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UNIQUENESS PROBLEM OF MEROMORPHIC MAPPINGS IN SEVERAL COMPLEX VARIABLES FOR MOVING TARGETS

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Abstract. Nevanlinna showed that for two nonconstant meromorphic functions on the complex plane, if they have the same inverse images counting multiplicities for four distinct values, then they coincide up to a Möbius transformation, and if they have the same inverse images for five distinct values, then they coincide. Fujimoto and Smiley extended Nevanlinna's uniqueness theorems to the case of meromorphic mappings of several complex variables into the complex projective space for hyperplanes. Recently, Motivated by Ru Min and Stoll's accomplishment of the second main theorem for moving targets, Li Baoqin and Shirosaki proved some uniqueness theorems of entire functions in several complex variables and meromorphic functions in one complex variable, respectively, for moving targets. Using the techniques of value distribution theory in several complex variables, we prove some uniqueness theorems of meromorphic mappings of several complex variables into the complex projective space for moving targets.

1. Introduction. Using the second main theorem of value distribution theory, Nevanlinna [9] proved the following uniqueness theorems of meromorphic functions.

THEOREM 1.A (Nevanlinna [9]). *Let f and g be two nonconstant meromorphic functions on the complex plane $\boldsymbol{C}$. If there are four distinct values $a_i \in P^1(\boldsymbol{C})$ $(\cong \boldsymbol{C} \cup \{\infty\})$, $i = 1, \dots, 4$, such that $f(z) - a_i$ and $g(z) - a_i$ have the same zeros counting multiplicities for each i, then f and g coincide up to a Möbius transformation.*

THEOREM 1.B (Nevanlinna [9]). *Let f and g be two nonconstant meromorphic functions on the complex plane $\boldsymbol{C}$. If there are five distinct values $a_i \in P^1(\boldsymbol{C})$ $(\cong \boldsymbol{C} \cup \{\infty\})$, $i = 1, \dots, 5$, such that $f(z) - a_i$ and $g(z) - a_i$ have the same zeros regardless of multiplicities for each i, then $f = g$.*

Since then, there have been a number of papers (e.g., Fujimoto [1, 3], Gunderson [4], Ji [5], Li [8], Shirosaki [14] and Smiley [15]) working towards this kind of problems. In particular, Gunderson [4] gave a clever example to explain that for two nonconstant meromorphic functions f, g on the complex plane, if $f(z) - a_i$ and $g(z) - a_i$ have the same zeros regardless of multiplicities for four distinct values $a_i \in P^1(\boldsymbol{C})$, $i = 1, \dots, 4$, then f need not be a Möbius transformation of g. This means that the assumption "counting multiplicities" of Theorem 1.A cannot simply be relaxed. Using the techniques of value distribution theory in several complex variables, Fujimoto [1, 3] and Smiley [5] gave some extensions

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of Nevanlinna's uniqueness theorems to several complex variables and proved the following uniqueness theorems.

THEOREM 1.C (Fujimoto [1]). *Let H_i, $1 \leq i \leq 3N+1$, be $3N+1$ hyperplanes in $P^N(\boldsymbol{C})$ located in general position, and let f and g be two nonconstant meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ with $f(\boldsymbol{C}^n) \not\subset H_i$ and $g(\boldsymbol{C}^n) \not\subset H_i$ such that $\nu(f, H_i) = \nu(g, H_i)$ for $1 \leq i \leq 3N+1$, where $\nu(f, H_i)$ and $\nu(g, H_i)$ denote the pull-back of the divisors (H_i) on $P^N(\boldsymbol{C})$ by f and g, respectively. Then there is a projective linear transformation L of $P^N(\boldsymbol{C})$ such that $L(f) = g$.*

THEOREM 1.D (Fujimoto [3] and Smiley [15]). *Let H_i, $1 \leq i \leq 3N+2$, be $3N+2$ hyperplanes in $P^N(\boldsymbol{C})$ located in general position, and let f and g be two linearly nondegenerate meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$. Assume that*

(i) $f^{-1}(H_i) = g^{-1}(H_i)$ *for* $1 \leq i \leq 3N+2$,
(ii) $\dim f^{-1}(H_i \cap H_j) \leq n-2$ *for* $1 \leq i < j \leq 3N+2$, *and*
(iii) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{3N+2} f^{-1}(H_j)$.

Then $f = g$.

Recently, motivated by the accomplishment of the second main theorem of value distribution theory for moving targets (e.g., Ru and Stoll [11, 12] and Steinmetz [16]), Li [8] and Shirosaki [14] proved some unicity theorems for moving targets. However, Li [8] and Shirosaki [14] only studied entire functions on $\boldsymbol{C}^n$ and meromorphic functions on $\boldsymbol{C}$, respectively. Inspired by the idea in Fujimoto [1] and Shirosaki [14], in this paper we shall give some types of generalizations of Theorem 1.C and Theorem 1.D to the case of meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ for moving targets by establishing a weak Cartan-type second main theorem for moving targets.

