_{j+lq}(\alpha, s) x^{lq} \\ &= c_j(\alpha, s) x^j {}_nF_{n-1}\left(\frac{\alpha}{n} + \frac{j}{q}, \frac{\alpha+1}{n} + \frac{j}{q}, \dots, \frac{\alpha+n-1}{n} + \frac{j}{q};\right. \\ &\quad \left.\frac{\alpha+1}{p} + \frac{j}{q}, \dots, \frac{\alpha+p}{p} + \frac{j}{q}, \frac{1+j}{q}, \dots, \frac{q-1}{q}, \frac{q+1}{q}, \dots, \frac{q+j}{q};\ \frac{n^n}{p^p q^q} x^q\right). \end{aligned}$$

3. Global properties of solutions of $y^n + xy^p - 1 = 0$. Assume $s(s+1) \neq 0$. Put $\Delta(s) = \{x \mid |x| < |s^s/(s+1)^{s+1}|\}$. Then $\psi(\alpha, s, x)$ and $\psi'(s, x) = \partial\psi/\partial\alpha(0, s, x)$ are holomorphic in $\Delta(s)$ (Proposition 2.2 and Corollary 2.5).

LEMMA 3.1. *Assume $s \in \boldsymbol{R}$. Then we have $|\arg \psi(-1,s,x)| < \pi$, or equivalently, $|\mathrm{Im}\, \psi'(s,x)| < \pi$ in $\Delta(s)$.*

PROOF. Assume $|\mathrm{Im}\, \psi'(s,x_1)| = \pi$ for some $x_1 \in \Delta(s)$. From (2.5) and (2) of Corollary 2.5, we have

$$\exp(-s\psi'(s,x_1))(1-\exp(-\psi'(s,x_1))) = x_1\,.$$

This implies $\theta := \arg x_1 = (\pm s + 2n)\pi$ for some $n \in \boldsymbol{Z}$. Since $\mathrm{Im}\, \psi'(s,0) = 0$, there exist a positive number t_0 ($\leq |x_1|$) such that

$$|\mathrm{Im}\, \psi'(s, te^{i\theta})| < \pi \text{ for } 0 < t < t_0 \text{ and } |\mathrm{Im}\, \psi'(s, t_0 e^{i\theta})| = \pi\,.$$

Put $x_0 = t_0 e^{i\theta}$ and $b_0 = \psi(-1,s,x_0)$ (< 0). Since $y = \psi(-1,s,x)$ defines an open map, $\psi(-1,s,e^{i\theta}t)$ maps some open interval $(t_0-\delta, t_0+\delta)$ onto some open interval $(b_0-\delta', b_0+\delta'')$. This contradicts the choice of t_0. □

We assume $(p,q)=1$ and put $n = p+q$. Recall that $f_j(x)$, $0 \leq j \leq n-1$ given by (2.9) are the solutions of the equation (2.8). The equation (2.8) has multiple roots at

$$x_j := e\left(\frac{-p(1+2j)}{2n}\right)(p/n)^{-p/n}(q/n)^{-q/n}, \quad 0 \leq j \leq n-1\,, \tag{3.1}$$

where $e(x) = e^{2\pi i x}$ and at $x = \infty$. Note that $x = x_j$ are solutions of

$$(-p)^p q^q n^{-n} x^n = 1\,.$$

LEMMA 3.2. *At $x = x_j$, the equation* (2.8) *has a double root*

$$e((1+2j)/2n)(p/q)^{1/n} \tag{3.2}$$

and $n-2$ simple roots.

PROOF. The double root of the equation (2.8) is uniquely determined by (2.8) and $ny^{n-1} + pxy^{p-1} = 0$. □

We know that $f_j(x)$ are holomorphic in $\Delta := \Delta(-p/n) = \{x \mid |x| < (p/n)^{-p/n}(q/n)^{-q/n}\}$ and continuous in the closure $\bar{\Delta}$ of Δ.

Put

$$D_j = f_j(\bar{\Delta})\,. \tag{3.3}$$

Then we have $D_j = e(j/n)D_0$ and put $D_n = D_0$.

