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MEROMORPHIC FIRST INTEGRALS: SOME EXTENSION RESULTS

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Abstract. We present sufficient conditions of extending a meromorphic function which is defined outside an analytic compact curve in a complex surface. The function we deal with is a first integral for a holomorphic foliation in the whole surface. The key to extension is the study of singularities of the foliation on the complex curve.

1. Introduction. We consider a singular holomorphic foliation $\mathcal{F}$ in a complex surface M, equipped with a meromorphic first integral defined outside a compact complex curve S. We are basically concerned with the following question: under which conditions does $\mathcal{F}$ admit a meromorphic first integral in the entire surface M?

Proposition 2 asserts that when S is not $\mathcal{F}$-invariant, then the first integral extends to the whole M. To study the case where S is $\mathcal{F}$-invariant, some necessary hypotheses are set on the singularities of $\mathcal{F}$ in S: we assume that any singularity contained in S has no saddle-nodes in its desingularization. Such a singularity is called a *generalized curve*. We have:

THEOREM A. *Let $\mathcal{F}$ be a singular holomorphic foliation in a complex surface M admitting a meromorphic first integral h in $M \setminus S$, where S is a compact, smooth, connected complex curve. If some singularity of $\mathcal{F}$ in S is a non-dicritical generalized curve, then h extends to a meromorphic first integral for $\mathcal{F}$ in M.*

In the case where all singularities in S are dicritical (here, being *dicritical* means having an infinite number of separatrices), further hypothesis are set on the curve S:

THEOREM B. *Let $\mathcal{F}$ be a singular holomorphic foliation in a complex surface M admitting a meromorphic first integral h in $M \setminus S$, where S is a compact, smooth, connected complex curve with negative self-intersection number. If all singularities of $\mathcal{F}$ in S are generalized curves, then h extends to a meromorphic first integral defined in M.*

When S has non-negative self-intersection number, the extension is still possible if S contains an adequate amount of special dicritical singularities, which we call *ordinary dicritical*:

THEOREM C. *Let $\mathcal{F}$ be a singular holomorphic foliation in a complex surface M admitting a meromorphic first integral h outside a compact, smooth, connected complex curve S with self-intersection number $n \geq 0$. Suppose that the singularities of $\mathcal{F}$ in S are generalized*

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curves. If there are at least $n+1$ ordinary dicritical singularities in S, then h extends to a meromorphic first integral in M.

The basic tool for the proofs of Theorems A, B and C is Lemma 2, which is called Extension Lemma. It asserts that a meromorphic first integral in a neighborhood of one of the separatrices of a simple singularity extends to a neighborhood of the singularity. This, along with some results on the extension of meromorphic functions, transforms our problem into one of finding separatrices through the desingularization divisor.

Sections 5 and 6 are devoted to the situation where M is a complex projective space $\boldsymbol{C}P^n$. We study the problem in dimension two and then show how the problem in $\boldsymbol{C}P^n$ reduces to a two-dimensional one.

In Section 7 we give conditions upon that similar extension theorems apply to a foliation by curves in a complex manifold M of dimension n. Finally, in Section 8, we produce variants of Theorems A, B and C where we extend closed meromorphic one-forms defining a foliation in a complex surface. With some adaptations, the techniques of the previous sections also apply to this situation.

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2. Proofs of the main theorems. Let $\mathcal{F}$ be a singular holomorphic foliation defined in a complex surface M, that is, a two-dimensional complex manifold. By a singular holomorphic foliation we mean a holomorphic foliation outside an analytic set $s(\mathcal{F})$, the *singular set* of $\mathcal{F}$, of codimension two or greater. We remark that, as a consequence of Levi's extension theorem, a singular holomorphic foliation of dimension one is induced by a holomorphic vector field in a neighborhood of each point (see [L]). We say that a point $p \in s(\mathcal{F})$ is a reduced singularity if the eigenvalues λ_1 and λ_2 of the linear part of a vector field which defines $\mathcal{F}$ at p satisfy one of the following:

(i) $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_2/\lambda_1 \notin \boldsymbol{Q}^+$;

(ii) $\lambda_1 \neq 0, \lambda_2 = 0$ or $\lambda_1 \neq 0, \lambda_2 = 0$.

Singularities of type (i) are said to be *simple*. The special case in which $\lambda_1/\lambda_2 \in \boldsymbol{Q}^-$ is called a *resonance*. Singularities of type (ii) are called *saddle-nodes*.

A meromorphic function h is called a *meromorphic first integral* for $\mathcal{F}$ if its indeterminacy set is contained in $s(\mathcal{F})$ and its level curves contain the leaves of $\mathcal{F}$. Simple singularities which admit meromorphic first integrals are linearizable resonances, as the following results prove:

PROPOSITION 1. *Let p be a reduced singularity of $\mathcal{F}$ admitting a meromorphic first integral in some neighborhood. Then p is a resonance.*

PROOF. Suppose first that p is simple and non resonant. Then $\mathcal{F}$ is formally linearizable (see [CS1]); in formal coordinates at p, $\mathcal{F}$ is given by $\omega_p = xdy - \lambda ydx,\ \lambda \in \boldsymbol{C} - \boldsymbol{Q}$.

Write

$$F(x,y) = \sum_{m\geq m_0, n\geq n_0} a_{mn}x^m y^n\,,$$

the development in Laurent series of the (formal) first integral for $\mathcal{F}$. We have

$$\begin{aligned} 0 &= \omega_p \wedge dF \\ &= (xdy - \lambda y dx)\wedge\Big(\sum_{m\geq m_0, n\geq n_0} m a_{mn} x^{m-1} y^n dx + \sum_{m\geq m_0, n\geq n_0} n a_{mn} x^m y^{n-1} dy\Big) \\ &= -\sum_{m\geq m_0, n\geq n_0} (m+\lambda n) a_{mn} x^m y^n dx \wedge dy\,. \end{aligned}$$

Since $\lambda \notin \boldsymbol{Q}$, we must have $a_{mn} = 0$ for any $(m,n) \neq (0,0)$, contradicting the fact that F is non-constant.

