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VANISHING THEOREMS ON TORIC VARIETIES

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Abstract. We use Cox's description for sheaves on toric varieties and results about local cohomology with respect to monomial ideals to give a characteristic-free approach to vanishing results on toric varieties. As an application, we give a proof of a strong version of Fujita's Conjecture in the case of toric varieties. We also prove that every sheaf on a toric variety corresponds to a module over the homogeneous coordinate ring, generalizing Cox's result for the simplicial case.

Introduction. Our main goal in this article is to give a characteristic free approach to vanishing results on arbitrary toric varieties. We prove that the vanishing of a certain cohomology group depending on a Weil divisor is implied by the vanishing of the analogous cohomology group involving a higher multiple of that divisor. When the variety is complete and the divisor is $\boldsymbol{Q}$-Cartier, one recovers in this setting a theorem due to Kawamata and Viehweg. We apply these results to prove a strong form of Fujita's conjecture on a smooth complete toric variety.

Let X be a toric variety and $D_1, \dots, D_d$ the invariant Weil divisors on X, so that $\omega_X \simeq \mathcal{O}_X(-D_1 - \cdots - D_d)$. In the first part of the paper we deduce the following generalization of the theorem of Kawamata and Viehweg, for toric varieties.

THEOREM 0.1. *Let D be an invariant Weil divisor on X as above. Suppose that we have $E = \sum_{j=1}^{d} a_j D_j$, with $a_j \in \boldsymbol{Q}$ and $0 \leq a_j \leq 1$ such that for some integer $m \geq 1$, $m(D+E)$ is integral and Cartier. If for some $i \geq 0$ we have $H^i(\mathcal{O}_X(D + m(D+E))) = 0$, then $H^i(\mathcal{O}_X(D)) = 0$. In particular, if X is complete and there is E aforementioned such that $D+E$ is $\boldsymbol{Q}$-ample, then $H^i(\mathcal{O}_X(D)) = 0$ for every $i \geq 1$.*

As a particular case of this theorem, we see that if for some $m \geq 1$ and $L \in \mathrm{Pic}(X)$ we have $H^i(L^m(-D_{j_1} - \cdots - D_{j_r})) = 0$, then $H^i(L(-D_{j_1} - \cdots - D_{j_r})) = 0$. The cases $r = 0$ and $r = d$ of this assertion were known to hold by reduction to a field of positive characteristic. Over such a field X is Frobenius split and one concludes using arguments in Mehta and Ramanathan [MR]. The fact that X is Frobenius split will follow also from our results.

Our method yields other vanishing results as well. For example, we prove that if X is a smooth toric variety and $L \in \mathrm{Pic}(X)$ is such that for some $m \geq 1$ and $i \geq 0$ we have

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$H^i(\Omega_X^j \otimes L^m) = 0$, then $H^i(\Omega_X^j \otimes L) = 0$. By taking X complete and L ample, we thus recover a theorem of Bott, Steenbrink and Danilov.

In the second part of the paper we give some applications of vanishing theorems on toric varieties. Our main result is a proof in the toric case of a strong version of the following conjecture due to Fujita (see [La] for discussion and related results).

CONJECTURE 0.2. Let X be a smooth projective variety of dimension n and $L \in \mathrm{Pic}(X)$ an ample line bundle. Then $\omega_X \otimes L^{n+1}$ is globally generated and $\omega_X \otimes L^{n+2}$ is very ample.

When the ample line bundle L is generated by global sections, an argument of Ein and Lazarsfeld [EL] based on vanishing results proves the conjecture over a field of characteristic zero. Under the same hypothesis on L, the first assertion of the conjecture is proved in arbitrary characteristic by Smith [Sm]. On a smooth projective toric variety every ample line bundle is very ample (see [De]), so these results prove the conjecture in this setting.

We give a direct proof of a strengthened form of the conjecture in the case of a toric variety, where instead of making a conclusion about a power L^m, we make a statement about any line bundle L satisfying conditions on the intersection numbers with the invariant curves. We are able to replace ω_X by the negative of the sum of any set of D_i, and also improve the bound by one in the case when X is not the projective space. More precisely, we prove:

THEOREM 0.3. *Let X be an n-dimensional complete smooth toric variety, $L \in \mathrm{Pic}(X)$ a line bundle and $D_1, \dots, D_m$ distinct prime invariant divisors.*

(1) *If $(L \cdot C) \geq n$ for every invariant integral curve $C \subset X$, then $L(-D_1 - \cdots - D_m)$ is globally generated, unless $X \simeq \mathbf{P}^n$, $L \simeq \mathcal{O}(n)$ and $m = n+1$.*

(2) *If $(L \cdot C) \geq n+1$ for every invariant integral curve $C \subset X$, then $L(-D_1 - \cdots - D_m)$ is very ample, unless $X \simeq \mathbf{P}^n$, $L \simeq \mathcal{O}(n+1)$ and $m = n+1$.*

To obtain these results we use Cox's notion of homogeneous coordinate ring of X. When the fan defining X is nondegenerate (i.e., it is not contained in a hyperplane), this is a polynomial ring $S = k[Y_1, \dots, Y_d]$ together with a reduced monomial ideal B and with a grading in the class group of X which is compatible with the $\mathbf{Z}^d$-grading by monomials. In general we need to slightly adjust this definition, but we leave this generalization for the core of the paper. As in the case of projective space, each graded S-module P gives a quasi coherent sheaf $\tilde{P}$ on X and for each $i \geq 1$, the Zariski cohomology $H^i(X, \tilde{P})$ is the degree zero part of the local cohomology module $H_B^{i+1}(P)$. This idea has been used in [EMS] to give an algorithm for the computation of cohomology of coherent sheaves on a toric variety.

Our basic result says that if P is in fact $\mathbf{Z}^d$-graded and if the multiplication by Y_j is an isomorphism in certain $\mathbf{Z}^d$-degrees, then the same is true for $H_B^i(P)$. The main example is $P = S$ in which we get that the multiplication

$$\nu_{Y_j} : H_B^i(S)_\alpha \to H_B^i(S)_{\alpha+e_j}$$

is an isomorphism for every $\alpha = (\alpha_j) \in \mathbf{Z}^d$ such that $\alpha_j \neq -1$. In particular, $H_B^i(S)_\alpha$ depends only on the signs of the components of α. This case was used in [EMS] in order to describe the support of $H^i_B(S)$. Similar results for the Ext modules appear also in [Mu] and [Ya]. Our second example is that of the modules giving the sheaves Ω^i_X on a smooth toric variety. Using this result and the relation between the local cohomology of a module and the Zariski cohomology of the corresponding sheaf, we deduce the various vanishing theorems.

In the first section of the paper we summarize the construction in [Cox] for the homogeneous coordinate ring suitably generalized to be applicable also to toric varieties defined by a degenerate fan. All the results can be easily extended to this context. We prove that every quasicoherent sheaf on a toric variety comes from a graded module over the homogeneous coordinate ring, generalizing the result in [Cox] for the simplicial case. We describe the relation between the local cohomology of modules and the cohomology of the associated sheaves. This is used in the second section to prove the vanishing results described above.

In order to apply these results, we need numerical characterizations for ampleness and numerical effectiveness for the toric case and in the third section we provide these results. In the simplicial case, a toric Nakai criterion is given in [Oda]. We show that the result holds for an arbitrary complete toric variety. We also prove that $L \in \mathrm{Pic}(X)$ is globally generated if and only if $(L \cdot C) \geq 0$, for every integral invariant curve $C \subset X$. In particular, we see that L is globally generated if and only if it is numerically effective. These results have been recently obtained also by Mavlyutov in [Ma]. We mention a generalization in a different direction due to Di Rocco [DR] who proved that on a smooth toric variety, $L \in \mathrm{Pic}(X)$ is k-ample if and only if $(L \cdot C) \geq k$ for every invariant curve $C \subset X$.

As a consequence of the above results, we deduce that L is big and nef if and only if there is a map $\phi : X \to X'$ induced by a fan refinement (therefore ϕ is proper and birational) and $L' \in \mathrm{Pic}(X')$ ample such that $L \simeq \phi^*(L')$. This easily implies the version of Kawamata-Viehweg vanishing theorem for nef and big line bundles.