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2. Preliminaries and our results. Let $F(z)$ be a nonzero entire function on $\boldsymbol{C}^n$. For $a \in \boldsymbol{C}^n$, set $F(z) = \sum_{m=0}^{\infty} P_m(z-a)$, where the term $P_m(z)$ is either identically zero or a homogeneous polynomial of degree m. The number $\nu_F^0(a) := \min\{m; P_m \neq 0\}$ is said to be the zero-multiplicity of F at a. Set $|\nu_F^0| := \overline{\{z \in \boldsymbol{C}^n; \nu_F^0(z) \neq 0\}}$.

For $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$ we set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. For $r > 0$, define

$$B(r) = \{z \in \boldsymbol{C}^n; \|z\| < r\} \quad \text{and} \quad S(r) = \{z \in \boldsymbol{C}^n; \|z\| = r\}.$$

Let $d = \partial + \bar{\partial}$ and $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \bar{\partial})$. We write

$$v(z) = (dd^c\|z\|^2)^{n-1} \quad \text{and} \quad \sigma(z) = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{n-1}$$

for $z \in \boldsymbol{C}^n - \{0\}$.

Let $f : \boldsymbol{C}^n \to P^N(\boldsymbol{C})$ be a meromorphic mapping. We take holomorphic functions $f_0, f_1, \ldots, f_N$ on $\boldsymbol{C}^n$ such that $I_f := \{z \in \boldsymbol{C}^n; \ f_0(z) = f_1(z) = \cdots = f_N(z) = 0\}$ is of dimension at most $n-2$ and $f(z) = (f_0(z), f_1(z), \ldots, f_N(z))$ on $\boldsymbol{C}^n - I_f$ in terms of homogeneous coordinates on $P^N(\boldsymbol{C})$. We call such a representation $f = (f_0, f_1, \dots, f_N)$ a reduced representation of f. Since our notation is often independent of the choice of reduced representations, we shall identify f with its reduced representations in this paper. Set $\|f\| = (|f_0|^2 + \cdots + |f_N|^2)^{1/2}$. The order function of f is given by

$$T_f(r) = \int_{S(r)} \log \|f\| \sigma - \int_{S(1)} \log \|f\| \sigma \, .$$

A meromorphic mapping $a : \boldsymbol{C}^n \to P^N(\boldsymbol{C})$ is "small" with respect to the meromorphic mapping f of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ if $T_a(r) = o(T_f(r))$ as $r \to +\infty$. Let $a = (a_0, a_1, \dots, a_N)$ be a reduced representation of a. We define

$$m_{f,a}(r) = \int_{S(r)} \log \frac{\|f\|\|a\|}{|(f,a)|} \sigma - \int_{S(1)} \log \frac{\|f\|\|a\|}{|(f,a)|} \sigma$$

and

$$N_{f,a}(r) = \int_{S(r)} \log |(f,a)| \sigma - \int_{S(1)} \log |(f,a)| \sigma \, ,$$

where $(f,a) := \sum_{i=0}^N a_i f_i$. Then

$$N_{f,a}(r) = \int_1^r \frac{n(t)}{t^{2n-1}} dt,$$

where

$$n(t) := \begin{cases} \displaystyle\int_{|\nu^0_{(f,a)}| \cap B(t)} \nu^0_{(f,a)}(z) \upsilon & (n \geq 2) \, , \\ \displaystyle\sum_{|z| \leq t} \nu^0_{(f,a)}(z) & (n = 1) \, . \end{cases}$$

For a postive integer M, define

$$N^{[M]}_{f,a}(r) = \int_1^r \frac{n^{[M]}(t)}{t^{2n-1}} dt \, ,$$

where

$$n^{[M]}(t) := \begin{cases} \displaystyle\int_{|\nu^0_{(f,a)}| \cap B(t)} \min\{\nu^0_{(f,a)}(z), M\} \upsilon & (n \geq 2) \, , \\ \displaystyle\sum_{|z| \leq t} \min\{\nu^0_{(f,a)}(z), M\} & (n = 1) \, . \end{cases}$$

If F is a meromorphic function on $\boldsymbol{C}^n$ and $a \in \boldsymbol{C} \cup \{\infty\}$, then we adopt the standard notation for $m_F(r,a)$, $N_F(r,a)$ and etc. Thus we have

$$N_{f,a}(r) = N_{(f,a)}(r, 0)$$

for two meromorphic mappings f, a of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$. If $(f,a) \not\equiv 0$, then the first main theorem for moving targets in value distribution theory (see Ru and Stoll [11, 12]) states

$$T_f(r) + T_a(r) = m_{f,a}(r) + N_{f,a}(r)$$

for $r > 1$.