LEMMA 3.3.

$$\left(\frac{-1+2j}{n}\right)\pi \leq \arg y \leq \left(\frac{1+2j}{n}\right)\pi \quad \textit{for} \ \ y \in D_j\,, \tag{3.4}$$

$$D_j \cap D_{j+1} = \{f_j(x_j)\} = \{f_{j+1}(x_j)\} = \{e((1+2j)/2n)(p/q)^{1/n}\}\,, \tag{3.5}$$

and $D_j \cap D_k = \emptyset$ if $j-k \neq \pm 1$.

PROOF. The inequalities (3.4) follow from Lemma 3.1 and (2) of Corollary 2.5. These inequalities imply that $D_j \cap D_k = \emptyset$ if $j - k \neq \pm 1$. Since any element of $D_j \cap D_{j+1}$ is one of (3.2), we have

$$D_j \cap D_{j+1} = \{e((1+2j)/2n)(p/q)^{1/n}\}$$

from (3.4). From Lemma 3.2, (3.5) follows. □

COROLLARY 3.4. *Let γ_0 be a loop starting and ending at the origin and once surrounding x_0. Let $\gamma_j = e(-pj/n)\gamma_0$. Then, by the analytic continuation along γ_j, $f_j(x)$ and $f_{j+1}(x)$ are interchanged and other $f_k(x)$ are unchanged.*

PROOF. Assume γ_0 (hence any γ_j) acts trivially on $\{f_0, \dots, f_{n-1}\}$. Then $f_j(x)$ are entire functions. This contradicts Proposition 2.6. □

DEFINITION 3.1. Let E be a Fuchsian linear differential equation of rank n on $\boldsymbol{P}^1$. Let $Z = \boldsymbol{P}^1 - \{\text{singular points of } E\}$. Fix a base point $z_b \in Z$, and let V be the set of germs of holomorphic solutions of E at z_b. For any $\gamma \in \pi_1(Z, z_b)$ and $f \in V$, the analytic continuation $\gamma_* f$ of f along γ is again in V. We consider γ_* an element of $GL(V)$ and call the set $M(E)$ of all γ_* the *monodromy group* of E and $M(E)/(\text{its center})$ the *projective monodromy group* of E.

We say that $M(E)$ is (or E is) *reducible* if there exists a non trivial subspace V_1 of V which is invariant under the action of $M(E)$ and say $M(E)$ is (or E is) *irreducible* if $M(E)$ is not reducible.

We say that $M(E)$ is (or E is) *imprimitive* if V has a direct sum decomposition $V = V_1 + V_2 + \cdots + V_k$ such that any element of $M(E)$ induces a permutation of $\{V_1, V_2, \dots, V_k\}$.

Choose a basis $v_j(z),\ 1 \leq j \leq n$ of V. Then we have a holomorphic map

$$v(z) = [v_1(z) : v_2(z) : \cdots : v_n(z)]$$

of a neighbourhood of z_b into $\boldsymbol{P}^{n-1}$. By taking analytic continuations of v, we have a multi-valued map (again denoted by) v of Z into $\boldsymbol{P}^{n-1}$ which we call a *Schwarz map* of E.

Remark 3.1. If the Schwarz map has a single-valued inverse map π_E, then the projective monodromy group of E is isomorphic to the covering transformation group of π_E.

The map of Δ to $\boldsymbol{P}^{n-1}$ defined by $[f_0(x) : f_1(x) : \cdots : f_{n-1}(x)]$ is extended to a multi-valued map of $\boldsymbol{C} - \{x_0, \dots, x_{n-1}\}$ to $\boldsymbol{P}^{n-1}$ by the analytic continuations. Take the closure of its image in $\boldsymbol{P}^{n-1}$, which we denote by $X_{n,p}$.

PROPOSITION 3.5. *Let $\sigma_k(y) = \sigma_k(y_0, y_1, \dots, y_{n-1})$ be the elementary symmetric function of degree k. Then we have the equality*

$$X_{n,p} = \{[y_0 : y_1 : \cdots : y_{n-1}] \in \boldsymbol{P}^{n-1} \mid \sigma_k(y) = 0,\ 1 \leq k \leq n-1,\ k \neq q\}. \tag{3.6}$$

Put

$$\pi_{n,p}([y_0 : y_1 : \cdots : y_{n-1}]) = (-1)^n \frac{p^p q^q}{n^n} \frac{(\sigma_q(y_0, \dots, y_{n-1}))^n}{(\sigma_n(y_0, \dots, y_{n-1}))^q}. \tag{3.7}$$

Then $z = \pi_{n,p}(y)$ *defines an* $n! : 1$ *rational map of* $X_{n,p}$ *to* $\boldsymbol{P}^1$ *satisfying*

$$\pi_{n,p}([f_0(x) : f_1(x) : \cdots : f_{n-1}(x)]) = (-p)^p q^q n^{-n} x^n \,. \tag{3.8}$$

The branch points of this map are $z = 0, 1, \infty$ *with the ramification indices* $n, 2, pq$, *respectively. The covering transformation group is isomorphic to the symmetric group* S_n *of order* $n!$.