The fourth section is devoted to the above generalization (in this context) of Fujita's Conjecture and some related results. The proof goes by induction on the dimension of X, by taking the restriction to the invariant prime divisors. The result which allows the induction says that for every $l \geq 1$, if L is a line bundle such that $(L \cdot C) \geq l$ for every invariant integral curve $C \subset X$, then for every invariant prime divisor D and every $C \subset X$ aforementioned, $(L(-D) \cdot C) \geq l - 1$. From the case $l = 1$ we see that if L is ample, then $L(-D)$ is globally generated. We conclude this section by proving a related result, which characterizes the situation in which L is ample and D is a prime invariant divisor, but $L(-D)$ is not ample.

A well-known ampleness criterion (see, for example, [Fu]) can be interpreted as saying that on a complete toric variety X, $L \in \mathrm{Pic}(X)$ is ample if and only if it is globally generated and the map induced by restrictions

$$H^0(L) \to \bigoplus_{x \in X_0} H^0(L|_{\{x\}})$$

is an epimorphism, where X_0 is the set of fixed points of X.

In the last section we generalize this property of ample line bundles under the assumption that X is smooth. We prove that the analogous map is still an epimorphism if we replace X_0

with any set of pairwise disjoint invariant subvarieties. In this case, the blowing-up $\tilde{X}$ of X along the union of these subvarieties is still a toric variety and we obtain the required surjectivity by applying to $\tilde{X}$ the results in the second section.

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1. The homogeneous coordinate ring of a toric variety. Let k be a fixed algebraically closed field of arbitrary characteristic. We will use freely the definitions and results on toric varieties from [Fu]. We first review the notation we are going to use.

Let $N \simeq \boldsymbol{Z}^n$ be a lattice and $M = \mathrm{Hom}(N, \boldsymbol{Z})$ the dual lattice. For a rational fan Δ in $V = N \otimes \boldsymbol{R}$, we have the associated toric variety $X = X(\Delta)$. For every $i \leq n$, the set of cones in Δ of dimension i is denoted by Δ_i. The torus $N \otimes_{\boldsymbol{Z}} k^*$ acts on X, and by an invariant subvariety of X we mean a subvariety which is invariant under this action.

The closed invariant subvarieties of X of dimension i are in bijection with the set Δ_{n-i}. For each cone $\tau \in \Delta$ we denote by $V(\tau)$ the corresponding subvariety. Recall that $V(\tau)$ is again a toric variety and $\tau_1 \subset \tau_2$ if and only if $V(\tau_2) \subset V(\tau_1)$. In particular, the prime invariant Weil divisors $D_1, \ldots, D_d$ on X correspond to the one dimensional cones in Δ. If X is smooth, then so is each $V(\tau)$.

Let V' be the vector space spanned by Δ, $N' = N \cap V'$ and $M' = \mathrm{Hom}(N', \boldsymbol{Z})$ its dual lattice. We have an exact sequence:

$$0 \to M' \to \mathrm{Div}_T(X) \to \mathrm{Cl}(X) \to 0\,,$$

where $\mathrm{Div}_T(X) = \bigoplus_{i=1}^{d} \boldsymbol{Z} D_i \simeq \boldsymbol{Z}^d$ is the group of invariant Weil divisors and $\mathrm{Cl}(X)$ is the class group of X.

We fix a decomposition $M \simeq M' \times \boldsymbol{Z}^e$, where e is the codimension of V' in V. We correspondingly have a decomposition $X \simeq X' \times (k^*)^e$, where X' is the toric variety defined by Δ in N'.

The homogeneous coordinate ring of X was introduced by Cox in [Cox] in the case when the fan Δ is not degenerate, i.e., is not contained in a hyperplane. We slightly generalize this notion in order to allow an arbitrary toric variety, following the suggestion in [Cox]. We first review some of the definitions and the results in that paper, all of which can be easily generalized to this context.

For each i with $1 \leq i \leq d$ we introduce an indeterminate Y_i, corresponding to the divisor D_i. We introduce also the indeterminates Y_j with $d+1 \leq j \leq d+e$, and the homogeneous coordinate ring of X is the ring $S = k[Y_1, \ldots, Y_d, Y_{d+1}^{\pm 1} \ldots, Y_{d+e}^{\pm 1}]$. Note that the decomposition $M \simeq M \times \boldsymbol{Z}^e$ corresponds to a decomposition $k[M] \simeq k[M'] \otimes k[Y_{d+1}^{\pm 1}, \ldots, Y_{d+e}^{\pm 1}]$.

For every effective divisor $D = \sum_{i=1}^{d} a_i D_i$, we write Y^D for the corresponding monomial $\prod_{i=1}^{d} Y_i^{a_i} \in S$. On the ring S we have a fine grading, the usual $\mathbf{Z}^{d+e}$-grading by monomials. However, in this section we will consider exclusively a coarse $\mathrm{Cl}(X)$-grading defined by

$$\deg\left(\prod_{i=1}^{d+e} Y_i^{a_i}\right) = \left[\sum_{i=1}^{d} a_i D_i\right] \in \mathrm{Cl}(X)\,.$$

In the ring S there is a distinguished ideal which is a reduced monomial ideal. For each cone $\sigma \in \Delta$ we put $D_{\hat{\sigma}} = \sum_{i;\,\tau_i \not\subset \sigma} D_i$, the sum being taken over the divisors corresponding to one dimensional cones outside σ and $Y^{\hat{\sigma}} = Y^{D_{\hat{\sigma}}}$. If $\Delta_{\max}$ is the set of maximal cones in Δ, then $B = (Y^{\hat{\sigma}} \mid \sigma \in \Delta_{\max})$.

As in the case of projective space, a graded S-module P gives a quasicoherent sheaf on X by the following procedure. X is covered by the affine toric varieties $U_\sigma = \mathrm{Spec}\, k[\sigma^\vee \cap M]$, for $\sigma \in \Delta$. Using the above decomposition of $k[M]$ and the argument in [Cox], we obtain canonical isomorphisms $k[\sigma^\vee \cap M] \simeq (S_{Y^{\hat{\sigma}}})_0$ for every $\sigma \in \Delta$, which are pairwise compatible. Therefore if P is a graded S-module, on the affine piece U_σ we can consider the quasicoherent sheaf defined by $(P_{Y^{\hat{\sigma}}})_0$. These sheaves glue together to give a quasicoherent sheaf $\tilde{P}$ on X. In this way we get an exact functor $P \to \tilde{P}$ from graded S-modules to quasicoherent sheaves. If P is finitely generated, then $\tilde{P}$ is coherent.

In particular, if $\alpha \in \mathrm{Cl}(X)$, $\mathcal{O}(\alpha)$ is defined to be $\widetilde{S(\alpha)}$. As in [Cox], if $\alpha = [D]$, then there is a natural isomorphism $\mathcal{O}(\alpha) \simeq \mathcal{O}(D)$. Moreover, we have an isomorphism of graded rings

$$S \simeq \bigoplus_{\alpha \in \mathrm{Cl}(X)} H^0(X, \mathcal{O}(\alpha))\,.$$

For a quasicoherent sheaf $\mathcal{F}$, we put $\mathcal{F}(\alpha) := \mathcal{F} \otimes \mathcal{O}(\alpha)$.

REMARK. In general, if P is a graded S-module, the natural morphism $\tilde{P} \otimes \mathcal{O}(\alpha) \to \widetilde{P(\alpha)}$ is not an isomorphism. However, it is an isomorphism if $\alpha \in \mathrm{Pic}(X)$. Indeed, by taking a graded free presentation of P, we can reduce ourselves to the case when $P = S(\beta)$ for some $\beta = [E]$. Since $\alpha = [D]$ with D locally invertible, $\mathcal{O}(\alpha)$ is invertible and the fact that the morphism $\mathcal{O}(D) \otimes \mathcal{O}(E) \to \mathcal{O}(D + E)$ is an isomorphism follows now directly from the definition.

We prove now that every quasicoherent sheaf is isomorphic to $\tilde{P}$ for some graded S-module P. This was proved in [Cox] under the assumption that X is simplicial. With a slightly different definition for the homogeneous coordinate ring it was proved more generally for toric varieties with enough effective invariant divisors by Kajiwara in [Ka].

THEOREM 1.1. *For every toric variety X and every quasicoherent sheaf $\mathcal{F}$ on X, there is a graded S-module P such that $\mathcal{F} \simeq \tilde{P}$.*

PROOF. We take $P = \bigoplus_{\alpha \in \mathrm{Cl}(X)} H^0(X, \mathcal{F}(\alpha))$, which is clearly a graded S-module. For simplicity, we will use the notation $P_\sigma = P_{Y^{\hat{\sigma}}}$.