Similar formal calculations employing Dulac's normal form ([CS1]) show that p cannot be a saddle-node. □

We say that a one-dimensional analytic set S is a *separatrix* through $p \in s(\mathcal{F})$ if $p \in S$ and S is invariant by $\mathcal{F}$. We remark that a simple singularity admits a pair of smooth separatrices. For a saddle node, we can assure the existence of one smooth separatrix (see [CS1]). In general, a singularity always admits at least one separatrix ([CS]).

LEMMA 1 (Linearization lemma). *Let $p \in s(\mathcal{F})$ and S a separatrix for $\mathcal{F}$ at p. Suppose that $\mathcal{F}$ admits a meromorphic first integral F in a neighborhood V of $S^* = S \setminus \{p\}$. Then the holonomy with respect to S is linearizable.*

PROOF. Let $\gamma : [0,1] \to S^*$ be a simple closed path around p. Let $q = \gamma(0) = \gamma(1)$ and Σ a small complex disc centered at q, contained in V and transversal to $\mathcal{F}$, provided with a complex coordinate w. If $h_\gamma : \Sigma \to \Sigma$ is the holonomy map associated to γ, then $F(h_\gamma(w)) = F(w)$ for any $w \in \Sigma$. Setting a new complex coordinate z in which $F|_\Sigma$ reads $F|_\Sigma(z) = z^n$, we have $(h_\gamma(z))^n = z^n$. Therefore, $h_\gamma(z) = e^{2\pi i k/l} z$, where $k, l \in \boldsymbol{Z}$ and $l|n$. □

LEMMA 2 (Extension lemma). *Let $p \in s(\mathcal{F})$ and S a separatrix for $\mathcal{F}$ at p. Suppose that $\mathcal{F}$ admits a meromorphic first integral F in a neighborhood V of $S^* = S \setminus \{p\}$. If p is a simple singularity, then F extends meromorphically to a neighborhood of p.*

PROOF. The previous Lemma and [MM] show that $\mathcal{F}$ is a linearizable resonance at p; there exists a system of coordinates (x,y) centered at p such that $\mathcal{F}$ is defined by $\omega = xdy - \lambda y dx$, $\lambda \in \boldsymbol{Q}^+$. Write $\lambda = -p/q$, $p, q \in \boldsymbol{N}$, $(p,q) = 1$. Suppose that S has the local equation $\{y = 0\}$. Developing F in the Laurent series

$$F(x,y) = \sum_{n\geq n_0} a_{mn}x^m y^n,$$

we have

$$0 = dF \wedge \omega = \sum_{n \geq n_0} (mq - np) a_{mn} x^m y^n dx \wedge dy.$$

We see that $a_{mn} \neq 0$ if and only if $mq - np = 0$, which occurs if and only if there exists $l \in \boldsymbol{Z}$ such that $m = lp$ and $n = lq$. It is then possible to rewrite

$$F(x, y) = \sum_{l \geq l_0} a_{lp,lq} (x^p y^q)^l$$

for some $l_0 \in \boldsymbol{Z}$. This shows that F extends meromorphically to a neighborhood of 0. □

Levi's extension theorem, which provides the extension of a meromorphic function defined in a Hartogs' domain to its holomorphic closure ([Siu]), allows us to prove the following:

LEMMA 3. *Let M be a complex surface and S a smooth, compact, connected complex curve. Suppose that h is a meromorphic function defined in $M \setminus S$. If h extends as a meromorphic function to $(M \setminus S) \cup V_p$, where V_p is a neighborhood of a point $p \in S$, then it extends meromorphically to M.*

PROOF. Let $\mathcal{W}$ be the union of the points $q \in S$ for which there exists a neighborhood V_q, $q \in V_q$, such that h extends meromorphically to $(M \setminus S) \cup V_q$. $\mathcal{W}$ is non-empty by hypothesis and open from its definition. Let us show that it is closed. Take $p_0 \in S$ in the closure of $\mathcal{W}$. This means that there exists a sequence $q_n \in \mathcal{W}$ such that $q_n \to p_0$. Chose a coordinate neighborhood U_{p_0} around p_0 , $\Phi = (x, y) : U_{p_0} \to \boldsymbol{C}^2$ a coordinate chart, such that $P := \Phi(U_{p_0})$ is a polydisc and $\Phi(S_0 \cap U_{p_0}) = \{y = 0\}$. Take n_0 sufficiently large so that $q_{n_0} \in U_{p_0}$. Then $P \setminus \{y = 0\} \cup \Phi(U_{p_0} \cap V_{q_{n_0}})$ is a Hartogs' domain. Levi's theorem assures that h extends meromorphically to U_{p_0}. Therefore, $p_0 \in \mathcal{W}$ and the result follows. □

Let $\mathcal{F}$ be a foliation in a complex surface M admitting a meromorphic first integral in $M \setminus S$, where S is a smooth, compact, connected complex curve. We are concerned with finding conditions for extending the meromorphic function to the whole M. First of all, if S is not $\mathcal{F}$-invariant, then extension is immediate:

PROPOSITION 2. *Let M be a complex surface with a singular holomorphic foliation $\mathcal{F}$ admitting a meromorphic first integral h in $M \setminus S$, where S is a smooth, connected complex curve. If S is not $\mathcal{F}$-invariant, then h extends to M as a meromorphic first integral for $\mathcal{F}$.*

PROOF. Let $p \in S$ be a regular point of $\mathcal{F}$ where the foliation is transversal to S. Choose a coordinate neighborhood U_p around p and $\Phi = (x, y) : U_p \to \boldsymbol{C}^2$ a coordinate chart such that $P := \Phi(U_p)$ is a polydisc, $\Phi(S \cap U_p) = \{y = 0\}$ and $\mathcal{F}|_{U_p}$ is a foliation with vertical leaves given by $dx = 0$. Since h is a first integral, we have that $h(x, y) = h(x)$ for $(x, y) \in P \setminus \{y = 0\}$. Therefore, h extends meromorphically to $\{y = 0\}$ by setting

$h(x, 0) = h(x)$. This yields the extension of h to S outside $s(\mathcal{F}) \cap S$ and the points of tangency between $\mathcal{F}$ and S. They constitute, however, a codimension two analytic set, and the meromorphic extension to them is straight. □

Suppose now that S is invariant by $\mathcal{F}$. From Proposition 1, it is reasonable to assume that the singularities of $\mathcal{F}$ over S do not have saddle-nodes in their desingularization; they are *generalized curves*, according to the definition in [CLS].