For any $q \geq N+1$, let $a_1, \dots, a_q$ be q "small" meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ with reduced representations $a_j = (a_{j0}, a_{j1}, \dots, a_{jN})$, $j = 1, \dots, q$. We say that $a_1, \dots, a_q$ are located in general position if for any $1 \leq j_0 < j_1 < \cdots < j_N \leq q$, $\det(a_{j_k l}) \not\equiv 0$. Let $\mathscr{M}_n$ be the field (over $\boldsymbol{C}$) of all meromorphic functions on $\boldsymbol{C}^n$. Let $\mathscr{R}(\{a_i\}_{i=1}^q) \subset \mathscr{M}_n$ be the smallest subfield over $\boldsymbol{C}$ which contains $\boldsymbol{C}$ and all a_{jk}/a_{jl} with $a_{jl} \not\equiv 0$, where $1 \leq j \leq q$ and $0 \leq k, l \leq N$. Define $\widetilde{\mathscr{R}}(\{a_i\}_{i=1}^q) \subset \mathscr{M}_n$ by the smallest subfield over $\boldsymbol{C}$ which contains all $h \in \mathscr{M}_n$ with $h^k \in \mathscr{R}(\{a_i\}_{i=1}^q)$ for some positive integer k. Then, for any $h \in \widetilde{\mathscr{R}}(\{a_i\}_{i=1}^q)$, it is easy to check $T_h(r) = O(\sum_{i=1}^q T_{a_i}(r)) = o(T_f(r))$ as $r \to +\infty$. Furthermore we call that f is not linearly degenerate over $\mathscr{R}(\{a_i\}_{i=1}^q)$ ($\widetilde{\mathscr{R}}(\{a_i\}_{i=1}^q)$) if $f_0, f_1, \dots, f_N$ are linearly independent over $\mathscr{R}(\{a_i\}_{i=1}^q)$ ($\widetilde{\mathscr{R}}(\{a_i\}_{i=1}^q)$ respectively).

Suppose that $R(r)$ and $S(r)$ are two positive functions for $r > 0$. "$R(r) \leq S(r)||$" ("$R(r) = S(r)||$") mean that $R(r) \leq S(r)$ ($R(r) = S(r)$ respectively) for all large r outside a set of finite Lebesgue measure. Assume that f and $\{a_i\}_{i=1}^q$, $q \geq N+1$, are meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ such that $\{a_i\}_{i=1}^q$ are in general position and "small" with respect to f. If f is not linearly degenerate over $\mathscr{R}(\{a_i\}_{i=1}^q)$, then the second main theorem for moving targets in value distribution theory (see Ru and Stoll [11, 12] and Shirosaki [13]) can be described as, for any $\varepsilon > 0$,

$$(q - N - 1 - \varepsilon)T_f(r) \leq \sum_{j=1}^{q} N_{f,a_j}(r) + o(T_f(r))|| \, .$$

Let $\mathscr{M}$ be the field (over $\boldsymbol{C}$) of all meromorphic functions on $\boldsymbol{C}$ and f a nonconstant meromorphic function on $\boldsymbol{C}$. Define $\Gamma_f := \{h \in \mathscr{M};\ T_h(r) = o(T_f(r))\ (r \to +\infty)\}$. Shirosaki [14] proved the following results.

THEOREM 2.A (Shirosaki [14]). *Let f, g be two nonconstant meromorphic functions on $\boldsymbol{C}$ such that $f(z) - a_i(z)$ and $g(z) - a_i(z)$ have the same zeros of the same multiplicities for four distinct $a_i \in \Gamma_f \cup \{\infty\}$, $i = 1, \dots, 4$. Then there exist $A, B, C, D \in \Gamma_f$ such that*

$$g = \frac{Af + B}{Cf + D}$$

with $AD - BC \not\equiv 0$.

THEOREM 2.B (Shirosaki [14]). *Let f, g be two nonconstant meromorphic functions on $\boldsymbol{C}$ such that $f(z) - a_i(z)$ and $g(z) - a_i(z)$ have the same zeros of the same multiplicities for five distinct $a_i \in \Gamma_f \cup \{\infty\}$, $i = 1, \dots, 5$. Then $f = g$.*

REMARK. Ye [18] claimed an extension of Theorem 2.A for meromorphic mappings of $\boldsymbol{C}^n$ into $P^m(\boldsymbol{C})$ for moving targets. Roughly speaking, Ye [18] claimed the following result: For any two meromorphic mappings of $\boldsymbol{C}^n$ into $P^m(\boldsymbol{C})$ sharing $2(m+1)$ "small" mappings in a certain sense, then there is a nonzero bilinear function vanishing on these two meromorphic mappings. But there are some mistakes in Ye's proof, e.g., the bottom line (which is a key step in his proof) of p. 526 in Ye [18] seems incorrect (note: counting multiplicities there) for

$m > 1$ and the conclusion (13) in Ye [18] is not proved in the case of $c_{j_o j_1} = 0$ (note: the assumption of Lemma 3.3 in Ye [18] is not satisfied in this case).

Using the idea in Fujimoto [1] and Shirosaki [14], we shall prove the following results.

THEOREM 2.1. *Let $f, g : \boldsymbol{C}^n \to P^N(\boldsymbol{C})$ be two nonconstant meromorphic mappings, and let $\{a_i\}_{i=1}^{3N+1}$ be "small" (with respect to f) meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ in general position such that f is not linearly degenerate over $\mathscr{R}(\{a_i\}_{i=1}^{3N+1})$. Assume that*

(i) *(f, a_i) and (g, a_i) have the same zeros of the same multiplicities for $1 \leq i \leq 3N+1$,*

(ii) $\dim\{z \in \boldsymbol{C}^n;\ \ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \leq n-2$ *for* $1 \leq i < j \leq 3N+1$, *and*

(iii) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{3N+1}\{z \in \boldsymbol{C}^n;\ \ (f(z), a_j(z)) = 0\}$.