For each $\sigma \in \Delta$, there are canonical maps

$$\phi_\sigma : (P_\sigma)_0 \to H^0(U_\sigma, \mathcal{F}),$$

defined as follows. If $s/Y^D \in (P_\sigma)_0$ such that $s \in H^0(X, \mathcal{F}(\alpha))$ and D is an effective divisor with $[D] = \alpha$ and $\operatorname{Supp} D \cap U_\sigma = \emptyset$, then $1/Y^D$ defines a section in $H^0(U_\sigma, \mathcal{O}(-\alpha))$ and $\phi_\sigma(s/Y^D) = (1/Y^D)s$ is the image of $(1/Y^D, s)$ by the canonical pairing

$$H^0(U_\sigma, \mathcal{O}(-\alpha)) \times H^0(X, \mathcal{F}(\alpha)) \to H^0(U_\sigma, \mathcal{F}).$$

These morphisms glue together to give $\phi : \tilde{P} \to \mathcal{F}$ (note that $\mathcal{F}$ is assumed to be quasicoherent). We will prove that ϕ is an isomorphism by proving that ϕ_σ is an isomorphism for each $\sigma \in \Delta$.

We first show that ϕ_σ is a monomorphism. Suppose that $\phi_\sigma(s/Y^D) = 0$ for some $s \in H^0(X, \mathcal{F}(\alpha))$ and D effective, $[D] = \alpha$.

We may assume that $\operatorname{Supp} D = \bigcup_{\tau_i \not\subset \sigma} V(\tau_i)$, and in this case we will prove that there is an integer $N \geq 1$ such that $Y^{ND}s = 0$ in $H^0(X, \mathcal{F}(\alpha + N\alpha))$. In fact, we will find for each $\tau \in \Delta$ an integer N_τ such that $Y^{N_\tau D}s|_{U_\tau} = 0$. Then it is clear that $N = \sum_\tau N_\tau$ satisfies the requirement.

From now on, we fix also $\tau \in \Delta$. Since $\sigma \cap \tau$ is a face of τ, we can write $\sigma \cap \tau = \tau \cap u^\perp$ for some $u \in \tau^\vee \cap M$. If for each $v \in M$, the corresponding element of $k[M]$ is denoted by χ^v, we consider the principal divisor $D_0 = \operatorname{div}(\chi^u)$. It is effective on U_τ, where its support corresponds to the one-dimensional cones $\tau_i \subset \tau \cap \sigma$.

We consider the restrictions of all the sections from above to U_τ: $s|_{U_\tau} \in H^0(U_\tau, \mathcal{F}(\alpha))$, $Y^D|_{U_\tau} \in H^0(U_\tau, \mathcal{O}(\alpha))$ and $(1/Y^D)|_{U_\sigma \cap U_\tau} \in H^0(U_{\sigma\cap\tau}, \mathcal{O}(-\alpha))$.

Since $\phi_\sigma(s/Y^D) = 0$ in $H^0(U_\sigma, \mathcal{F})$, we have that $s|_{U_\sigma} = 0 \in H^0(U_\sigma, \mathcal{F}(\alpha))$, as the image of $(Y^D, \phi_\sigma(s/Y^D))$ by the canonical pairing

$$H^0(U_\sigma, \mathcal{O}(\alpha)) \otimes H^0(U_\sigma, \mathcal{F}) \to H^0(U_\sigma, \mathcal{F}(\alpha)).$$

In particular, we have $s|_{U_{\sigma\cap\tau}} = 0$. But $U_\sigma \cap U_\tau = U_{\sigma\cap\tau} \subset U_\tau$ is a principal affine subset defined by $Y^{D_0} \in H^0(U_\tau, \mathcal{O}_X)$. Therefore, we get an integer $t \geq 1$ such that $Y^{tD_0}s = 0$ in $H^0(U_\tau, \mathcal{F}(\alpha))$.

If $a_{\tau'}$ and $a^0_{\tau'}$ are the coefficients of $V(\tau')$ in D and D_0, respectively, and N_τ is such that $N_\tau a_{\tau'} \geq t a^0_{\tau'}$ for every one-dimensional face $\tau' \subset \tau$ (by the form of D and D_0, we can choose such an N_τ), then $Y^{N_\tau D}s = 0$ in $H^0(U_\tau, \mathcal{F}(\alpha + N_\tau\alpha))$. This follows from the fact that if τ'' is a one-dimensional cone with $\tau'' \not\subset \tau$, then $\mathcal{O}(V(\tau''))|_{U_\tau}$ is invertible and $Y^{V(\tau'')}$ is an invertible section in it. This completes the proof of the fact that ϕ_σ is a monomorphism.

We prove now that ϕ_σ is an epimorphism. Let $t \in H^0(U_\sigma, \mathcal{F})$, and let $D = \sum_{\tau_i \not\subset \sigma} D_i$ and $\alpha = [D]$.

Using an analogous argument, we see that for each $\tau \in \Delta$, there is an integer N_τ such that $Y^{N_\tau D}t|_{U_{\sigma\cap\tau}} \in H^0(U_{\sigma\cap\tau}, \mathcal{F}(N_\tau\alpha))$ can be extended to a section in $H^0(U_\tau, \mathcal{F}(N_\tau\alpha))$. Indeed, with the notation and arguments we used before, we first find N'_τ such that $Y^{N'_\tau D_0}t$ can be extended to U_τ and then find N_τ, as claimed.

If we apply this to two cones $\tau_1, \tau_2 \in \Delta$ and take $N \geq N_{\tau_1}, N_{\tau_2}$, we see that $Y^{ND}t$ can be extended to both U_{τ_1} and U_{τ_2}, giving sections t_1 and t_2, respectively. Since $(t_1 - t_2)|_{U_{\sigma \cap \tau_1 \cap \tau_2}} = 0$, by applying to $\tau_1 \cap \tau_2$ the argument we used to show that ϕ_σ is a monomorphism, we find $N_{\tau_{12}}$ such that $Y^{N_{\tau_{12}}D} t_1 = Y^{N_{\tau_{12}}D} t_2$ on $U_{\tau_1} \cap U_{\tau_2}$.

This shows that for large enough N, we can extend $Y^{ND}t|_{U_{\sigma\cap\tau}}$ to $t_\tau \in H^0(U_\tau, \mathcal{F}(N\alpha))$ for every $\tau \in \Delta$ such that $t_{\tau_1}|_{U_{\tau\cap\tau_2}} = t_{\tau_2}|_{U_{\tau_1\cap\tau_2}}$ for every $\tau_1, \tau_2 \in \Delta$. Therefore t is in the image of ϕ_σ, which completes the proof. □

Using the same argument as in [Cox], we deduce the following corollary.

COROLLARY 1.2. *For every toric variety X and every coherent sheaf $\mathcal{F}$ on X, there is a finitely generated S-module P' such that $\mathcal{F} \simeq \tilde{P}'$.*

PROOF. With the notation in the proof of Theorem 1.1, we have seen that

$$\phi_\sigma : (P_\sigma)_0 \to H^0(U_\sigma, \mathcal{F})$$

is an isomorphism for every $\sigma \in \Delta$.

Since $\mathcal{F}$ is coherent, this implies that we can find a finitely generated graded submodule $P' \subset P$ such that $(P'_\sigma)_0 = (P_\sigma)_0$ for every $\sigma \in \Delta$. It is clear that this P' satisfies the assertion of the corollary. □

As in the case of projective space, the cohomology of the sheaf $\tilde{P}$ can be expressed as the local cohomology of the module P at the irrelevant ideal B.

PROPOSITION 1.3. *Let P be a graded S-module. Then there exist an isomorphism of graded modules*

$$H_B^{i+1}(P) \simeq \bigoplus_{\alpha\in \mathrm{Cl}(X)} H^i(X, \widetilde{P(\alpha)})$$

for every $i \geq 1$ and an exact sequence

$$0 \to H_B^0(P) \to P \to \bigoplus_{\alpha\in \mathrm{Cl}(X)} H^0(X, \widetilde{P(\alpha)}) \to H_B^1(P) \to 0\,.$$

PROOF. X is covered by the affine open subsets U_σ, $\sigma \in \Delta_{\max}$, and we compute the cohomology of $\tilde{P}$ as Cech cohomology with respect to this cover.

On the other hand, we can compute the local cohomology of P at B, using the direct limit of Koszul complexes on the powers of the generators of $B = (Y_{\hat{\sigma}} \mid \sigma \in \Delta_{\max})$ (see [Ei], Appendix 4.1).