PROOF. Denote by $\hat{X}_{n,p}$ the set of common zeros of $\sigma_k,\ 0 \le k \le n-2,\ k \ne q$. From Bezout's theorem, $\pi_{n,p}|_{\hat{X}_{n,p}}$ is an $n! : 1$ map of $\hat{X}_{n,p}$ to $\boldsymbol{P}^1$. From Corollary 2.7, we have $X_{n,p} \subset \hat{X}_{n,p}$, that is, $X_{n,p}$ is an irreducible component of $\hat{X}_{n,p}$. From Corollary 2.7, (3.8) holds and from Corollary 3.4, we know that S_n acts on each fiber of $\pi_{n,p}|_{X_{n,p}}$. Consequently, we must have $\hat{X}_{n,p} = X_{n,p}$.

The equality (3.8) implies that the ramification index is n at $z = 0$. From Corollary 3.4, the index at $z = 1$ is 2. From Proposition 2.6, we know that the ramification index at $z = \infty$ is pq. This completes the proof. □

The statement (2) of the following corollary is proved in [B-H, Proposition 2.6].

COROLLARY 3.6. (1) *If* $p < n-1$, *then* $\psi(-1/n, -p/n, \varepsilon_n^k x)$, $0 \le k \le n-1$, *are solutions of a differential equation* ${}_{n-1}E_{n-2}$, *the projective monodromy group of which is isomorphic to the symmetric group* S_n *of order* $n!$. *Any* $n-1$ *of the above solutions are linearly independent.*

(2) *The projective monodromy group of*

$$ {}_{n-1}E_{n-2}\left(\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n};\ \frac{1}{p}, \ldots, \frac{p-1}{p}, \frac{1}{q}, \ldots, \frac{q-1}{q}\right) \tag{3.9}$$

is isomorphic to S_n.

PROOF. Proof of (1). Assume $p < n-1$ or equivalently $q > 1$. Put $\alpha = -1/n$ and $s = -p/n$. Let q^* be the integer such that

$$1 \le q^* \le n-1 \quad \text{and} \quad qq^* \equiv 1 \bmod n \,.$$

Then $p^* := n - q^*$ also satisfies $pp^* \equiv 1 \bmod n$. For $k = p$ or q, put $d_k = (kk^* - 1)/n$. Note $q^* > 1$ and $d_q > 0$ because $q > 1$. We easily have $c_{q^*}(\alpha, s) = 0$, and hence $\varphi_{q^*}(\alpha, s, x) = 0$ (see Proposition 2.8). Since

$$(-\alpha + d_p)/p = (\alpha + q - d_q)/q = p^*/n \,,$$

we have

$$\begin{aligned} &\varphi_0(\alpha, s, x) \\ &\quad = {}_{n-1}F_{n-2}\Bigg(\frac{-\alpha}{p}, \ldots, \frac{-\alpha+p-1}{p}, \frac{\alpha}{q}, \ldots, \widehat{\frac{\alpha+q-d_q}{q}}, \ldots, \frac{\alpha+q-1}{q}; \\ &\qquad\qquad \frac{n-1}{n}, \ldots, \widehat{\frac{p^*}{n}}, \ldots, \frac{1}{n}; z\Bigg), \end{aligned}$$

where $z = (-1)^p p^p q^q n^{-n} x^n$ as before. By the same way, we know that $\{\varphi_j \mid 0 \le j \le n-1,\, j \ne q^*\}$ forms a system of fundamental solutions of

$$ \tag{3.10} {}_{n-1}E_{n-2}\Big(\frac{-\alpha}{p}, \dots, \frac{-\alpha+p-1}{p}, \frac{\alpha}{q}, \dots, \frac{\widehat{\alpha+q-d_q}}{q}, \dots, \frac{\alpha+q-1}{q}; \frac{n-1}{n}, \dots, \frac{\widehat{p^*}}{n}, \dots, \frac{1}{n}\Big). $$

The equalities (2.19) imply that $\psi(-1/n, -p/n, \varepsilon_n^k x)$, $0 \le k \le n-1$, are solutions of (3.10) and moreover any $n-1$ of them are linearly independent. Since the projective monodromy group of (3.10) is isomorphic to the covering transformation group of $\pi_{n,p}$, which is isomorphic to S_n from Proposition 3.5. This completes the proof of (1).