Let $\pi : \tilde{M} \to M$ be a sequence of blow-ups that desingularizes $s(\mathcal{F}) \cap S$ (see [Sei]). We consider the desingularization divisor $D = \pi^{-1}(S) = \bigcup_{i=0}^{n} P_i$, where $P_0 = \pi^*(S)$ is the strict transform of S and $\bigcup_{i=1}^{n} P_i = \pi^{-1}(s(\mathcal{F}) \cap S)$ are the projective lines associated to the blow-ups. Let $\tilde{\mathcal{F}}$ be the foliation induced in $\tilde{M}$ and $\tilde{h} = h \circ \pi$ its meromorphic first integral defined in $\tilde{M} \setminus D$. Among $P_1, \ldots, P_n$ there are perhaps some non-invariant lines. By the previous proposition, $\tilde{h}$ automatically extends to these lines outside their intersection with other invariant lines.

Let $\tilde{D}$ be the set of invariant curves in D. We decompose $\tilde{D} = \bigcup_{j=0}^{k} D_j$, where each D_j is connected and $D_i \cap D_j = \emptyset$ if $i \neq j$. D_0 is taken to be the component which contains P_0. Our job is now reduced to searching separatrices through each D_j which are not contained in D_j. Since we are dealing with generalized curves, this is equivalent to the existence of a singularity of $\tilde{\mathcal{F}}$ outside a corner. Suppose such a separatrix exists at a point p contained in some component D_{j_0}. Denote by S_0 the separatrix and by P_{i_0} the line which contains p. Since p is not a saddle-node and a meromorphic first integral is defined in a neighborhood of $S_0 \setminus \{p\}$, by applying Extension Lemma 2, it is possible to extend $\tilde{h}$ to a neighborhood of p. Lemma 3 allows us to extend $\tilde{h}$ to $P_{i_0} \setminus \{q_1, \ldots, q_l\}$, the points of intersection of P_{i_0} with other lines in D_{j_0}. Now we apply the same process and extend $\tilde{h}$ to a neighborhood of each q_j and, as a consequence, to the lines which contain them. This procedure is repeated until $\tilde{h}$ is extended throughout D_{j_0}.

Next we show that it is always possible to find a separatrix through $D_1, \ldots, D_n$. We do not always assure the existence of a separatrix through D_0. However, some conditions may be given so that this occurs.

Let M be a complex surface and $\mathcal{F}$ a singular holomorphic foliation. The *algebraic multiplicity* (or simply the *multiplicity*) of $\mathcal{F}$ at $p \in M$, denoted by $m_p(\mathcal{F})$, is the lowest order of the terms appearing in the Taylor series of ω_p, some holomorphic one-form which gives the foliation at p. Let S be a smooth separatrix through p. Choose a local system of coordinates (x, y) at p such that $S = \{y = 0\}$ and $\omega_p = p(x, y)dx + q(x, y)dy$ is a defining one-form for $\mathcal{F}$. The *tangent multiplicity* of $\mathcal{F}$ and S at p, $m_p(\mathcal{F}, S)$, is the order of $q(x, 0)$ at $x = 0$. If S is one of the separatrices of a simple singularity, or the strong separatrix of a saddle node, then $m_p(\mathcal{F}, S) = 1$. We also have that p is a regular point if and only if $m_p(\mathcal{F}, S) = 0$.

Let $\pi : \tilde{M} \to M$ be a sequence of blow-ups starting at $p \in M$ and $D = \pi^{-1}(p)$ the associated divisor. It is proved in [CLS] that

$$m_p(\mathcal{F}) + 1 = \sum_{q \in P \subset D} (\rho(P)) m_q^*(\mathcal{F}, P),$$

where

$$m_q^*(\mathcal{F}, P) = \begin{cases} m_q(\mathcal{F}, P) & \text{if } q \text{ is not a corner}\,, \\ m_q(\mathcal{F}, P) - 1 & \text{if } q \text{ is a corner} \end{cases}$$

and $\rho(P)$ is a weight associated to P. For our purposes, it is sufficient to know that $\rho(P) = 1$ when P is associated to the first blow-up.

LEMMA 4. *Let p be a singularity of a singular holomorphic foliation $\mathcal{F}$ admitting a smooth separatrix S. Suppose that p is a generalized curve. Then p admits another separatrix distinct from S.*

PROOF. If p is dicritical, there is nothing to prove. Suppose that p admits a finite number of separatrices. If p is already reduced, then it is simple and has two transversal smooth separatrices. If p is not reduced, we desingularize it and prove by induction in the number of blow-ups.

Suppose first that one blow-up desingularizes $\mathcal{F}$. Denote by P the projective line introduced, by $\tilde{S}$ the strict transform of S (which is smooth and transversal to P), and set $p_0 = P \cap \tilde{S}$. If there exists another singularity of $\tilde{\mathcal{F}}$ in P, it is reduced and has a separatrix transversal to P. So, let us examine the case where p_0 is a unique singularity in P. It is reduced and has P and $\tilde{S}$ as the set of its separatrices. We have

$$m_p(\mathcal{F}) + 1 = m_{p_0}(\tilde{\mathcal{F}}, P) = 1\,,$$

which implies $m_{p_0} = 0$, an absurdity.