Since for $\sigma_1, \ldots, \sigma_t \in \Delta_{\max}$, $\bigcap_{i=1}^t U_{\sigma_i} = U_\sigma$, where $\sigma = \bigcap_{i=1}^t \sigma_i$ and

$$H^0(U_\sigma, \widetilde{P(\alpha)}) = (P(\alpha)_{Y^{\hat{\sigma}}})_0 = (P_{Y_{\hat{\sigma}_1},\ldots,Y_{\hat{\sigma}_t}})_\alpha\,,$$

we conclude as in the case of the projective space (see [Ei], Appendix 4.1). □

NOTE. In the situation in Proposition 1.3, suppose that P is in fact a $\mathbf{Z}^{d+e}$-graded S-module, so that the corresponding sheaf $\tilde{P}$ is equivariant with respect to the torus action. In this case the local cohomology module $H_B^{i+1}(P)$ is $\mathbf{Z}^{d+e}$-graded, too, and under the isomorphism in Proposition 1.3 this finer decomposition of $H_B^{i+1}(P)$ corresponds to the eigenspace decomposition of the Zariski cohomology of the different twists $\widetilde{P(\alpha)}$.

2. Vanishing theorems. We keep the notation from the previous section. However, from now on we consider on S the fine $\mathbf{Z}^{d+e}$ grading by monomials and all S-modules are assumed to be $\mathbf{Z}^{d+e}$-graded. Note that this implies that the associated sheaf is equivariant with respect to the torus action. The canonical basis of $\mathbf{Z}^{d+e}$ will be denoted by $f_1, \ldots, f_{d+e}$.

For every subset $I \subset \mathbf{Z}$ and every graded S-module P, we will say that P is I-homogeneous if for every $\alpha = (\alpha_j) \in \mathbf{Z}^{d+e}$ with $\alpha_j \notin I$, the multiplication by Y_j:

$$\nu_{Y_j} : P_\alpha \to P_{\alpha+f_j}$$

is an isomorphism. Our main example is S, which is obviously $\{-1\}$-homogeneous.

PROPOSITION 2.1. *If P is an I-homogeneous S-module, then $H_B^i(P)$ is I-homogeneous.*

PROOF. We compute the local cohomology module as the cohomology of a Cech-type complex (see, for example, [Ei], Appendix 4.1). Let us temporarily denote the generators of B by $m_1, \ldots, m_t$. For a subset $L \subset \{1, \ldots, t\}$, let m_L be the least common multiple of $\{m_l \mid l \in L\}$. Since it is enough to prove the assertion at the level of complexes, we have to check that for every $\alpha \in \mathbf{Z}^{d+e}$ with $\alpha_j \notin I$, the multiplication by Y_j:

$$\mu_{Y_j} : (P_{m_L})_\alpha \to (P_{m_L})_{\alpha+f_j}$$

is an isomorphism.

This is obvious if $Y_j | m_L$. Suppose now that $Y_j \nmid m_L$. Then the assertion is clear once we notice that in this case, if $m/m_L^s \in P_{m_L}$, then $\deg(m/m_L^s)_j = \deg(m)_j$, so that we can apply the fact that P is I-homogeneous. □

We consider now an example of $\{-1, 0\}$-homogeneous S-modules. These are the modules which define the exterior powers Ω_X^i of the cotangent sheaf. For simplicity, in this case we will assume that X is smooth.

It is shown by Batyrev and Cox in [BC] that if the fan defining X is nondegenerate, then the cotangent bundle on X appears in an Euler sequence:

$$0 \to \Omega_X^1 \to \bigoplus_{j=1}^{d} \mathcal{O}_X(-D_j) \to \mathcal{O}_X^{d-n} \to 0\,.$$

In general, we have $X \simeq X' \times (k^*)^e$ with X' as above and $\Omega_X^1 \simeq p_1^*(\Omega_{X'}^1) \oplus \mathcal{O}_X^e$. Therefore we can include Ω_X^1 in an exact sequence:

$$0 \to \Omega_X^1 \to \left(\bigoplus_{j=1}^{d} \mathcal{O}_X(-D_j)\right) \oplus \mathcal{O}_X^e \to \mathcal{O}_X^{d-n+e} \to 0\,.$$

We consider the graded morphism inducing the epimorphism in the second exact sequence:

$$E = \left(\bigoplus_{j=1}^{d} S(-f_j) \right) \oplus S^e \to F = S^{d-n+e} .$$

For each $i \geq 1$, let M_i be the kernel of the induced map $\bigwedge^i E \to \bigwedge^{i-1} E \otimes F$.

LEMMA 2.2. *With the above notation, we have*

(i) $\tilde{M}_i \simeq \Omega^i_X$.

(ii) M_i *is* $\{-1, 0\}$*-homogeneous.*

PROOF. (i) The assertion follows easily from the above mentioned result of Batyrev and Cox and the fact that in the Euler sequence all the sheaves are locally free.

(ii) Since M_i is a submodule of $\bigwedge^i E$, which is free, the multiplication by Y_j on M_i is injective.

Let $\alpha = (\alpha_j) \in \mathbf{Z}^{d+e}$, $\alpha_j \notin \{-1, 0\}$. Since $\bigwedge^{i-1} E \otimes F$ is free, the surjectivity of the map $\nu_{Y_j} : (M_i)_\alpha \to (M_i)_{\alpha+f_j}$ follows from the surjectivity of the analogous map for $\bigwedge^i E$. The latter is surjective since $\bigwedge^i E$ is a direct sum of modules of the form $S(-f_{j_1} - \cdots - f_{j_r})$ with $r \leq i$ and $j_1 < \cdots < j_r$. □

PROPOSITION 2.3. *Let X be an arbitrary toric variety.*

(i) *If P is a $\{-1, 0\}$-homogeneous S-module and $L \in \mathrm{Pic}(X)$ is such that $H^i(\tilde{P} \otimes L^m) = 0$ for some $i \geq 0$ and $m \geq 1$, then $H^i(\tilde{P} \otimes L) = 0$. In particular, if X is projective and $L \in \mathrm{Pic}(X)$ is ample, then $H^i(\tilde{P} \otimes L) = 0$ for all $i \geq 1$.*

(ii) *Let P be a $\{-1\}$-homogeneous S-module such that for every $\alpha \in \mathrm{Cl}(X)$, $\widetilde{P(\alpha)} \simeq \tilde{P} \otimes \mathcal{O}(\alpha)$. Suppose that $D \in \mathrm{Div}_T(X)$ and that there is $E = \sum_{j=1}^{d} a_j D_j$ with $a_j \in \boldsymbol{Q}$ and $0 \leq a_j \leq 1$ such that $m(D + E)$ is integral and Cartier for some integer $m \geq 1$. If $H^i(\tilde{P} \otimes \mathcal{O}_X(D + m(D + E))) = 0$ for some $i \geq 0$, then $H^i(\tilde{P} \otimes \mathcal{O}_X(D)) = 0$. In particular, if X is projective and we have E aforementioned such that $D + E$ is $\boldsymbol{Q}$-ample, then $H^i(\tilde{P} \otimes \mathcal{O}_X(D)) = 0$ for all $i \geq 1$.*

PROOF. (i) If $L = \mathcal{O}(\alpha)$, $H^i(\tilde{P} \otimes L) = H^i(\widetilde{P(\alpha)})$ (see the remark in the first section). We will restrict ourselves to the case $i \geq 1$ in order to apply the isomorphism in Proposition 1.3. When $i = 0$, one can give a similar argument using the exact sequence in that proposition.

As already mentioned, we have

$$H^i(\tilde{P} \otimes L) \simeq \bigoplus_{\underline{b}} H^{i+1}_B(P)_{\underline{b}} ,$$

where the direct sum is taken over those $\underline{b} = (b_1, \ldots, b_{d+e}) \in \mathbf{Z}^{d+e}$ such that $[\sum_{i=1}^{d} b_i D_i] = \alpha$. Since by hypothesis $H^i(\tilde{P} \otimes L^m) = 0$, for every $\underline{b}$ with $[\sum_{i=1}^{d} b_i D_i] = \alpha$ we have $H^i_B(P)_{m\underline{b}} = 0$. Proposition 2.1 implies that

$$H^i_B(P)_{\underline{b}} \simeq H^i_B(P)_{m\underline{b}} ,$$

which proves the first assertion.

In the case of an ample line bundle L on a projective toric variety, we have $H^i(\tilde{P}\otimes L^m) = 0$ for $i \geq 1$ and $m \gg 0$, so that we are in the previous situation.