Proof of (2). In (3.9), p and q are symmetric so that we can remain the assumption of $p < n-1$. Put $r = (-\alpha + d_p)/p = (\alpha + q - d_q)/q = p^*/n$. Then, from Lemma 2.10, the equation (3.10) has the special solution

$$ z^{-r}{}_{n-1}F_{n-2}\Big(r, r+\frac{1}{n}, \dots, \widehat{r+\frac{q^*}{n}}, \dots, r+\frac{n-1}{n};\ 1+\frac{d_p}{p}, \dots, 1+\frac{1}{p}, \frac{p-1}{p}, \dots, \frac{1+d_p}{p}, 1+\frac{q-d_q}{q}, \dots, 1+\frac{1}{q}, \frac{q-1}{q}, \dots, \frac{q-d_q-1}{q}; 1/z\Big). $$

Thus the projective monodromy groups of (3.9) and (3.10) are mutually isomorphic, proving (2). This completes the proof. □

4. Schwarz map of a family of imprimitive ${}_nE_{n-1}$.

Assume $(p,q) = 1$ and put

$$ n = p+q\,, \quad s = -p/n\,, \quad z = (-p)^p q^q n^{-n} x^n\,, \quad \varepsilon_k = e(1/k) = e^{2\pi i/k}\,. $$

For an integer $m \ge 2$, put $\alpha = -1/(mn)$ and define

$$ \tag{4.1} f_j^{(1/m)}(x) = \varepsilon_{mn}^j \psi(\alpha, s, \varepsilon_n^{pj} x) \quad 0 \le j \le n-1\,, $$

which is a m-th root of $f_j(x)$. The following lemma is an immediate consequence of the definition (4.1) of $f_j^{(1/m)}$.

LEMMA 4.1. *We have*

$$ f_j^{(1/m)}(e(p/n)x) = e(-1/(mn)) f_{j+1}^{(1/m)}(x), \quad \textit{for } 0 \le j \le n-2\,, $$
$$ f_{n-1}^{(1/m)}(e(p/n)x) = e((n-1)/(mn)) f_0^{(1/m)}(x)\,. $$

When we consider $f_j^{(1/m)}(x)$ as a multi-valued function of z, we denote it by $f_j^{(1/m)}(z)$.

LEMMA 4.2. $f_j^{(1/m)}(z)$, $0 \le j \le n-1$, *are linearly independent solutions of differential equation* (1.1).

PROOF. Since $c_j(\alpha, s) \ne 0$, for $0 \le j \le n-1$, Corollary 2.9 proves the lemma. □

Similar to (3.3), we put

$$D_j^{(1/m)} = f_j^{(1/m)}(\bar{\Delta}).$$

Then we have $D_j^{(1/m)} = e(j/(mn))D_0^{(1/m)}$ and can prove the following lemma and its corollary from Lemma 3.3 and Corollary 3.4.

LEMMA 4.3.

$$\begin{aligned} D_j^{(1/m)} \cap D_{j+1}^{(1/m)} &= \{f_j^{(1/m)}(x_j)\} = \{f_{j+1}^{(1/m)}(x_j)\} \\ &= \{e((1+2j)/(2mn))(p/q)^{1/n}\},\ 0 \le j \le n-2, \\ D_{n-1}^{(1/m)} \cap e(1/m)D_0^{(1/m)} &= \{f_{n-1}^{(1/m)}(x_{n-1})\} = \{e(1/m)f_0^{(1/m)}(x_{n-1})\} \\ &= \{e((2n-1)/(2mn))(p/q)^{1/n}\}. \end{aligned}$$

COROLLARY 4.4. *Let γ_j be the loop defined in Corollary 3.4. For $0 \le j \le n-2$, by the analytic continuation along γ_j, $f_j^{(1/m)}(x)$ and $f_{j+1}^{(1/m)}(x)$ are interchanged and other $f_k^{(1/m)}(x)$ are unchanged; by that along γ_{n-1}, $f_{n-1}^{(1/m)}(x)$ and $e(1/m)f_0^{(1/m)}(x)$ are interchanged and other $f_k^{(1/m)}(x)$ are unchanged.*