Suppose now that $n > 1$ is the number of blow-ups necessary to desingularization and that the result is already proved for singularities which desingularize in less than n steps. Let us perform a first blow-up at p, introducing P, $\tilde{S}$ and p_0 as above. If there exists a singularity $q \in P$, distinct from p_0, then the induction hypothesis applies to assure the existence of a separatrix through q distinct from P. It remains to consider the case where the only singularity in P is p_0, having P and $\tilde{S}$ as the set of its separatrices. However, according to [CLS], a generalized curve having exactly two transversal smooth separatrices is reduced. The argument of the preceding paragraph applies here to achieve a contradiction. □

REMARK 1. Lemma 4 may be false if S is not smooth. For instance, take $p = (0, 0) \in \boldsymbol{C}^2$, $S : x^2 - y^3 = 0$ and $\mathcal{F} : d(x^2 - y^3) = 2x dx - 3y^2 dy = 0$. p is a generalized curve having S as its unique separatrix.

At this point, we are ready to prove Theorem A:

PROOF OF THEOREM A. We suppose that S is $\mathcal{F}$-invariant, since the other case was already proved. Applying Lemma 4, we extend h to $S \setminus \{p_1, \ldots , p_n\}$, where $p_1, \ldots , p_n$ are the other singularities of $\mathcal{F}$ in S. Since these points form a codimension two analytic set, h extends through them, yielding a meromorphic first integral for $\mathcal{F}$ defined in M. □

Remark that the conclusion of the theorem implies that all singularities of $\mathcal{F}$ in S are generalized curves.

Let M be a complex surface. Let $S = \bigcup_{i=1}^{n} S_i \subset M$ be a finite union of compact complex curves. The matrix $M_S = (s_{ij})_{1\le i,j\le n}$, where $s_{ij} = S_i \cdot S_j$, is called the *intersection matrix* associated to S. Notice that M_S is symmetric and has real entries.

Observe that if $M_0 \in M_n(\boldsymbol{R})$ is symmetric and $Q \in M_n(\boldsymbol{R})$ is non-singular, then M_0 is negative definite if and only if $Q^t M_0 Q$ is. As a consequence, a permutation of columns followed by the corresponding permutation of lines of a negative definite, symmetric, real matrix yields a negative definite, symmetric, real matrix. This means that the negative definiteness of the intersection matrix of a curve is independent from the enumeration associated to its components. The following is proved in [La]:

THEOREM 1. *Let $\pi : \tilde{M} \to M$ be a sequence of a finite number of blow-ups at $p \in M$ and $D = \pi^{-1}(p)$, $D = \bigcup_{i=1}^{n} P_i$, where P_i are projective lines. Then the intersection matrix M_D is negative definite.*

We establish now a connection between the negative definiteness of the intersection matrix M_S and the existence of separatrices through a divisor S.

Let $S = \bigcup_{i=1}^{n} S_i$ be a union of complex curves in a complex surface M. To S we associate a graph Γ_S constructed in the following way: The set of vertices $V_{\Gamma_S} = \{V_1, \dots, V_n\}$ corresponds bijectively to the set of components of S; to each point in $S_i \cap S_j$ we define an edge connecting V_i and V_j. We have the following proposition:

PROPOSITION 3 ([C]). *Let M be a complex surface with a singular holomorphic foliation $\mathcal{F}$. Let $S = \bigcup_{i=1}^{m} S_i$ be a union of $\mathcal{F}$-invariant compact smooth complex curves. Suppose that the singularities of $\mathcal{F}$ in S are non-dicritical and*

(i) *The associated graph Γ_S is a tree*,

(ii) *M_S is negative definite.*

Then, there exists a separatrix through S.

LEMMA 5. *Let $M_0 \in M_n(\boldsymbol{R})$ be a symmetric negative-definite matrix. If $M_1 \in M_{n_1}(\boldsymbol{R})$ is a submatrix of M_0 in its diagonal, then M_1 is negative-definite.*

PROOF. We may suppose that M_0 has the form

$$M_0 = \begin{pmatrix} M_1 & N^t \\ N & M_2 \end{pmatrix},$$

where $M_2 \in M_{n-n_1}(\boldsymbol{R})$ and $N \in M_{(n-n_1)\times n_1}(\boldsymbol{R})$. If $\mathbf{v} \in \boldsymbol{R}^{n_1}$, $\mathbf{v} \neq 0$, then we have

$$\mathbf{v} M_1 \mathbf{v}^t = (\mathbf{v}, 0) M_0 (\mathbf{v}, 0)^t < 0,$$

since M_0 is negative definite. This accomplishes the proof. □

Suppose that M carries a singular holomorphic foliation $\mathcal{F}$. Let π be a sequence of blow-ups that desingularizes $p \in s(\mathcal{F})$ and $D = \pi^{-1}(p)$ the associated divisor. Denote by $\tilde{D}$ the union of all invariant lines in D. Write $\tilde{D} = \bigcup_{i=1}^{n} D_i$, where each D_i is a connected set coposed by union of projective lines and $D_i \cap D_j = \emptyset$ if $i \neq j$. We have

PROPOSITION 4. *There exists a separatrix through each D_i.*

PROOF. In fact, after renumbering the projective lines in D if necessary, each M_{D_i} will be a submatrix in the diagonal of M_D. The result follows from the fact that M_D is negative definite. □

We are at the point of proving Theorem B:

PROOF OF THEOREM B. When S is not $\mathcal{F}$-invariant, the result is already proved. If S is $\mathcal{F}$-invariant, perform the desingularization of $s(\mathcal{F}) \cap S$. Denote by π the sequence of blow-ups. Easy calculations show that blowing up a divisor with negative definite intersection matrix yields a divisor with negative definite intersection matrix. The proof of Proposition 4. shows that a divisor contained in a larger divisor with negative definite intersection matrix also has negative definite intersection matrix. Since we depart from a curve S with negative self-intersection number, these facts show that the largest connected set containing $\pi^*(S)$ composed by the union of invariant curves of $\pi^{-1}(S)$ has negative definite intersection matrix. This assures that it is crossed by a separatrix. It is therefore possible to extend h to M. □

We remark that Theorem B may be proved through more general results. A divisor with negative definite intersection matrix may be blown down to a complex surface having normal singularities ([La], Theorem 4.9 and Proposition 4.6). On the other hand, a theorem of Levi assures the extension of a meromorphic function defined outside a codimension-two variety in a normal complex space ([N], Theorem VII-4). The proof we present here has a virtue of relying on properties of foliated surfaces.