Then there exists an $(N+1) \times (N+1)$ matrix L with elements in $\widetilde{\mathscr{R}}(\{a_i\}_{i=1}^{3N+1})$ and $\det(L) \not\equiv 0$ such that

$$L(z)\begin{pmatrix} f_0(z) \\ f_1(z) \\ \vdots \\ f_N(z) \end{pmatrix} = \begin{pmatrix} g_0(z) \\ g_1(z) \\ \vdots \\ g_N(z) \end{pmatrix},$$

where $(f_0, f_1, \dots, f_N)$ and $(g_0, g_1, \dots, g_N)$ are some reduced representations of f and g, respectively.

THEOREM 2.2. *Let $f, g : \boldsymbol{C}^n \to P^N(\boldsymbol{C})$ be two nonconstant meromorphic mappings, and let $\{a_i\}_{i=1}^{3N+2}$ be "small" (with respect to f) meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ in general position such that f is not linearly degenerate over $\widetilde{\mathscr{R}}(\{a_i\}_{i=1}^{3N+2})$. Assume that*

(i) *(f, a_i) and (g, a_i) have the same zeros of the same multiplicities for $1 \leq i \leq 3N+2$,*

(ii) $\dim\{z \in \boldsymbol{C}^n;\ \ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \leq n-2$ *for* $1 \leq i < j \leq 3N+2$, *and*

(iii) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{3N+2}\{z \in \boldsymbol{C}^n;\ (f(z), a_j(z)) = 0\}$.

Then $f = g$.

REMARK. The assumption (ii) of Theorem 2.1 and Theorem 2.2 can be replaced by the following condition: $\{z \in \boldsymbol{C}^n;\ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\}$ $(i \neq j)$ are at most $(n-1)$-dimensional analytic sets such that the counting functions for their $(n-1)$-dimensional analytic components are "small" with respect to f. Thus Theorem 2.A and Theorem 2.B are special cases of Theorem 2.1 and Theorem 2.2 when $n = N = 1$, respectively.

Now we shall present an outline of our proof of the main results. By Cartan's second main theorem with truncated counting function for hyperplanes (e.g., see (6.2) in Fujimoto [2], (3.B.40) in p. 169 of Kobayashi [6] or (5.6) of Vitter [17]), we easily obtain Theorem 1.D (e.g., see Fujimoto [3] and Smiley [15]). Since the Cartan-type second main theorem for moving targets is not proved yet even in the case of $n = N = 1$ (cf. Li [8] and Shirosaki

[14]) and it seems impossible to dominate $\sum_{j=1}^{3N+2} N_{f,a_j}(r)$ by $N(T_f(r)+T_g(r))$ for $N>1$ under the assumption of Theorem 2.2, we do not know whether Theorem 2.2 in the case of $N>1$ can be derived from the second main theorem for moving targets (see Ru and Stoll [11, 12] and Shirosaki [13]) by Smiley's argument in [15]. Thus in this paper we mainly follow the technique of Fujimoto [1] and Shirosaki [14], and our idea here is heavily based on the framework of Borel's Lemma. We first extend the classical Borel's lemma to the case of moving targets. But the second main theorem for moving targets (see Ru and Stoll [11, 12] and Shirosaki [13]) seems to be not sufficient for us to prove that our case is suitable to the generalized Borel's lemma. In order to overcome the difficulty, we establish a weak Cartan-type second main theorem for moving targets which can be used to prove that our object satisfies the assumption of the generalized Borel's lemma. Finally, we use a combinatorial conclusion to finish our proof.

The extension of Theorem 1.D to the case of moving targets is conjectured as follows:

CONJECTURE 2.C. *Let $f, g : \boldsymbol{C}^n \to P^N(\boldsymbol{C})$ be two nonconstant meromorphic mappings, and let $\{a_i\}_{i=1}^{3N+2}$ be "small" (with respect to f) meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ in general position such that f and g are not linearly degenerate over $\mathscr{R}(\{a_i\}_{i=1}^{3N+2})$. Assume that*

(i) *(f, a_i) and (g, a_i) have the same zeros regardless of multiplicities for $1 \le i \le 3N+2$,*

(ii) *$\dim\{z \in \boldsymbol{C}^n;\ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \le n-2$ for $1 \le i < j \le 3N+2$, and*

(iii) *$f(z) = g(z)$ on $\bigcup_{j=1}^{3N+2}\{z \in \boldsymbol{C}^n;\ (f(z), a_j(z)) = 0\}$.*

Then $f = g$.

REMARK. If all a_i $(i = 1, \dots, 3N+2)$ are constants, then Conjecture 2.C is nothing but Theorem 1.D.