From Lemma 4.2, a Schwarz map of (1.1) is given by

$$z \in \boldsymbol{P}^1 - \{0,1,\infty\} \longmapsto [f_0^{(1/m)}(z) : f_1^{(1/m)}(z) : \cdots : f_{n-1}^{(1/m)}(z)]. \tag{4.2}$$

We denote the closure of its image by $X_{n,p}^{(1/m)}$, which is an irreducible curve in $\boldsymbol{P}^{n-1}$.

THEOREM 4.5. *Assume $(p,q)=1$ and put $n=p+q$, $s=-p/n$ and $\alpha = -1/(mn)$, $m \ge 2$. Then we have the equality*

$$X_{n,p}^{(1/m)} = \{[y_0 : y_1 : \cdots : y_{n-1}] \in \boldsymbol{P}^{n-1} \mid \sigma_k(y_0^m, y_1^m, \ldots, y_{n-1}^m) = 0, 1 \le k \le n-1,\ k \ne q\}, \tag{4.3}$$

where σ_k is the elementary symmetric function of degree k. Put

$$\pi_{n,p}^{(1/m)}([y_0 : y_1 : \cdots : y_{n-1}]) = (-1)^n \frac{p^p q^q}{n^n} \frac{\left(\sigma_q(y_0^m, y_1^m, \ldots, y_{n-1}^m)\right)^n}{\left(\sigma_n(y_0^m, y_1^m, \ldots, y_{n-1}^m)\right)^q}. \tag{4.4}$$

Then $z = \pi_{n,p}^{(1/m)}(y)$ defines an $m^{n-1}n! : 1$ rational map of $X_{n,p}^{(1/m)}$ to $\boldsymbol{P}^1$ satisfying

$$\pi_{n,p}^{(1/m)}([f_0^{(1/m)}(x) : f_1^{(1/m)}(x) : \cdots : f_{n-1}^{(1/m)}(x)]) = (-p)^p q^q n^{-n} x^n. \tag{4.5}$$

The branch points of this map are $z = 0, 1, \infty$ with ramification indices $n, 2, mpq$, respectively.

PROOF. We denote the right hand side of (4.3) by $\hat{X}_{n,p}^{(1/m)}$ for the moment. Since

$$(f_j^{(1/m)}(x))^m = f_j(x),$$

we have, from Proposition 3.5, $X_{n,p}^{(1/m)} \subset \hat{X}_{n,p}^{(1/m)}$. By Bézout's theorem, $\pi_{n,p}^{(1/m)}$ is an $m^{n-1}n!$: 1 map of $\hat{X}_{n,p}^{(1/m)}$ to $\boldsymbol{P}^1$ and from (3.8) it satisfies (4.5). On the other hand, $\pi_{n,p}^{(1/m)}$ restricted to $X_{n,p}^{(1/m)}$ has $m^{n-1}n!$ points in any generic fiber because the covering transformation group of $X_{n,p}^{(1/m)}$ includes S_n from Corollary 4.4 and multiplication of $e(1/m)$ to coordinate y_{n-1} from Lemma 4.1. Hence we have $X_{n,p}^{(1/m)} = \hat{X}_{n,p}^{(1/m)}$. The ramification index at $z = \infty$ is mpq from Proposition 2.6. This completes the proof. □

COROLLARY 4.6. *Let* $\alpha = -1/(mn)$, $m \geq 2$, *then the differential equation* (1.1) *has imprimitive finite irreducible projective monodromy group of order* $m^{n-1}n!$.

PROOF. The order of the projective monodromy group of (1.1) is equal to the degree of $\pi_{n,p}^{(1/m)}$, which is $m^{n-1}n!$ from the above theorem. Let Γ_0 and Γ_1 be loops once surrounding $z = 0$ and $z = 1$, respectively. From Lemma 4.1 and Corollary 4.4, both Γ_0 and Γ_1 induce permutations on the set $\{\langle f_j^{(1/m)} \rangle \mid 0 \leq j \leq n-1\}$ of one dimensional subspaces $\langle f_j^{(1/m)} \rangle$. Hence the monodromy group of (1.1) is imprimitive.