In the following lines we make an attempt to extend a meromorphic first integral through a smooth complex curve with non-negative self-intersection number.

Let p be a non-reduced singularity of $\mathcal{F}$ in an invariant curve S, which is smooth at p. A *linear chain* at p (with respect to S) (see [CS]) is a sequence of blow-ups performed in the following way: Let π_1 be a blow-up at p and $P_1 = \pi_1^{-1}(p)$. If $p_1 = \pi_1^*(S) \cap P_1$ is reduced, then the linear chain at p_1 is π_1. If p_1 is non-reduced, then make another blow-up π_2 at p_1 and, if necessary, successive blow-ups at the corners, until all of them are reduced; the linear chain at p consists of the composition $\pi_n \circ \ldots \circ \pi_1$ of these blow-ups. We make the following definition:

DEFINITION 1. Let p be a singularity of a germ of holomorphic foliation $\mathcal{F}$ admitting a germ of smooth separatrix S. We say that p is an *ordinary dicritical singularity* if the desingularization of p has one non-invariant projective line lying in the divisor associated to the first linear chain with respect to S.

EXAMPLE 1. Let S_1 and S_2 be two smooth algebraic curves in $\boldsymbol{C}P^2$. Choose an affine plane $\boldsymbol{C}P^2 \setminus L_\infty$ such that L_∞ does not intersect $S_1 \cap S_2$. Let $p_1(x, y) = 0$ and $p_2(x, y) = 0$ be irreducible polynomial equations for S_1 and S_2 in $\boldsymbol{C}P^2 \setminus L_\infty$. Let $\mathcal{F}$ be the foliation in $\boldsymbol{C}P^2$ induced by $\omega(x, y) = p_2^2 d(p_1/p_2) = p_1 dp_2 - p_2 dp_1 = 0$. Then $S_1 \cap S_2$ is composed by dicritical singularities of $\mathcal{F}$ which are ordinary dicritical with respect to both S_1 and S_2. We remark that if S_1 and S_2 are transversal, then, by Bezout's theorem, $S_1 \cap S_2$ has degree(S_1)degree(S_2) points. In particular, if degree$(S_1) <$ degree(S_2), then S_1 contains more than $(\text{degree}(S_1))^2 = S_1 \cdot S_1$ ordinary dicritical singularities.

The above definition explains the statement of Theorem C, which we prove now:

PROOF OF THEOREM C. We prove by induction in the intersection number of S. Suppose first $S \cdot S = 0$. Let $p \in S$ be an ordinary dicritical singularity. If, in the sequence of blow-ups that produces the linear chain from p, a dicritical line intersects the strict transform of S, then, at this moment, this will have negative self-intersection number. Theorem C applies to this case. Otherwise, we will reach the following situation: The strict transform $\tilde{S}$ of S will have self-intersection number at most -2, while the intersection number of the projective line P (intersecting S) will be -1. The intersection matrix associated to the divisor $\tilde{S} \cup P$ will clearly be negative definite. Further steps in the desingularization process will take this to a divisor with negative definite intersection matrix.

Suppose now that $S \cdot S = n > 0$ and the result is valid for curves with self-intersection number less than n. We may suppose that all $n+1$ ordinary dicritical singularities lie in the second case of the previous paragraph. Otherwise we reduce to a curve of smaller intersection number and apply the induction hypothesis. After an appropriate sequence of blow-ups, we reach the situation where $\tilde{S}$ has self-intersection number at most $n - 2(n+1) = -n-2$ and $P_i \cdot P_i = -1$ for $i = 1, \ldots, n+1$ (each P_i is a projective line intersecting $\tilde{S}$ belonging to the first linear chain of one of the singularities related above). The divisor $D = \tilde{S} \cup P_1 \cup \ldots \cup P_{n+1}$ has the following $(n+2) \times (n+2)$ intersection matrix

$$M_D = \begin{pmatrix} \tilde{S} \cdot \tilde{S} & 1 & \ldots & 1 \\ 1 & -1 & \ldots & 0 \\ & & \ldots & \\ 1 & 0 & & -1 \end{pmatrix},$$

which is negative definite. This concludes the proof. □

3. Some Consequences. We present in this section several situations where Theorems A, B and C apply.

COROLLARY 1. *Let $\mathcal{F}$ be a parabolic foliation on $\boldsymbol{C}P^2$ whose leaves are proper outside some algebraic invariant curve $S \subset \boldsymbol{C}P^2$. Assume that the singularities of $\mathcal{F}$ along S satisfy the hypothesis of Theorem A, B or C. Then $\mathcal{F}$ exhibits a rational first integral.*

PROOF. A theorem of Suzuki ([Su]) implies that $\mathcal{F}$ admits a meromorphic first integral on $\boldsymbol{C}P^2 \setminus S$, since S is a Stein manifold. □

COROLLARY 2. *Let X be a polynomial vector field on $\boldsymbol{C}^2$. Suppose that the orbits of X have total finite curvature and are complete for the Euclidean metric on $\boldsymbol{C}^2$* (this implies that the line at infinity, l_∞, is invariant). *If there are no affine invariant lines for X and if the singularities of the corresponding projective foliation on $\boldsymbol{C}P^2$ are as in Theorem A, we conclude that X admits a rational first integral and its orbits are contained in algebraic curves.*

PROOF. A well-known theorem of Osserman on minimal surfaces assures that each orbit is a parabolic Riemann surface ([W]), so that $\mathcal{F}$ is parabolic. According to [Sc] the fact that the total curvature is finite also implies that the orbits are properly embedded in $\boldsymbol{C}^2$. The result then follows from the corollary above. □

COROLLARY 3. *Let $\mathcal{F}$ and $\mathcal{F}_1$ be projective foliations on $\boldsymbol{C}P^2$. Assume that $\mathcal{F}$ is a pencil by algebraic curves of genus $g \geq 2$, and that there exists some analytic automorphism $T : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ that conjugates $\mathcal{F}$ and $\mathcal{F}_1$ on $\boldsymbol{C}^2$. Assume also that the singularities of $\mathcal{F}_1$ along the line at infinity are as in Theorem A. Then $\mathcal{F}_1$ admits a rational first integral and T is algebraic.*