(ii) We proceed similarly. Using our hypothesis on P and Proposition 1.3, for every $i \geq 1$ we have

$$H^i(\tilde{P} \otimes \mathcal{O}(\alpha)) \simeq H^i(\widetilde{P(D)}) \simeq \bigoplus_{\underline{b}} H_B^{i+1}(P)_{\underline{b}} \,,$$

where the direct sum is taken over those $\underline{b} = (b_1, \ldots, b_{d+e}) \in \mathbf{Z}^{d+e}$ such that $[\sum_{i=1}^d b_i D_i] = [D]$.

Using again the hypothesis on P and the fact that $m(D+E)$ is Cartier (see the remark in the first section), we get

$$H^i((P(D+m(D+E)))\tilde{\ }) = 0\,.$$

We fix some $\underline{b} \in \mathbf{Z}^{d+e}$ with $[\sum_{i=1}^d b_i D_i] = [D]$. We have to prove that $H_B^{i+1}(P)_{\underline{b}} = 0$. If $\underline{b}' = \underline{b} + m(\underline{b} + \underline{a})$, where $\underline{a} = (a_1, \ldots, a_d, 0 \ldots, 0)$, then $[\sum_{i=1}^d b_i' D_i] = [D + m(D+E)]$, and therefore $H_B^{i+1}(P)_{\underline{b}'} = 0$.

Proposition 2.1 implies that in order to complete the proof, it is enough to show that $b_j \geq 0$ if and only if $(m+1)b_j + ma_j \geq 0$. This follows easily from the fact that $0 \leq a_j \leq 1$. □

We apply Proposition 2.3 in conjunction with Lemma 2.2 for $P = M_i$ and for $P = S$.

THEOREM 2.4. (i) (Bott-Steenbrink-Danilov) *If X is a smooth toric variety and $L \in \mathrm{Pic}(X)$ is such that $H^i(\Omega_X^j \otimes L^m) = 0$ for some $m \geq 1$ and $i \geq 0$, then $H^i(\Omega_X^j \otimes L) = 0$. In particular, if X is projective and $L \in \mathrm{Pic}(X)$ ample, then $H^i(\Omega_X^j \otimes L) = 0$ for every $i \geq 1$.*

(ii) *Let X be an arbitrary toric variety, $D \in \mathrm{Div}_T(X)$ and $E = \sum_{j=1}^d a_j D_j$, with $a_j \in \mathbf{Q}$ and $0 \leq a_j \leq 1$ such that for some integer $m \geq 1$ we have $m(D+E)$ integral and Cartier. If $H^i(\mathcal{O}_X(D + m(D+E))) = 0$, then $H^i(\mathcal{O}_X(D)) = 0$. In particular, if X is projective and there is E aforementioned such that $D+E$ is $\mathbf{Q}$-ample, then $H^i(\mathcal{O}_X(D)) = 0$ for all $i \geq 1$.*

REMARK. As pointed out by the referee, in the case $P = S$ the assertion in Proposition 2.1 can be proved also via the combinatorial description of the cohomology of a sheaf of fractional ideals (see for example [KKMS], p. 42). More precisely, the graded components $H_B^{i+1}(S)_\alpha$ and $H_B^{i+1}(S)_{\alpha+f_j}$ (or, equivalently, the corresponding eigenspaces of $H^i(X, \mathcal{O}(\alpha))$ and $H^i(X, \mathcal{O}(\alpha + f_j))$) can be described as simplicial cohomology groups of certain subsets of $\mathbf{R}^n$. The assertion can be proved by showing that these spaces are homotopically equivalent. Note that the case $P = S$ is enough to give the statement of Proposition 2.4 (ii).

If $D = \sum_{j=1}^d b_j D_j$ is a $\mathbf{Q}$-divisor, we define

$$\lceil D \rceil := \sum_{j=1}^d \lceil b_i \rceil D_j\,,$$

where for any real number x, $\lceil x \rceil$ is the integer defined by $x \leq \lceil x \rceil < x + 1$. Similarily, we define

$$\lfloor D \rfloor := \sum_{j=1}^{d} \lfloor b_j \rfloor D_j ,$$

where for every x, $\lfloor x \rfloor$ is the integer defined by $x - 1 < \lfloor x \rfloor \leq x$. K_X denotes the canonical divisor $-\sum_{j=1}^{d} D_j$ so that $\omega_X = \mathcal{O}(K_X)$.

COROLLARY 2.5. *Let X be a projective toric variety.*

(i) (Kawamata-Viehweg) *If $D = \sum_{j=1}^{d} b_j D_j$ is a **Q**-Cartier ample **Q**-divisor, then $H^i(\mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for every $i \geq 1$.*

(ii) *If D is as above, then $H^i(O_X(\lfloor D \rfloor)) = 0$ for every $i \geq 1$.*

(iii) *Let $L \in \mathrm{Pic}(X)$ be an ample bundle. If $D_{j_1}, \ldots, D_{j_r}$ are distinct prime invariant divisors, then $H^i(L(-D_{j_1} - \cdots - D_{j_r})) = 0$ for every $i \geq 1$.*

PROOF. All these are particular cases of Theorem 2.4 (ii). □

REMARK. In the proof of Fujita's Conjecture we will use the assertion in Corollary 2.5 for smooth varieties. As the referee pointed out, when X is smooth it is possible to prove this assertion directly, by induction on dimension and descending induction on r, as for $r = d$ this is just Kodaira's vanishing theorem.

As we mentioned in the Introduction, some particular cases of the above results can be proved by reducing the problem to a toric variety X over a field of positive characteristic p and prove that such a variety is Frobenius split. This means that if F is the Frobenius morphism, then the canonical morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ has a left inverse. With the description for the cohomology we used above this can be seen as follows.

First of all, by embedding X as an open subvariety of a complete toric variety, we may suppose that X is complete. Next, by taking a toric resolution of singularities, we may suppose that X is also smooth (see [MR]). Moreover, an argument in that paper shows that in this case, if $\dim(X) = n$, then X is Frobenius split if and only if the morphism

$$f : H^n(\omega_X) \to H^n(\omega_X^p)$$

induced by the Frobenius morphism is not trivial. But $H^n(\omega_X) \simeq H_B^{n+1}(S)_{(-1,\ldots,-1)} \simeq k$, all the other components being zero. On the other hand,

$$H^n(\omega_X^p) \simeq \bigoplus_{[\sum(a_j+p)D_j]=0} H_B^{n+1}(S)_{\underline{a}}$$

has by Proposition 2.1 the component $H_B^{n+1}(S)_{(-p,\ldots,-p)}$ canonically isomorphic with $H_B^{n+1}(S)_{(-1,\ldots,-1)}$ and therefore with k. It is easy to see that via these identifications, the corresponding component of f is just the Frobenius morphism of k, and therefore f is nonzero.

For a different approach to Frobenius splitting in the toric context and other applications we refer to Buch, Thomsen, Lauritzen and Mehta [BTLM].

3. Ample and numerically effective line bundles. Our main goal in this section is to give the condition for a line bundle to be ample or nef (i.e., numerically effective) in terms of the intersection with the invariant curves. For ampleness, this is the toric Nakai criterion which is proved in [Oda] in the smooth case and is stated also in the simplicial case. We obtain also a similar condition for the nef property, both the results holding for arbitrary complete toric varieties. In particular, we will see that on such a variety, a line bundle is nef if and only if it is globally generated. With a different proof, these results have been obtained also by Mavlyutov in [Ma]. We use the ideas in [Oda] together with the description for the intersection with divisors in the non-smooth case from [Fu].

We will apply these results to show that a line bundle L on X which is big and nef is a pull-back of an ample line bundle on X', for a proper birational equivariant map of toric varieties $\phi : X \to X'$. Recall that a line bundle L on X is called nef if for every curve $C \subset X$, $(L \cdot C) \geq 0$.

THEOREM 3.1. *If X is a complete toric variety and $L \in \mathrm{Pic}(X)$, the following are equivalent*:

(i) *L is globally generated.*

(ii) *L is nef.*

(iii) *For every invariant integral curve $C \subset X$, $(L \cdot C) \geq 0$.*

PROOF. (i) ⇒ (ii) is true in general and (ii) ⇒ (iii) follows from the definition.

We now prove the implication (iii) ⇒ (i). Let D be an invariant Cartier divisor such that $L \simeq \mathcal{O}(D)$. Recall that there is a function $\psi = \psi_D : N \otimes \boldsymbol{R} \to \boldsymbol{R}$ associated with D which is linear on each cone $\sigma \in \Delta$. It is defined in the following way: if $D|_{U_\sigma} = \mathrm{div}(\chi^{-u_\sigma})|_{U_\sigma}$, then $\psi|_\sigma = u_\sigma|_\sigma$ (the notation is that used in the first section).