3. Some lemmas. To prove our results, we need some preparations. Let G be a torsion free abelian group and $A = (a_1, \dots, a_q)$ a q-tuple of elements a_i in G. Let $q \ge r > s > 1$. We say that the q-tuple A have the property $(P_{r,s})$ if any r elements $a_{l(1)}, \dots, a_{l(r)}$ in A satisfy the condition that for any given $i_1, \dots, i_s$ $(1 \le i_1 < \cdots < i_s \le r)$, there exist $j_1, \dots, j_s$ $(1 \le j_1 < \cdots < j_s \le r)$ with $\{i_1, \dots, i_s\} \ne \{j_1, \dots, j_s\}$ such that $a_{l(i_1)} \cdots a_{l(i_s)} = a_{l(j_1)} \cdots a_{l(j_s)}$.

PROPOSITION 3.1 (Fujimoto [1]). *Let G be a torsion free abelian group and $A = (a_1, \dots, a_q)$ a q-tuple of elements a_i in G. If A has the property $(P_{r,s})$ for some r, s with $q \ge r > s > 1$, then there exist $i_1, \dots, i_{q-r+2}$ with $1 \le i_1 < \cdots < i_{q-r+2} \le q$ such that $a_{i_1} = a_{i_2} = \cdots = a_{i_{q-r+2}}$.*

PROPOSITION 3.2 (Ye [18]). *Suppose that $h_0, h_1, \dots, h_m$ $(m \ge 1)$ are nowhere vanishing entire functions on $\boldsymbol{C}^n$ and $b_0, b_1, \dots, b_m$ are nonzero meromorphic functions on $\boldsymbol{C}^n$*

with

$$T_{b_k}(r) = o(T(r)) + O(1)||$$

as $r \to +\infty$ *for* $k = 0, 1, \dots, m$, *where* $T(r) := \sum_{k=0}^{m} T_{h_k}(r)$. *Assume that*

$$b_0h_0 + b_1h_1 + \cdots + b_mh_m = 1\,.$$

Then $b_0h_0, b_1h_1, \dots, b_mh_m$ *are linearly dependent over* $\boldsymbol{C}$.

REMARK. The assumption in Proposition 3.2 need not imply that $h_0, h_1, \dots, h_m$ are linearly dependent over $\boldsymbol{C}$ (cf. Lemma 6.1.20 in Noguchi and Ochiai [10] and Theorem 3.3 in Shirosaki [14]). For example, let $(h_0(z), h_1(z), h_2(z)) := (1, e^{e^z}, e^{z+e^z})$ and $(b_0(z), b_1(z), b_2(z)) := (1, -e^z, 1)$ for $z \in \boldsymbol{C}$. Then the assumption in Proposition 3.2 is satisfied. But h_0, h_1, h_2 are not linearly dependent over $\boldsymbol{C}$.

PROPOSITION 3.3. *Suppose that* $h_0, h_1, \dots, h_m$ $(m \geq 2)$ *are nowhere vanishing entire functions on* $\boldsymbol{C}^n$ *and* $b_0, b_1, \dots, b_m$ *are nonzero meromorphic functions on* $\boldsymbol{C}^n$ *with*

$$T_{b_i/b_j}(r) = o(T_{h_{rst}}(r)) + O(1)|| \quad (0 \leq i < j \leq m)$$

as $r \to +\infty$ *for* $0 \leq r, s, t \leq m$ *with* $r \neq s, s \neq t, t \neq r$, *where* $h_{rst} := (h_r, h_s, h_t)$ *is a holomorphic mapping of* $\boldsymbol{C}^n$ *into* $P^2(\boldsymbol{C})$. *Assume that*

$$b_0h_0 + b_1h_1 + \cdots + b_mh_m = 0\,.$$

Then there exists a decomposition of indices

$$\{0, 1, \dots, m\} = I_1 \cup I_2 \cup \cdots \cup I_l$$

such that

(i) *every* I_k *contains at least two indices,*
(ii) *for* $i, j \in I_k$, b_ih_i/b_jh_j *is constant,*
(iii) *for* $i \in I_p$ *and* $j \in I_q$ $(p \neq q)$, b_ih_i/b_jh_j *is not constant, and*
(iv) *for every* I_k, $\sum_{j \in I_k} b_jh_j = 0$.

REMARK. Clearly, if $n = 1$ and b_k, $k = 0, 1, \dots, m$, are constants, then Proposition 3.3 is nothing but the classical Borel Lemma (cf. Theorem 1.1 in p. 186 of Lang [7] and Corollary 6.1.25 of Noguchi and Ochiai [10]).

The proof of Proposition 3.3 is similar to that of Corollary 6.1.25 of Noguchi and Ochiai [10]. In fact, by Proposition 3.2 we can easily get Proposition 3.3. So we omit the proof here.

The following weak Cartan-type second main theorem for moving targets is crucial to proving our main results in this paper.