PROOF. First we observe that $\mathcal{F}_1$ admits a meromorphic first integral and therefore a rational first integral by Theorem A. Therefore T is an analytic automorphism of $\boldsymbol{C}^2$ that takes algebraic curves into algebraic curves. Since the algebraic curves involved have genus $g \geq 2$ it follows from a result of Kizuka ([K]) that T must be algebraic. □

4. Examples. We give some examples where there are obstructions to extend a meromorphic first integral.

EXAMPLE 2. Consider the foliation $\mathcal{F}$ in $\boldsymbol{C}P^2$ induced by

$$\omega = dy - (a(x)y + b(x))dx = 0\,,$$

where $a(x)$ and $b(x)$ are polynomials. Let $A(x)$ be a primitive for $a(x)$ and $B(x)$ a primitive for $b(x)/\exp(A(x))$. The meromorphic function

$$F(x, y) = \frac{y}{\exp(A(x))} - \exp(B(x))$$

is a first integral for $\mathcal{F}$ in $\boldsymbol{C}P^2 \setminus L_\infty$. All singularities of $\mathcal{F}$ are contained in L_∞. We have the following cases:

(i) If $\text{degree}(a) < \text{degree}(b)$, then $s(\mathcal{F})$ consists of a single point at $L_\infty \cap \overline{\{x = 0\}}$. It is a non-reduced singularity, giving rise to a saddle-node by a single blow-up.

(ii) If $\text{degree}(a) \geq \text{degree}(b)$, then the crossing $L_\infty \cap \overline{\{x = 0\}}$ is also a non-reduced singularity, which produces a saddle-node after one blow-up. In this case, L_∞ contains another singularity, which is a saddle-node.

The above example does not admit a rational first integral, since it contains saddle-nodes in L_∞ (see Proposition 1).

EXAMPLE 3. The following construction is carried out by means of the techniques of [L]. We construct a surface M_0 provided with a foliation $\mathcal{F}_0$, having an invariant projective line P_0 such that $P_0 \cdot P_0 = -1$, with two singularities p_1 and p_2, both of them are linearizable with index $-1/2$ with respect to P_0. We also construct a surface M_1 provided with a foliation $\mathcal{F}_1$, having an invariant projective line P_1 such that $P_1 \cdot P_1 = -1$, with a linearizable singularity q_1 with index -2 with respect to P_1, and a second singularity r_1, which is radial. We define M_2 to be a copy of M_1. Similarly, define $\mathcal{F}_2$ to be the foliation in M_2, P_2 the invariant projective line, q_2 and r_2 the singularities.

We glue a neighborhood of P_0 in M_0 with a neighborhood of P_1 in M_1 by identifying the local models of $\mathcal{F}_0$ in p_1 and $\mathcal{F}_1$ in q_1, and with a neighborhood of P_2 in M_2 by identifying the local models of $\mathcal{F}_0$ in p_2 and $\mathcal{F}_2$ in q_2. The result is a complex surface M with a foliation $\mathcal{F}$ having $P_0 \cup P_1 \cup P_2$ as an invariant divisor.

Blow up r_1 and r_2, giving rise to dicritical lines $\tilde{L}_1$ and $\tilde{L}_2$. Denote by $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2$ and $\tilde{\mathcal{F}}$ the strict transforms of P_0, P_1, P_2 and $\mathcal{F}$, respectively. Choosing a point $s_1 \in \tilde{L}_1$, we provide $\tilde{L}_1 \setminus \{s_1\}$ with a complex coordinate z such that $\tilde{P}_1 \cap \tilde{L}_1$ corresponds to $z = 0$. We define a holomorphic function H in $\tilde{L}_1 \setminus \{z = 0\}$ in the coordinate z by $H(z) = \exp(1/z)$. H may be extended to a first integral for $\tilde{\mathcal{F}}$ in a neighborhood of $\tilde{L}_1$ outside $\tilde{P}_1$ by following the leaves of $\tilde{\mathcal{F}}$. Similarly, we extend H to a neighborhood of $\tilde{P}_1$ outside $\tilde{P}_1 \cup \tilde{P}_0$ and then to a neighborhood of $\tilde{P}_0$ outside $\tilde{P}_0 \cup \tilde{P}_1 \cup \tilde{P}_2$. Carrying out the same construction starting from $\tilde{L}_2$, we will have, by symmetry, a meromorphic first integral h defined in a neighborhood $\tilde{P}_0 \cup \tilde{P}_1 \cup \tilde{L}_1 \cup \tilde{P}_2 \cup \tilde{L}_2$ outside $\tilde{P}_0 \cup \tilde{P}_1 \cup \tilde{P}_2$. If we blow down $\tilde{L}_1, \tilde{P}_1$ and $\tilde{L}_2, \tilde{P}_2$, then the result will be a foliation $\mathcal{G}$ in a complex surface with an invariant line P such that $P \cdot P = 1$, having two dicritical singularities and admitting a meromorphic first integral outside P. This does not extend to P. Notice that these singularities are not ordinary dicritical with respect to P, according to our definition. Considering the foliation $\tilde{\mathcal{F}}$ and the complex curve $\tilde{P}_0 \cup \tilde{P}_1 \cup \tilde{P}_2$, we have an example where theorem A fails when the curve in question is singular.