A well-known result (see [Fu], Section 3.3) says that L is globally generated if and only if ψ is convex. Recall that $\dim(X) = n$. To prove that ψ is convex, it is enough to prove that for every $\sigma_1, \sigma_2 \in \Delta_n$ with $\dim(\sigma_1 \cap \sigma_2) = n-1$, $\psi|_{\sigma_1 \cup \sigma_2}$ is convex, i.e., for every $x \in \sigma_1$, $y \in \sigma_2$ and $t \in [0, 1]$ such that $tx+(1-t)y \in \sigma_1 \cup \sigma_2$, we have $\psi(tx+(1-t)y) \geq t\psi(x)+(1-t)\psi(y)$.

It is clear, therefore, from the definition of ψ that it is enough to prove that for each σ_1, σ_2 as above and each $D_i = V(\tau_i)$, with $\tau_i \subset \sigma_2 \setminus \sigma_1$ a one-dimensional cone,

$$u_{\sigma_2}(v_i) \leq u_{\sigma_1}(v_i)\,,$$

where v_i is the primitive vector of τ_i.

Let $D = \sum_{j=1}^{d} a_j D_j$. Note that by definition, if $D_j = V(\tau_j)$, $\tau_j \subset \sigma$, then $u_\sigma(v_j) = -a_j$. For σ_1 and σ_2 as above, let $\tau = \sigma_1 \cap \sigma_2$. Our hypothesis gives $(D \cdot V(\tau)) \geq 0$. By definition, $(D + \mathrm{div}(\chi^{u_{\sigma_1}}))|_{U_{\sigma_1}} = 0$. Therefore

$$D + \mathrm{div}(\chi^{u_{\sigma_1}}) = \sum_{\tau_i \subset \sigma_2 \setminus \sigma_1} b_i D_i + \cdots\,,$$

where we wrote down only the divisors corresponding to cones in $\sigma_1 \cup \sigma_2$. Since $a_i = -u_{\sigma_2(v_i)}$ for $\tau_i \subset \sigma_2$, we get

$$b_i = u_{\sigma_1}(v_i) - u_{\sigma_2}(v_i)\,,$$

if $\tau_i \subset \sigma_2 \setminus \sigma_1$.

On the other hand, let us denote by $\overline{e}$ the generator of the one-dimensional lattice N/N_τ such that the classes of the primitive vectors of τ_i for $\tau_i \subset \sigma_2 \setminus \sigma_1$ are positive multiples of $\overline{e}$. Here N_τ denotes the subgroup of N generated by $N \cap \tau$. If for every τ_i aforementioned we write $\overline{v}_i = c_i \overline{e}$, then the intersection formula in [Fu], Section 5.1 shows that

$$(D + \mathrm{div}(\chi^{u_{\sigma_1}}) \cdot V(\tau)) = b_i/c_i \ ,$$

for every $\tau_i \subset \sigma_1 \setminus \sigma_2$. Since $0 \leq (D \cdot V(\tau)) = b_i/c_i$ and $c_i > 0$, we deduce that $b_i \geq 0$ for every τ_i aforementioned. From the formula for b_i we see that the proof is complete. □

REMARK. The equivalence between (i) and (ii) above can be deduced also from the result of Reid from [Re], which says that every effective one dimensional cycle on X is rationally equivalent to an effective sum of invariant curves.

THEOREM 3.2 (Toric Nakai criterion). *If X is a complete toric variety, with* $\dim(X) = n$ *and* $L \in \mathrm{Pic}(X)$, *then the following are equivalent*:

(i) *L is ample.*

(ii) *For every invariant integral curve $C \subset X$, $(L \cdot C) > 0$.*

PROOF. The proof of the relevant implication (ii) ⇒ (i) is the same as the above proof for the implication (iii) ⇒ (i). We have just to use the fact that $L = \mathcal{O}(D)$ is ample if and only if ψ_D is strictly convex and to replace all the inequalities by strict inequalities. □

Recall that a line bundle $L \in \mathrm{Pic}(X)$ is called big if for a certain multiple L^m, the rational map it defines: $\phi_{L^m} : X \to \mathbf{P}^N$ has the image of maximal dimension $n = \dim(X)$.

PROPOSITION 3.3. (i) *If X is a complete toric variety of dimension n and $L \in$* $\mathrm{Pic}(X)$ *is a line bundle which is globally generated and big*, *then* $\dim \phi_L(X) = n$.

(ii) *$L \in \mathrm{Pic}(X)$ is globally generated and big if and only if there is a fan Δ' such that Δ is a refinement of Δ' and $L' \in \mathrm{Pic}(X')$ ample*, *where $X' = X(\Delta')$*, *and that if $f : X \to X'$ is the map induced by the refinement*, $f^*(L') \simeq L$.

PROOF. Let us fix an invariant Cartier divisor D such that $L \simeq \mathcal{O}(D)$. If ψ_D is the function which appeared in the proof of Theorem 3.1, it defines an associated convex polytope

$$P_D = \{u \in M \otimes \boldsymbol{R} \mid u \geq \psi_D \text{ on } N \otimes \boldsymbol{R}\}\,.$$

If L is globally generated, then $\dim \phi_L(X) = \dim P_D$ (see [Fu], Section 3.4). But $P_{mD} = mP_D$, so that $\dim \phi_L(X) = \dim \phi_{L^m}(X)$, which completes the proof of (i).

Since a map as in (ii) is birational, the "if" part in (ii) is trivial. Let us suppose now that L is globally generated and big. By the above argument, $P = P_D$ is an n-dimensional convex polytope. Such a polytope defines a complete fan Δ' and an ample Cartier divisor D' on $X' = X(\Delta')$. The cones in Δ' are in a one-to-one correspondence, reversing inclusions, with the faces of P: for a face Q of P we have the cone

$$C_Q = \{v \in N \otimes \boldsymbol{R} \mid \langle u, v\rangle \leq \langle u', v\rangle \text{ for all } u \in Q, u' \in P\}\,.$$

For every $\sigma \in \Delta_n$, u_σ is a vertex of P. Indeed, it is the intersection of P with

$$\{u \in M \otimes \boldsymbol{R} \mid \langle u, v_i \rangle \geq \psi_D(v_i) \text{ for } v_i \in \sigma\}\,.$$

In fact, every vertex of P is of this form. Indeed, if u_0 is a vertex of P, then there is $v \in N \otimes \boldsymbol{R}$ such that $\langle u_0, v \rangle < \langle u, v \rangle$, for all $u \in P \setminus \{u_0\}$. In particular, we have $\psi_D(v) = \langle u_0, v \rangle$. If $\sigma \in \Delta_n$ is such that $v \in \sigma$, then $\langle u_0, v \rangle = \langle u_\sigma, v \rangle$, so that $u_0 = u_\sigma$.

Now it is easy to check that

$$C_{u_\sigma} = \bigcup_{\tau \in \Delta_n, u_\tau = u_\sigma} \tau\,.$$

Therefore Δ is a refinement of Δ'. Moreover, the ample divisor D' on X' is defined by

$$\psi_{D'}(v) = \min_{\sigma \in \Delta_n} \langle u_\sigma, v \rangle = \psi_D(v)\,.$$

It follows that if $f : X \to X'$ is the map induced by the refinement, $f^*(D') = D$, which completes the proof. □

It is easy to see that using the results of this section, we can extend the form of the Kawamata-Viehweg theorem we obtained in the previous section to the case of a divisor which is big and nef. For the proof, however, we have to assume that the divisor is Cartier.

THEOREM 3.4 (Kawamata-Viehweg). *If X is a projective toric variety and $L \in$ $\operatorname{Pic}(X)$ is a line bundle which is nef and big, then $H^i(\omega_X \otimes L) = 0$ for every $i \geq 1$.*

PROOF. Since L is a line bundle, the duality theorem gives

$$H^i(X, \omega_X \otimes L) \simeq H^{n-i}(X, L^{-1})\,,$$

where $n = \dim(X)$ (see [Fu], Section 4.4).

Using Theorem 3.1 and Proposition 3.3, we get a morphism $f : X \to X'$, induced by a fan refinement, and $L' \in \operatorname{Pic}(X')$ ample such that $f^*(L') \simeq L$. But then

$$H^{n-i}(X, L^{-1}) \simeq H^{n-i}(X', {L'}^{-1}) \simeq H^i(X', \omega_{X'} \otimes L') = 0\,,$$

by Corollary 2.5. □

COROLLARY 3.5. *Let X be a complete toric variety and L a line bundle on X. If the base locus of L is nonempty, then it contains an integral invariant curve $C \subset X$.*

PROOF. This is an immediate consequence of Theorem 3.1, since for an integral curve $C \subset X$, if C is not contained in the base locus of L, then $(L \cdot C) \geq 0$. □

4. Fujita's conjecture on toric varieties. The main result of this section is the following strong form of Fujita's Conjecture in the toric case.