PROPOSITION 3.4. *Assume that* f *and* $\{a_i\}_{i=1}^{q}$ $(q \geq N + 1)$ *are meromorphic mappings of* $\boldsymbol{C}^n$ *into* $P^N(\boldsymbol{C})$ *such that* $\{a_i\}_{i=1}^{q}$ *are in general position and "small" with respect to* f. *If* f *is not linearly degenerate over* $\mathscr{R}(\{a_i\}_{i=1}^{q})$, *then, for any* $\varepsilon > 0$, *there exists a positive integer* M *such that*

$$(q - N - 1 - \varepsilon)T_f(r) \leq \sum_{j=1}^{q} N_{f,a_j}^{[M]}(r) + o(T_f(r))||\,.$$

PROOF. Let $f=(f_0, f_1, \dots, f_N)$ and $a_i=(a_{i0}, a_{i1}, \dots, a_{iN})$ be reduced representations of f and a_i, respectively. Let p be a positive integer. Let $\mathscr{L}(p)$ be the vector space generated over $\boldsymbol{C}$ by

$$\left\{ \prod_{1\le i\le q, 0\le j,k\le N} \left(\frac{a_{ij}}{a_{ik}}\right)^{p_{ijk}}; \ a_{ik}\not\equiv 0 \text{ and } p_{ijk} \text{ non-negative integers with } \sum_{1\le i\le q,0\le j,k\le N} p_{ijk}=p \right\}.$$

Then $\mathscr{L}(p)\subset\mathscr{L}(p+1)$. Thus we can take a basis $\{b_1, b_2, \dots, b_t\}$ of $\mathscr{L}(p+1)$ such that $\{b_1, b_2, \dots, b_s\}$ is a basis of $\mathscr{L}(p)$, where $s=\dim\mathscr{L}(p)$ and $t=\dim\mathscr{L}(p+1)$.

(i) if $n=1$, then we have (see (12) in Shirosaki [13])

$$s(q-N-1)T_f(r)\le s\sum_{j=1}^{q} N_{f,a_j}(r)-N_W(r,0)+(N+1)(t-s)T_f(r)+o(T_f(r))||,$$

where $W := \text{Wronski}(b_1f_0, \dots, b_1f_N, b_2f_0, \dots, b_2f_N, \dots, b_tf_0, \dots, b_tf_N)$. By (4) in Shirosaki [13] we have

$$s\sum_{j=1}^{q} N_{f,a_j}(r)-N_W(r,0)\le s\sum_{j=1}^{q} N_{f,a_j}^{[(N+1)t]}(r)+o(T_f(r)).$$

Therefore

$$(q-N-1)T_f(r)\le \sum_{j=1}^{q} N_{f,a_j}^{[(N+1)t]}(r)+(N+1)\left(\frac{t}{s}-1\right)T_f(r)+o(T_f(r))||.$$

Since $\liminf_{p\to\infty} t/s=1$, we have Proposition 3.4 in the case of $n=1$.

(ii) if $n>1$, we only need a little modification in W, and the proof of (i) can be carried over to the case of $n>1$ (see Proposition 4.3 and Proposition 4.10 in Fujimoto [2] or Lemma 3.2 in Ye [18] for references). So we omit the proof here. The proof of Proposition 3.4 is finished.

4. Proof of main results. Let $f, g : \boldsymbol{C}^n \to P^N(\boldsymbol{C})$ be two nonconstant meromorphic mappings with reduced representations $f=(f_0, f_1, \dots, f_N)$ and $g=(g_0, g_1, \dots, g_N)$, respectively. Let $\{a_j\}_{j=1}^{2N+2}$ be $2N+2$ "small" (with respect to f) meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ in general position with reduced representations $a_j=(a_{j0}, a_{j1}, \dots, a_{jN})$, $j=1, \dots, 2N+2$, such that f is not linearly degenerate over $\mathscr{R}(\{a_i\}_{i=1}^{2N+2})$. Assume that

(i) (f, a_i) and (g, a_i) have the same zeros of the same multiplicities for $1\le i\le 2N+2$,

(ii) $\dim\{z\in\boldsymbol{C}^n;\ (f(z), a_i(z))=(f(z), a_j(z))=0\}\le n-2$ for $1\le i<j\le 2N+2$, and

(iii) $f(z)=g(z)$ on $\bigcup_{j=1}^{2N+2}\{z\in\boldsymbol{C}^n;\ (f(z), a_j(z))=0\}$.

Define $h_i := (f, a_i)/(g, a_i)$, $i = 1, \dots, 2N+2$. Then each h_i is a nowhere vanishing entire function on $\boldsymbol{C}^n$. Although each h_i is dependent on the choice of reduced representions of f and g, the ratio $h_p/h_q = (f, a_p)/(g, a_p) \cdot (g, a_q)/(f, a_q)$ is uniquely determined independent of any choice of reduced representions of f, g, a_p and a_q. By the definition we have

$$\sum_{k=0}^{N} a_{ik} f_k - h_i \sum_{k=0}^{N} a_{ik} g_k = 0 \quad (i = 1, \dots, 2N+2)\,.$$

Therefore

$$\det(a_{i0}, \dots, a_{iN}, a_{i0}h_i, \dots, a_{iN}h_i;\ \ 1 \le i \le 2N+2) = 0\,.$$