Since neither $(-\alpha + k)/p - l/n$ nor $(\alpha + k)/q - l/n$ is an integer for any integers k and l, (1.1) is irreducible from Proposition 3.3 of [B-H]. □

COROLLARY 4.7. *For any positive integer* m, n *and* q *satisfying* $1 \leq q \leq n-1$ *and* $(n, q) = 1$, *the algebraic set*

$$\{[y_0 : y_1 : \cdots : y_{n-1}] \in \boldsymbol{P}^{n-1} \mid \sigma_k(y_0^m, y_1^m, \ldots, y_{n-1}^m) = 0,\ 1 \leq k \leq n-1,\ k \neq q\}$$

is irreducible.

PROOF. The statement is true for $m = 1$ from Proposition 3.5 and for $m \geq 2$ from Theorem 4.4. □

5. $\psi(\alpha, -1/3, x)$**.** In this section, we give several results concerning to $\psi(\alpha, -1/3, x)$.

5.1. A proof of Cardano's formula.

LEMMA 5.1.

$$\psi(-1/2, -1/2, x) = \frac{-x + \sqrt{x^2 + 4}}{2}, \tag{5.1}$$

$$\psi(-1, 1, x) = \frac{1 + \sqrt{1 - 4x}}{2}. \tag{5.2}$$

PROOF. From (2.17) and (2.18), we have

$$\psi(-1/2, -1/2, x) = {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; -\frac{1}{4}x^2\right) - \frac{1}{2}x\, {}_2F_1\left(1, 0; \frac{3}{2}; -\frac{1}{4}x^2\right).$$

Since ${}_2F_1(a, b; b; x) = (1 - x)^{-a}$, (5.1) is proved.

If $k \geq 1$, then we have

$$\begin{aligned} c_k(-1,1) &= -(k,k-1)/k! \\ &= -k(k+1)\cdots(2k-2)/k! = -(2k-2)!/(k!(k-1)!) \\ &= -1\cdot 3\cdots(2k-3)2^{k-1}/k! = -(1/2,k-1)2^{2k-2}/k! \\ &= (-1/2,k)4^k/(2k!)\,. \end{aligned}$$

Hence we have (5.2). □

LEMMA 5.2.

$$\begin{aligned} &\psi(-1/3,-1/3,x) \\ &= \left(\frac{1}{2}\left(1+\frac{4}{27}x^3\right)^{1/2}+\frac{1}{2}\right)^{1/3} - \frac{1}{3}x\left(\frac{1}{2}\left(1+\frac{4}{27}x^3\right)^{1/2}+\frac{1}{2}\right)^{-1/3} \\ &= \left(\frac{1}{2}\left(1+\frac{4}{27}x^3\right)^{1/2}+\frac{1}{2}\right)^{1/3} - \left(\frac{1}{2}\left(1+\frac{4}{27}x^3\right)^{1/2}-\frac{1}{2}\right)^{1/3}, \end{aligned} \tag{5.3}$$

where cube roots take positive values if x is a positive small number.

PROOF. From (2.17) and (2.18), we have

$$\begin{aligned} &\psi(-1/3,-1/3,x) \\ &= {}_3F_2\left(\frac{1}{3},-\frac{1}{6},\frac{1}{3};\frac{2}{3},\frac{1}{3};-\frac{4}{27}x^3\right) - \frac{1}{3}x\,{}_3F_2\left(\frac{2}{3},\frac{1}{6},\frac{2}{3};\frac{4}{3},\frac{2}{3};-\frac{4}{27}x^3\right) \\ &= {}_2F_1\left(-\frac{1}{6},\frac{1}{3};\frac{2}{3};-\frac{4}{27}x^3\right) - \frac{1}{3}x\,{}_2F_1\left(\frac{1}{6},\frac{2}{3};\frac{4}{3};-\frac{4}{27}x^3\right), \end{aligned}$$

which is equal to, from Remark 2.1,

$$\begin{aligned} &\varphi_0(-1/3,1/1;-x^3/27) - 1/3\,x\,\varphi_0(1/3,1/1;-x^3/27) \\ &=\psi(-1/3,1;-x^3/27) - 1/3\,x\,\psi(1/3,1;-x^3/27) \\ &=[\psi(-1,1;-x^3/27)]^{1/3} - 1/3\,x\,[\psi(-1,1;-x^3/27)]^{-1/3} \\ &=\left[\frac{1+\sqrt{1+4x^3/27}}{2}\right]^{1/3} - \frac{1}{3}x\left[\frac{1+\sqrt{1+4x^3/27}}{2}\right]^{-1/3} \end{aligned}$$