THEOREM 4.1. *Let X be an n-dimensional projective smooth toric variety, $L \in$ $\operatorname{Pic}(X)$ a line bundle and $D_1, \ldots, D_m$ distinct prime invariant divisors.*

(i) *If $(L \cdot C) \geq n$ for every invariant intergral curve $C \subset X$, then $L(-D_1 - \cdots - D_m)$ is globally generated, unless $X \simeq \mathbf{P}^n$, $L \simeq \mathcal{O}(n)$ and $m = n + 1$.*

(ii) *If* $(L \cdot C) \geq n+1$ *for every invariant integral curve* $C \subset X$, *then* $L(-D_1 - \cdots - D_m)$ *is very ample*, *unless* $X \simeq \mathbf{P}^n$, $L \simeq \mathcal{O}(n+1)$ *and* $m = n+1$.

In particular, we have the following corollary.

COROLLARY 4.2. *Let X be an n-dimensional projective smooth toric variety and* $L \in \operatorname{Pic}(X)$.

(i) *If* $(L \cdot C) \geq n$ *for every invariant integral curve* $C \subset X$, *then* $\omega_X \otimes L$ *is globally generated, unless* $(X, L) \simeq (\mathbf{P}^n, \mathcal{O}(n))$.

(ii) *If* $(L \cdot C) \geq n+1$ *for every invariant integral curve* $C \subset X$, *then* $\omega_X \otimes L$ *is very ample, unless* $(X, L) \simeq (\mathbf{P}^n, \mathcal{O}(n+1))$.

We prove Theorem 4.1, using the numerical conditions for L to be globally generated or ample, as well as the vanishing result in Corollary 2.5 (iii). The proof goes by induction on the dimension of X, based on the following proposition.

PROPOSITION 4.3. *Let X be a projective smooth toric variety with* $\dim(X) = n$, $L \in \operatorname{Pic}(X)$ *and* $l \geq 1$ *an integer. If* $(L \cdot C) \geq l$ *for every invariant integral curve* $C \subset X$, *then for every prime invariant divisor D and every C aforementioned,* $(L(-D) \cdot C) \geq l-1$.

We first deal with the case $l = 1$ of this proposition in the lemma below.

LEMMA 4.4. *Let X be a projective smooth toric variety*, $\dim(X) = n$. *If* $L \in \operatorname{Pic}(X)$ *is ample and D is an invariant prime divisor*, *then* $L(-D)$ *is globally generated.*

PROOF OF LEMMA 4.4. We prove the lemma by induction on n. For $n = 1$, $X = \mathbf{P}^1$ and the assertion is clear. If $n \geq 2$ and $L(-D)$ is not globally generated, since the base locus of $L(-D)$ is invariant, we can choose a fixed point x in this locus.

Let D' be a prime divisor distinct from D and containing x. By Corollary 2.5 (iii), the restriction map

$$H^0(L(-D)) \to H^0(L(-D)|_{D'})$$

is surjective. On the other hand, D' is a smooth toric variety of dimension $n-1$ and $D \cap D'$ is either empty or a prime invariant divisor on D'. Therefore the restriction map

$$H^0(L(-D)|_{D'}) \to H^0(L(-D)|_x)$$

is also surjective.

Since the composition of the above maps is surjective, we get a contradiction to the assumption that x is in the base locus of $L(-D)$. □

We now give the proof of the proposition for an arbitrary $l \geq 1$.

PROOF OF PROPOSITION 4.3. We make induction on n, the case $n = 1$ being trivial. Note that since $l \geq 1$, L is ample.

Let us assume now that $n = 2$. Clearly, it is enough to prove that $(L(-D) \cdot D) \geq l-1$. Since $(L(-D) \cdot D) = (L \cdot D) - (D^2)$, we may restrict ourselves to the case $(D^2) \geq 2$. From the description of the selfintersection numbers in terms of the fan Δ, it follows easily that if D' and D'' are the divisors whose rays are adjacent to the ray corresponding to D, then $(D'^2) \leq 0$ or $(D''^2) \leq 0$.

But if, for example, $(D'^2) \leq 0$, then $L(-(l-1)D')$ is ample, so that Lemma 4.4 implies that $L(-(l-1)D' - D)$ is globally generated and therefore

$$0 \leq (L(-(l-1)D' - D) \cdot D) = (L(-D) \cdot D) - (l-1)\,,$$

which completes the case $n = 2$.

Suppose now that $n \geq 3$ and let $\tau \in \Delta_{n-1}$ be such that $C = V(\tau)$. We can choose a prime invariant divisor D' such that $D' \neq D$ and $C \subset D'$. Therefore $(L(-D) \cdot C) = (L(-D)|_{D'} \cdot C)$, and we may clearly restrict to the case when $D \cap D' \neq \emptyset$, so that it is a prime invariant divisor on D'. We apply the induction hypothesis for $L|_{D'}$; note that for every integral invariant curve $C' \subset D'$,

$$(L|_{D'} \cdot C') = (L \cdot C') \geq l\,.$$

This concludes the proof. □

We can now prove the strong form of Fujita's conjecture for the toric case.

PROOF OF THEOREM 4.1. (i) It is clear that we may assume $n \geq 2$ and $X \not\simeq \mathbf{P}^n$. We make induction on n. If $L(-D_1 - \cdots - D_m)$ is not globally generated, then

$$(L(-D_1 - \cdots - D_m) \cdot V(\tau)) < 0$$

for some $\tau \in \Delta_{n-1}$. We will show that this asumption implies $X \simeq \mathbf{P}^n$, a contradiction.

We can immediately restrict ourselves to the following situation: $2 \leq m \leq n+1$, D_1 and D_2 are the divisors corresponding to the rays spanning together with τ maximal cones and $D_3, \ldots, D_{n+1}$ are the divisors containing $V(\tau)$.

Claim. We have $m = n+1$, $D_i \simeq \mathbf{P}^{n-1}$ for every i, $1 \leq i \leq n+1$, and $D_i \cap D_j \neq \emptyset$ for every $i \neq j$.

Fix i such that $i \leq m$. Since $n \geq 2$, our hypothesis and Proposition 4.3 imply that $L(-D_i)$ is ample. Hence Corollary 2.5 (iii) shows that the restriction map

$$H^0(L(-D_1 - \cdots - D_m)) \to H^0(L(-D_1 - \cdots - D_m)|_{D_i})$$

is surjective. Since $V(\tau) \subset \operatorname{Bs} L(-D_1 - \cdots - D_m)$, it follows that $L(-D_1 - \cdots - D_m)|_{D_i}$ is not globally generated.

Another application of Proposition 4.3 gives $(L(-D_i) \cdot C) \geq n-1$ for every integral invariant curve $C \subset X$. In particular, $(L(-D_i)|_{D_i} \cdot C') \geq n-1$ for every integral invariant curve $C' \subset D_i$. From the induction hypothesis we get $D_i \simeq \mathbf{P}^{n-1}$, $m = n+1$ and $D_i \cap D_j \neq \emptyset$ for $j \neq i$.

It is now easy to see that $X \simeq \mathbf{P}^n$. The claim implies that if $D_i = V(\tau_i)$,$1 \leq i \leq n+1$, and if τ_0 is any other one-dimensional cone in Δ, then τ_0 and τ_i do not span a cone in Δ for any i. From this it follows that the only one-dimensional cones in Δ are $\tau_1, \ldots, \tau_{n+1}$. Since X is smooth, it follows that $X \simeq \mathbf{P}^n$.

(ii) Since an ample line bundle on a complete smooth toric variety is very ample (see [De]), it is enough to prove that if $L(-D_1 - \cdots - D_m)$ is not ample, then $X \simeq \mathbf{P}^n$. Again we may assume $n \geq 2$.

If $L(-D_1 - \cdots - D_m)$ is not ample, then there exists an invariant integral curve $C \subset X$ such that

$$(L(-D_1 - \cdots - D_m) \cdot C) \leq 0\,.$$

As above, we may assume that D_1 and C correspond to cones in Δ spanning together a maximal cone.