EXAMPLE 4. Let G be the group of Möbius maps generated by $g(z) = z/(z+1)$. Let T be a complex torus, α and β the generators of $\pi_1(T)$ and $\Phi : \pi_1(T) \to G$ the homomorphism such that $\Phi(\alpha) = g$, $\Phi(\beta) = g$. We make the suspension of this homomorphism, that is, we build a complex fiber bundle E with base T and fiber $\overline{\mathbf{C}}$ and a holomorphic foliation $\mathcal{F}$ in E transversal to the fibers such that the holonomy of $\mathcal{F}$ in a fiber is given by Φ (see [CL]). $\mathcal{F}$ admits a meromorphic first integral in $E \setminus E_0$, where $E_0 \simeq T$ is the null section, constructed in the following way: Let z be a complex coordinate in a fixed fiber F_0 such that the generator of the holonomy group is written as $g(z) = z/(z+1)$. $H(z) = \exp(2\pi i/z)$ is holomorphic outside $\{z = 0\}$ and satisfies $H(g(z)) = H(z)$ for $z \neq 0$. Therefore, by following the leaves of $\mathcal{F}$, we may extend H to a holomorphic first integral h for $\mathcal{F}$ defined outside E_0. Of course, h does not extend to E_0. Notice that the obstruction for the extension is the existence of a map in the holonomy with respect to E_0 which has the structure of a flower, which implies that its orbits acummulate in the origin (see [C1]).

EXAMPLE 5. In this example we follow the construction of Riccati foliations with given holonomy, as done in [L]. Let G be the group of Möbius maps generated by $f_1(z) = -z$ and $f_2(z) = z/(z + j)$, where $j = \exp(2\pi i/3)$. G is non-abelian and its generators satisfy $f_1^2 = f_2^3 = (f_1 \circ f_2)^6 = \mathrm{id}$. The function $H(z) = \mathcal{P}'(1/z)^2$, where $\mathcal{P}$ is the Weierstrass function, is meromorphic in $\overline{\boldsymbol{C}} \setminus \{z = 0\}$ and satisfies $H(f(z)) = H(z)$ for $f \in G$ (see [F], Section VII-II). We build a fiber bundle $P : E \to \overline{\boldsymbol{C}}$ with fiber $\overline{\boldsymbol{C}}$ and a singular holomorphic foliation $\mathcal{F}$ in E with three invariant vertical fibers, F_0, F_1 and F_2, transversal to the fibers in $E \setminus (F_0 \cup F_1 \cup F_2)$. Let $E_0 \simeq \overline{\boldsymbol{C}}$ be the null section. For a fixed fiber $F \neq F_0, F_1, F_2$, with a complex coordinate z ($\{z = 0\} = F \cap E_0$), the holonomy map corresponding to a loop in E_0 around $p_1 = P(F_1)$ is given by f_1, while f_2 is the holonomy map associated to a loop around $p_2 = P(F_2)$. The holonomy map associated to a loop around $p_0 = P(F_0)$ is $(f_1 \circ f_2)^{-1}$. We obtain a meromorphic first integral h for $\mathcal{F}$ defined outside $E_0 \cup F_0 \cup F_1 \cup F_2$ by extending the function H defined in $F \setminus \{z = 0\}$ by following the leaves of $\mathcal{F}$. In a neighborhood $V_i \times \overline{\boldsymbol{C}}$ of F_i, with coordinates (x_i, z_i), $(x_i, \hat{z}_i)$, where $\hat{z}_i = 1/z_i$ (the fibers correspond to the equations $x_i = c$ and p_i corresponds to $(x_i, z_i) = (0, 0)$), $\mathcal{F}$ is given by the equations

$$\omega_i(x, z_i) = \alpha_i z_i dx + x_i dz_i = 0\,,$$
$$\hat{\omega}_i(x, z_i) = -\alpha_i \hat{z}_i dx + x_i d\hat{z}_i = 0\,,$$

where $\alpha_0 = 6$, $\alpha_1 = 2$, $\alpha_2 = 3$. Since $i_{p_i}(\mathcal{F}, E_0) = -1/\alpha_i$, we have that $c(E_0) = \sum_{i=1}^{3} i_{p_i}(\mathcal{F}, E_0) = -1$. It is therefore possible to blow down E_0 by a map $\pi : E \to \hat{E} \simeq \boldsymbol{C}P^2$. The foliation $\pi_*\mathcal{F}$ has a meromorphic first integral outside the lines $\pi_* F_0$, $\pi_* F_1$ and $\pi_* F_2$. This does not extend to $\boldsymbol{C}P^2$ and the obstruction lies once again in the existence of a map in the holonomy of $\mathcal{F}$ with respect to E_0 which has a structure of flower (for instance, $[f_1, f_2] = f_1 \circ f_2 \circ f_1^{-1} \circ f_2^{-1} = z/(1 - 2z)$). Notice that all the singularities of $\pi_*\mathcal{F}$ are generalized curves.

5. Foliations in $\boldsymbol{C}P^2$.

In this section we study foliations in $\boldsymbol{C}P^2$ which admit a meromorphic first integral h defined in $\boldsymbol{C}P^2 \setminus S$, where S is a smooth algebraic curve. We remark that meromorphic functions in $\boldsymbol{C}P^2$ are rational, that is, they are given by quotients of polynomial functions. We have the following:

PROPOSITION 5. *Let S be an algebraic curve invariant by a foliation $\mathcal{F}$ in $\boldsymbol{C}P^2$ with a rational first integral h. Then S contains a dicritical singularity.*

PROOF. We suppose $h(x, y) = p(x, y)/q(x, y)$, where p and q are non-constant polynomials. Without loss of generality, we may suppose that S is irreducible. Take $\{f(x, y) = 0\}$ to be an irreducible polynomial equation defining S. The foliation $\mathcal{F}$ is defined by

$$p(x, y) - \lambda q(x, y) = 0, \quad \lambda \in \boldsymbol{C}\,. \tag{1}$$

Since S is invariant and irreducible, there exists $\lambda_0 \in \boldsymbol{C}$ such that f divides $p - \lambda_0 q$; there exists a polynomial g such that

$$f(x, y)g(x, y) = p(x, y) - \lambda_0 q(x, y)\,. \tag{2}$$

Substituting (2) in (1), we have the following set of equations:

$$f(x,y)g(x,y) - (\lambda - \lambda_0)q(x,y) = 0, \ \lambda \in \boldsymbol{C}. \qquad (3)$$

Choose a point p in the intersection of $\{q = 0\}$ and $\{f = 0\}$. This is a dicritical singularity for $\mathcal{F}$. In fact, assuming that it lies in the affine plane in question (otherwise simply perform an appropriate change of coordinates), (3) gives an infinite number of algebraic curves through p. □