Let $\mathscr{I}$ be the set of all combinations $I = (i_1, \dots, i_{N+1})$ with $1 \le i_1 < \cdots < i_{N+1} \le 2N+2$ of indices $1, 2, \dots, 2N+2$. For any $I = (i_1, \dots, i_{N+1}) \in \mathscr{I}$, define

$$\{I\} := \{i_1, \dots, i_{N+1}\}\,, \quad h_I := h_{i_1} \cdots h_{i_{N+1}}$$

and

$$\begin{aligned} A_I := (-1)^{(N+1)(N+2)/2+i_1+\cdots+i_{N+1}} &\det(a_{i_r l};\ \ 1 \le r \le N+1, 0 \le l \le N) \\ \times \det(a_{j_s l};\ \ 1 \le s \le N+1, 0 \le l \le N)\,, \end{aligned}$$

where $J = (j_1, \dots, j_{N+1}) \in \mathscr{I}$ such that $\{I\} \cup \{J\} = \{1, 2, \dots, 2N+2\}$. Then we have

$$\sum_{I \in \mathscr{I}} A_I h_I = 0\,,$$

where $A_I \not\equiv 0$ by $\{a_i\}$ being in general position and $A_I/A_J \in \mathscr{R}(\{a_i\}_{i=1}^{2N+2})$ by the definition of $\mathscr{R}(\{a_i\})$ for any $I, J \in \mathscr{I}$.

Since $f(z) = g(z)$ on $\bigcup_{j=1}^{2N+2} \{z \in \boldsymbol{C}^n;\ \ (f(z), a_j(z)) = 0\}$ and

$$\dim\{z \in \boldsymbol{C}^n;\ \ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \le n-2$$

for $1 \le i < j \le 2N+2$, we have

$$h_p(z)/h_q(z) = 1$$

for $z \in \bigcup_{j \ne p,q;\, j=1}^{2N+2} \{z \in \boldsymbol{C}^n;\ \ (f(z), a_j(z)) = 0\}$ outside an analytic set of dimension $\le n-2$ and then, for distinct $I, J \in \mathscr{I}$, we have

$$N_{h_I/h_J}(r, 1) \ge \sum_{k \notin \{I\} \cup \{J\} - \{I\} \cap \{J\}} N_{f, a_k}^{[1]}(r)\,.$$

For distinct $I, J, K \in \mathscr{I}$, set $h_{IJK} := (h_I, h_J, h_K)$ a holomorphic mapping of $\boldsymbol{C}^n$ into $P^2(\boldsymbol{C})$. Then, by Theorem (5.2.29) in Noguchi and Ochiai [10], we have

$$\begin{aligned}
3T_{h_{IJK}}(r) &\geq T_{h_I/h_J}(r) + T_{h_J/h_K}(r) + T_{h_K/h_I}(r) + O(1) \\
&\geq N_{h_I/h_J}(r,1) + N_{h_J/h_K}(r,1) + N_{h_K/h_I}(r,1) + O(1) \\
&\geq \sum_{k \notin \{I\} \cup \{J\} - \{I\} \cap \{J\}} N^{[1]}_{f,a_k}(r) + \sum_{k \notin \{J\} \cup \{K\} - \{J\} \cap \{K\}} N^{[1]}_{f,a_k}(r) \\
&\quad + \sum_{k \notin \{K\} \cup \{I\} - \{K\} \cap \{I\}} N^{[1]}_{f,a_k}(r) + O(1) \\
&\geq \sum_{k=1}^{2N+2} N^{[1]}_{f,a_k}(r) + O(1) \\
&\geq \frac{1}{M} \sum_{k=1}^{2N+2} N^{[M]}_{f,a_k}(r) + O(1) \\
&\geq \frac{1}{M}(N+1-\varepsilon)T_f(r) - o(T_f(r))|| \\
&\geq \frac{N}{M} T_f(r)||,
\end{aligned}$$

where ε $(0 < \varepsilon < 1/2)$ and M are given by Proposition 3.4 (note: it is easy to check $(\{I\}\cup\{J\}-\{I\}\cap\{J\})^c \cup (\{J\}\cup\{K\}-\{J\}\cap\{K\})^c \cup (\{K\}\cup\{I\}-\{K\}\cap\{I\})^c = \{1, \ldots, 2N+2\}$ here). Thus

$$T_{A_P/A_Q}(r) = o(T_{h_{IJK}}(r))|| \quad (r \to +\infty)$$

for any $P, Q, I, J, K \in \mathscr{I}$ with $P \neq Q$, $I \neq J$, $J \neq K$ and $K \neq I$. Therefore, for any $I \in \mathscr{I}$, by Proposition 3.3 there exists $J \in \mathscr{I}$ with $I \neq J$ such that $A_I h_I = c A_J h_J$ for a nonzero constant c. So $h_I/h_J = cA_J/A_I \in \mathscr{R}(\{a_i\}_{i=1}^{2N+2})$.