By Proposition 4.3, we may apply (i) to $L(-D_1)$ and conclude that if $X \not\simeq \mathbf{P}^n$, then $L(-2D_1 - D_2 - \cdots - D_m)$ is globally generated. In particular,

$$(L(-2D_1 - D_2 - \cdots - D_m) \cdot C) \geq 0\,,$$

so that

$$(L(-D_1 - \cdots - D_m) \cdot C) \geq 1\,,$$

a contradiction. □

We conclude this section by giving two results with the same flavour as those proved above. By Lemma 4.4, if L is ample, then $L(-D)$ is globally generated for every integral invariant divisor. The case $X = \mathbf{P}^n$, $L = \mathcal{O}(1)$ shows that this is optimal. The next proposition gives the condition under which for L ample we get $L(-D_1 - D_2)$ globally generated for distinct divisors D_1 and D_2 as above.

PROPOSITION 4.5. *Let X be a projective smooth toric variety with* $\dim(X) = n$, $L \in \mathrm{Pic}(X)$ *ample and D_1, D_2 distinct prime invariant divisors. Then $L(-D_1 - D_2)$ is not globally generated if and only if there is an $(n-1)$-dimensional cone $\tau \in \Delta$ such that if τ_1, τ_2 are the one dimensional cones correponding to D_1 and D_2, then (τ, τ_1) and (τ, τ_2) span cones in Δ_n and $(L \cdot C) = 1$, where $C = V(\tau)$.*

PROOF. The "if" part is clear, since in this case we have $(L(-D_1 - D_2) \cdot V(\tau)) = -1$.

Suppose now that $L(-D_1 - D_2)$ is not globally generated. We prove the proposition by induction on n. The case $n = 1$ is trivial, and therefore we may assume $n \geq 2$. Let $x \in X$ be a fixed point in the base locus of $L(-D_1 - D_2)$.

Suppose first that there is an invariant prime divisor $D \neq D_1, D_2$ such that $x \in D$. We apply the induction hypothesis for the smooth toric variety D, the line bundle $L|_D$ and the prime invariant divisors $D \cap D_1$ and $D \cap D_2$. By Corollary 2.5 (iii), the restriction map

$$H^0(L(-D_1 - D_2)) \to H^0(L(-D_1 - D_2)|_D)$$

is surjective, so that our hypothesis on x and D implies that x is in the base locus of $L(-D_1 - D_2)|_D$. Lemma 4.4 implies that $D \cap D_1$ and $D \cap D_2$ are nonempty. If $D = V(\tau_0)$, then by induction we find a cone τ' in the fan $\mathrm{Star}(\tau_0)$ of D. This corresponds to a cone $\tau \in \Delta$ which satisfies the requirements of the proposition.

Therefore it remains to consider the case when, for every fixed point x in the base locus of $L(-D_1 - D_2)$ and every divisor D containing x, we have $D = D_1$ or $D = D_2$. Clearly this implies $n = 2$ and the fact that the base locus consists of a point, the corresponding cone being generated by the rays defining D_1 and D_2. But this contradicts Corollary 3.5 and the proof is complete. □

As a consequence of Proposition 4.3 we get that if $(L \cdot C) \geq 2$ for every integral invariant curve on X, then $L(-D)$ is ample for every prime invariant divisor D. The next result makes this more precise by giving the condition for an ample line bundle L and a prime invariant divisor D to have $L(-D)$ not ample.

PROPOSITION 4.6. *Let X be a complete smooth toric variety with* $\dim(X) = n$, $L \in \mathrm{Pic}(X)$ *an ample line bundle and $D = V(\tau_0)$ a prime invariant divisor. Then $L(-D)$ is not ample if and only if there is $\tau \in \Delta_{n-1}$ such that $\langle \tau, \tau_0 \rangle \in \Delta_n$ and* $(L \cdot V(\tau)) = 1$.

PROOF. It is clear that if there exists τ as above, then $(L(-D) \cdot V(\tau)) = 0$, so that $L(-D)$ is not ample.

Suppose now that $L(-D)$ is not ample and therefore there exists $\tau' \in \Delta_{n-1}$ such that $(L(-D) \cdot V(\tau')) \leq 0$. Since $L(-D)$ is globally generated by Lemma 4.4, we must have $(L(-D) \cdot V(\tau')) = 0$.

We must have $(D \cdot V(\tau')) \neq 0$, and therefore we deduce that either $\langle \tau_0, \tau' \rangle \in \Delta_n$ or $V(\tau') \subset D$. In the first case, we have $(L \cdot V(\tau')) = 1$ and may take $\tau = \tau'$.

If $V(\tau') \subset D$, we choose a divisor $D_1 = V(\tau_1)$ such that $\langle \tau_1, \tau' \rangle \in \Delta_n$. Then $(L(-D - D_1) \cdot V(\tau')) < 0$, and Proposition 4.5 implies that there is $\tau \in \Delta_{n-1}$ such that $\langle \tau_0, \tau \rangle \in \Delta_n$ and $(L \cdot V(\tau)) = 1$. □

5. Sections of ample line bundles. In this section we fix a globally generated line bundle L on a complete toric variety X and an invariant divisor D such that $L \simeq \mathcal{O}(D)$. Since L is globally generated, for each maximal cone σ there is a unique $u_\sigma \in M$ such that $\mathrm{div}(\chi^{u_\sigma}) + D$ is effective and zero on U_σ. Equivalently, for each maximal cone σ, there is a nonzero section $s_\sigma \in H^0(X, L)$, unique up to scalars, which is an eigenvector with respect to the torus action and whose restriction to U_σ is everywhere nonzero.

A well-known ampleness criterion (see [Fu], Section 3.4) says that L is ample if and only if $u_\sigma \neq u_\tau$ (or, equivalently, $k\, s_\sigma \neq k\, s_\tau$) for $\sigma \neq \tau$. From the unicity of the sections s_σ, this is equivalent to the fact that if $\sigma \neq \tau$, then $s_\sigma|_{U_\tau}$ vanishes at some point. But in that case, it must vanish at the unique fixed point x_τ of U_τ.

We consider the following map whose components are given by the restriction maps:

$$\phi : H^0(L) \to \bigoplus_{\sigma \in \Delta_{\max}} H^0(L|_{\{x_\sigma\}})\,.$$

Since ϕ is an equivariant map under the torus action, the discussion above shows that L is ample if and only if ϕ is surjective.

Our goal in this section is to extend this property of ample line bundles in the case when X is smooth to a set of higher dimensional subvarieties which are pairwise disjoint. More precisely, we have the following

THEOREM 5.1. *Let X be a projective smooth toric variety and $L \in \mathrm{Pic}(X)$ an ample line bundle. If $Y_1, \dots, Y_r \subset X$ are integral invariant subvarieties such that $Y_i \cap Y_j = \emptyset$ for $i \neq j$ and*

$$\psi : H^0(L) \longrightarrow \bigoplus_{i=1}^{r} H^0(L|_{Y_i})$$

is induced by restrictions, then ψ is surjective.

PROOF. Let $Y = \bigcup_{i=1}^{r} Y_i$. In order to prove that

$$\psi : H^0(L) \to H^0(L|_Y)$$

is surjective, it is enough to prove that $H^1(L \otimes \mathcal{I}_{Y/X}) = 0$.

Let $\pi : \tilde{X} \to X$ be the blowing-up of X along Y and E the exceptional divisor. Then

$$H^1(X, L \otimes \mathcal{I}_{Y/X}) \simeq H^1(\tilde{X}, \pi^* L \otimes \mathcal{O}(-E)) .$$

Since X is smooth, the blowing-up of X along an integral invariant subvariety is still a smooth toric variety ([Ew]). Since $Y_i \cap Y_j = \emptyset$ for $i \neq j$, π is a composition of such transformations, and therefore $\tilde{X}$ is a toric variety. Moreover, from the description in [Ew] it follows that if $E_i = \pi^{-1}(Y_i)$, then E_i is an invariant prime divisor on $\tilde{X}$ and $E = \sum_{i=1}^{r} E_i$.

Since L is ample, Proposition 7.10 in [Ha] implies that there is an integer $s \geq 1$ such that $\pi^*(L^s) \otimes \mathcal{O}(-E)$ is ample on $\tilde{X}$. We choose an invariant divisor D on $\tilde{X}$ such that $\pi^* L \simeq \mathcal{O}(D)$. Then $D - (1/s)E = D - (1/s)\sum_{i=1}^{r} E_i$ is $\boldsymbol{Q}$-ample and $\lfloor D - (1/s)E \rfloor = D - E$. Now Corollary 2.5 gives $H^1(\tilde{X}, \pi^* L \otimes \mathcal{O}(-E)) = 0$. □

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