due to (5.2). This proves the lemma. □

THEOREM 5.3 (Cardano). *The equation*

$$X^3+3pX-2q=0$$

has roots

$$\varepsilon_3^m\big(q+\sqrt{p^3+q^2}\,\big)^{1/3}+\varepsilon_3^{2m}\big(q-\sqrt{p^3+q^2}\,\big)^{1/3}, \quad 0\leq m\leq 2\,, \tag{5.4}$$

where $\varepsilon_3=e^{2\pi i/3}$ and cube roots must be chosen such that

$$\big(q+\sqrt{p^3+q^2}\,\big)^{1/3}\big(q-\sqrt{p^3+q^2}\,\big)^{1/3}=-p\,. \tag{5.5}$$

PROOF. Theorem follows from Lemma 5.2 and Proposition 2.6. □

5.2. A uniformization of $\psi(-1/12, -1/3, x)$.

LEMMA 5.4. *Let $s = -p/n$. Then for any α, we have*

$$\prod_{j=0}^{n-1} \psi(\alpha, s, \varepsilon_n^j x) = 1\,. \tag{5.6}$$

PROOF. From (2.19), we have

$$\psi(\alpha, s, \varepsilon_n^j x) = \sum_{k=0}^{n-1} \varepsilon_n^{jk} \varphi_k(\alpha, s, x)\,.$$

First we note

$$\varphi_0(0, s, x) = 1\,, \quad \frac{\partial \varphi_0}{\partial \alpha}(0, s, x) = 0 \quad \text{and} \quad \varphi_k(0, s, x) = 0 \quad \text{for } k \geq 1\,.$$

Put $f(\alpha) = \prod_{j=0}^{n-1} \psi(\alpha, s, \varepsilon_n^j x)$. Then $f(0) = 1$ and

$$\begin{aligned}
\frac{df}{d\alpha}\bigg|_{\alpha=0} &= \sum_{k=0}^{n-1} \frac{\partial \psi}{\partial \alpha}(\alpha, s, \varepsilon_n^k x) \prod_{j \neq k} \psi(\alpha, s, \varepsilon_n^j x)\bigg|_{\alpha=0} = \sum_{k=0}^{n-1} \frac{\partial \psi}{\partial \alpha}(\alpha, s, \varepsilon_n^k x)\bigg|_{\alpha=0} \\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \varepsilon_n^{jk} \frac{\partial \varphi_j}{\partial \alpha}\bigg|_{\alpha=0} = \left(\sum_{j=1}^{n-1} \frac{\partial \varphi_j}{\partial \alpha}\bigg|_{\alpha=0}\right)\left(\sum_{k=0}^{n-1} \varepsilon_n^{jk}\right) \\
&= 0.
\end{aligned}$$

Since $f(\alpha + \beta) = f(\alpha) f(\beta)$, we have $f(\alpha) = f(0) \exp(\alpha\, df(0)/d\alpha)$. This proves (5.6). □

Let $\alpha = -1/(3m)$ and put $y_j = f_j^{(1/m)}(\alpha, -1/3, x)$ for $j = 0, 1, 2$ (as for $f_j^{(1/m)}$, see (4.1)). Then, from (4.3), (4.4) and (4.5), we have

$$y_0^m + y_1^m + y_2^m = 0\,, \quad \pi_{3,1}^{(1/m)}([y_0 : y_1 : y_2]) = \frac{(y_0^{2m} + y_1^{2m} + y_2^{2m})^3}{54(y_0 y_1 y_2)^{2m}} = -\frac{4}{27}x^3\,. \tag{5.7}$$

Let

$$J(\tau) = 12^{-3} h^{-2}(1 + 744h^2 + 196884h^4 + 21493760h^6 + \cdots)\,, \quad h = e^{\pi i \tau}$$

be the elliptic modular function defined on the upper half plane.