Let $\mathscr{H}^*$ be the abelian multiplication group of all nowhere vanishing entire functions on $\boldsymbol{C}^n$. Define $\mathscr{T} \subset \mathscr{H}^*$ by the smallest subgroup which contains all $f \in \mathscr{H}^*$ with $f^k \in \mathscr{R}(\{a_i\}_{i=1}^q)$ for some positive integer k. So we have $\mathscr{H}^* \cap \mathscr{R}(\{a_i\}_{i=1}^q) \subset \mathscr{T} \subset \widetilde{\mathscr{R}}(\{a_i\}_{i=1}^q)$. Then the multiplication group $G := \mathscr{H}^*/\mathscr{T}$ is a torsion free abelian group, and the q-tuple of elements in G represented by $(h_1, \ldots, h_q)$ has the property $(P_{2N+2,N+1})$ by the above argument. Define $f_i \sim f_j$ if $f_i/f_j \in \widetilde{\mathscr{R}}(\{a_i\}_{i=1}^q)$ for $f_i, f_j \in \mathscr{H}^*$. Then by Proposition 3.1 we have proved the following proposition.

PROPOSITION 4.1. *Let $f, g : \boldsymbol{C}^n \to P^N(\boldsymbol{C})$ be two nonconstant meromorphic mappings, and let $\{a_i\}_{i=1}^q$ $(q \geq 2N+2)$ be q "small" (with respect to f) meromorphic mappings of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ in general position such that f is not linearly degenerate over $\mathscr{R}(\{a_i\}_{i=1}^q)$. Assume that*

(i) *(f, a_i) and (g, a_i) have the same zeros of the same multiplicities for $1 \leq i \leq q$,*

(ii) *$\dim\{z \in \boldsymbol{C}^n;\ (f(z), a_i(z)) = (f(z), a_j(z)) = 0\} \leq n-2$ for $1 \leq i < j \leq q$, and*

(iii) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{q}\{z \in \boldsymbol{C}^n;\ (f(z), a_j(z)) = 0\}$.
Given the reduced representations of f, g, a_i, *define* $h_i(z) := (f, a_i)/(g, a_i)$ *for* $1 \leq i \leq q$. *Then there exist* i_k, $1 \leq k \leq q - 2N$, *with* $1 \leq i_1 < \cdots < i_{q-2N} \leq q$ *such that* $h_{i_1} \sim h_{i_2} \sim \cdots \sim h_{i_{q-2N}}$.

In order to prove Theorems 2.1 and 2.2, we define

$$A := \begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1N} \\ a_{20} & a_{21} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(N+1)0} & a_{(N+1)1} & \cdots & a_{(N+1)N} \end{pmatrix}$$

and

$$H := \begin{pmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{N+1} \end{pmatrix}.$$

PROOF OF THEOREM 2.1. Since $q = 3N+1$, by Proposition 4.1 and a suitable change of the reduced representations, without loss of generality, we may assume $h_1, h_2, \dots, h_{N+1} \in \widetilde{\mathscr{R}}(\{a_i\}_{i=1}^{3N+1})$. Then

$$A \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix} = HA \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{pmatrix}.$$

This immediately implies Theorem 2.1.

PROOF OF THEOREM 2.2. Since $q = 3N+2$, by Proposition 4.1 and a suitable change of the reduced representations, without loss of generality, we may assume $h_1, h_2, \dots, h_{N+2} \in \widetilde{\mathscr{R}}(\{a_i\}_{i=1}^{3N+2})$. Then

$$A \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix} = HA \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{pmatrix}$$

and

$$(a_{(N+2)0}, \dots, a_{(N+2)N}) \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix} = h_{N+2}(a_{(N+2)0}, \dots, a_{(N+2)N}) \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{pmatrix}.$$

Therefore

$$(a_{(N+2)0}, \dots, a_{(N+2)N}) \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix} = h_{N+2}(a_{(N+2)0}, \dots, a_{(N+2)N}) A^{-1} H^{-1} A \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix}.$$

Since f is not linearly degenerate over $\widetilde{\mathscr{R}}(\{a_i\}_{i=1}^{3N+2})$, we have

$$(a_{(N+2)0}, \dots, a_{(N+2)N}) = h_{N+2}(a_{(N+2)0}, \dots, a_{(N+2)N}) A^{-1} H^{-1} A\,.$$

Thus

$$(a_{(N+2)0}, \dots, a_{(N+2)N}) A^{-1} \begin{pmatrix} h_1 - h_{N+2} & 0 & \cdots & 0 \\ 0 & h_2 - h_{N+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{N+1} - h_{N+2} \end{pmatrix} = 0\,.$$

Let

$$(a_{(N+2)0}, \dots, a_{(N+2)N}) = (b_0, \dots, b_N) \begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1N} \\ a_{20} & a_{21} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(N+1)0} & a_{(N+1)1} & \cdots & a_{(N+1)N} \end{pmatrix}.$$

Since $\{a_i\}_{i=1}^{N+2}$ is in general position, we have $b_i \not\equiv 0$ $(i = 0, \dots, N)$. These mean that $h_i = h_{N+2},\ i = 1, \dots, N+1$. So $f = g$. The proof of Theorem 2.2 is finished.

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