LEMMA 5.5. *On the upper half plane $\{\tau \mid \mathrm{Im}(\tau) > 0\}$, we have a single-valued function $x = x(\tau)$ so that $J(\tau) = -4x^3/27$ and that $x \geq 0$ for $\tau = e(1/3) + ti$ with $t \geq 0$.*

PROOF. The assertion holds because $J(\tau) \leq 0$ on the half line $\tau = e(1/3) + ti$ with $t \geq 0$ and because $J(\tau)$ has only triple zeros. □

Now we have the following theorem.

THEOREM 5.6. *Let* $m = 4, n = 3, p = 1$ *and* $\alpha = -1/(mn)$, $s = -p/n$. *Let* $f_j^{(1/m)}(x)$, $j = 0, 1, 2$ *be solutions of* (1.3) *defined by* (4.1). *Let* $x = x(\tau)$ *be the single-valued function in the previous lemma. Then we have*

$$
\begin{aligned}
&f_0^{(1/4)}(x(\tau)) = C\vartheta_2(0,\tau)\,, \quad f_1^{(1/4)}(x(\tau)) = C\vartheta_0(0,\tau)\,,\\
&f_2^{(1/4)}(x(\tau)) = e(1/8)C\vartheta_3(0,\tau)\,,
\end{aligned} \tag{5.8}
$$

where $h = e^{\pi i \tau}$, $H_0 = \prod_{k=1}^{\infty}(1 - h^{2k})$ *and* $C = 2^{-1/3}e(1/24)h^{-1/12}H_0^{-1}$.

PROOF. Let $C_4 = \{[y_0 : y_1 : y_2] \in \boldsymbol{P}^2 \,|\, y_0^4 + y_1^4 + y_2^4 = 0\}$. Then

$$\pi_{3,1}^{(1/4)} : C_4 \to \boldsymbol{P}^1$$

satisfies, from (5.7),

$$\pi_{3,1}^{(1/4)}([y_0 : y_1 : y_2]) = \frac{(y_0^8 + y_1^8 + y_2^8)^3}{54(y_0y_1y_2)^8}\,.$$

It is well-known (see, for example [Akh]) that

$$\pi_{3,1}^{(1/4)}([\vartheta_2(0,\tau) : \vartheta_0(0,\tau) : e(1/8)\vartheta_3(0,\tau)]) = J(\tau)\,. \tag{5.9}$$

This together with the equality (5.6) implies that both

$$[f_0^{(1/4)} : f_1^{(1/4)} : f_2^{(1/4)}] \quad \text{and} \quad [\vartheta_2(0,\tau) : \vartheta_0(0,\tau) : e(1/8)\vartheta_3(0,\tau)]$$

belong to the same fiber $(\pi_{3,1}^{(1/4)})^{-1}(J(\tau))$. Hence for some fourth roots $\varepsilon, \varepsilon'$ of 1 and some function $C' = C'(\tau)$, we have

$$\{f_0^{(1/4)}, f_1^{(1/4)}, f_2^{(1/4)}\} = \{C'\vartheta_2(0,\tau), C'\varepsilon\vartheta_0(0,\tau), C'\varepsilon' e(1/8)\vartheta_3(0,\tau)\}\,.$$

If we put $\tau = (-1 + \sqrt{3}i)/2 + ti$ and let t to $+\infty$, then $z = J(\tau) < 0$ goes to $-\infty$. Since, from (5.3),

$$f_j^{(1/4)} = \varepsilon_{12}^j 2^{-1/12}((\sqrt{1 - J(\tau)}) + 1)^{1/3} - \varepsilon_3^j(\sqrt{1 - J(\tau)} - 1)^{1/3})^{1/4}\,,$$

we have (5.8) for some function $C = C(\tau)$ of τ. Since $\vartheta_2(0,\tau)\vartheta_0(0,\tau)\vartheta_3(0,\tau) = 2h^{1/4}H_0^3$ ([Akh]), C takes the value in the statement of the theorem. □

REMARK 5.1. We dealt with the case of $m = 4$ because we used the identity

$$\vartheta_0^4(0,\tau) + \vartheta_2^4(0,\tau) - \vartheta_3^4(0,\tau) = 0$$

in the proof.

COROLLARY 5.7. *Let a multi-valued function* $f(z)$ *be a solution of*

$${}_3E_2(1/12, -1/24, 11/24; 1/3, 2/3)\,.$$

Then $f(J(\tau))$ *turns out to be single-valued and a linear combination of* $C\vartheta_j(0,\tau)$, $j = 0, 2, 3$, *where* C *is as in Theorem* 5.6.

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