Let us suppose that a foliation $\mathcal{F}$ in $\boldsymbol{C}P^2$ admits a meromorphic first integral in $\boldsymbol{C}P^2 \setminus S$, where S is a smooth algebraic curve. Theorem A applies to this case if there exists a non-dicritical generalized curve in S. As a consequence of this theorem and the preceding result, we have

COROLLARY 4. *Let $\mathcal{F}$ be a singular holomorphic foliation in $\boldsymbol{C}P^2$ admitting a meromorphic first integral outside some smooth algebraic curve S. Suppose that a singularity of $\mathcal{F}$ in S is a generalized curve. Then $\mathcal{F}$ has a dicritical singularity in S.*

PROOF. Let $p \in s(\mathcal{F}) \cap S$ be a generalized curve. If it is non-dicritical, Theorem A says that $\mathcal{F}$ has a rational first integral. Proposition 5 then assures the existence of a dicritical singularity in S. □

6. Foliations in $\boldsymbol{C}P^n$ of codimension 1. Let $\mathcal{F}$ be a codimension one singular holomorphic foliation in $\boldsymbol{C}P^n$, $n \geq 3$. Suppose that $\mathcal{F}$ admits a meromorphic first integral outside some smooth hypersurface S. This n-dimensional case can be handled by reducing it to a two-dimensional problem.

Let $H \subset \boldsymbol{C}P^n$ be an m-dimensional complex plane, $2 \leq m \leq n$. We say that H is in *general position* with respect to $\mathcal{F}$ if H is not $\mathcal{F}$-invariant and $s(\mathcal{F}) \cap H$ is a codimension two analytic set. The proof of the following proposition is adapted from Lemma 5 in [CLS1]:

PROPOSITION 6. *Let $\mathcal{F}$ be a singular holomorphic foliation in $\boldsymbol{C}P^n$ and $H \subset \boldsymbol{C}P^n$ a hyperplane in general position with $\mathcal{F}$. Then $\mathcal{F}$ admits a rational first integral if and only if $\mathcal{F}|_H$ does.*

PROOF. The “only if" part of the proof is straightforward. Let us prove the opposite implication. It is enough to build a meromorphic first integral for $\mathcal{F}_V$, where V is an open neighborhood of H. Since $\boldsymbol{C}P^n \setminus H$ is a Stein manifold, it extends to $\boldsymbol{C}P^n$ ([Siu]). Let f be a meromorphic first integral for $\mathcal{F}|_H$. Take $p \in H$ a regular point for $\mathcal{F}$. It is possible to find a sufficiently small neighborhood W_p of p and a holomorphic coordinate chart $\Psi : W_p \to \Delta$, where $\Delta \subset \boldsymbol{C}^n$ is a polydisc, such that:

(i) $\Psi(H \cap W_p) = \{z_n = 0\} \cap \Delta$,

(ii) $\Psi_*(\mathcal{F})$ is given by $dz_1 = 0$.

Let $\tilde{f}_p = f \circ \Psi^{-1}|_{\Delta \cap \{z_n = 0\}}$. This extends naturally to a meromorphic function defined in Δ, which we still call $\tilde{f}_p$, by setting $\tilde{f}_p(z_1, \ldots, z_n) = \tilde{f}_p(z_1, \ldots, z_{n-1}, 0)$. This is a first integral for $\Psi_*(\mathcal{F})$. We define $f_p = \tilde{f}_p \circ \Psi$.

Notice that, if $W_p \cap W_q \neq \emptyset$, p and q being regular points for $\mathcal{F}$, then we have $f_p|_{W_p\cap W_q} = f_q|_{W_p\cap W_q}$. This follows easily from the identity principle for meromorphic functions. Let $W = \bigcup_{p\in H\setminus s(\mathcal{F})} W_p$. W is a neighborhood of $H \setminus s(\mathcal{F})$, where $\mathcal{F}$ admits a meromorphic first integral, which we call f_W. All we have to do is extending f_W to a neighborhood of $H \cap s(\mathcal{F})$. Since H is in general position with respect to $\mathcal{F}$, $H \cap s(\mathcal{F})$ is a codimension two analytic set in H. Let $p \in H \cap s(\mathcal{F})$. It is possible to find a neighborhood V_p of p, a change of coordinates Φ such that $\Phi(p) = 0$, $\Phi(V_p) = \Delta_1 \times D$ and $\Phi^{-1}((\Delta_1 \setminus \Delta_2) \times D) \subset W \cap V_p$, where $\Delta_2 \subset \Delta_1 \subset \boldsymbol{C}^{n-1}$ are polydiscs and $D \subset \boldsymbol{C}$ is a disc, all of which centered in the origin. $(\Delta_1 \setminus \Delta_2) \times D$ is a Hartogs' domain whose holomorphic closure is $\Delta_1 \times D$. Levi's theorem then allows us to extend f_W to V_p. The result is a meromorphic first integral F defined in V, the neighborhood of H consisting of $W \bigcup_{p\in s(\mathcal{F})\cap H} V_p$. □

It is proved in [CLS1] that the set of hyperplanes in general position with respect to a foliation $\mathcal{F}$ in $\boldsymbol{C}P^n$, $n \geq 3$, is generic in the set of all hyperplanes.

We can apply the above facts to reduce the extension problem in dimension n to a problem in dimension two. We find a sequence of linear subspaces $H_2 \subset \ldots \subset H_{n-1} \subset H_n = \boldsymbol{C}P^n$, where each H_i is a linear subspace of dimension i, transversal to $H_{i+1} \cap S$, and in general position with respect to $\mathcal{F}|_{H_{i+1}}$, for $i = 2, \ldots, n-1$ ($H_n = \boldsymbol{C}P^n$). Choosing each H_i in such a way that the meromorphic first integral for $\mathcal{F}$ is non-constant over it, $H_2 \simeq \boldsymbol{C}P^2$ will be provided with a foliation $\mathcal{F}|_{H_2}$ which admits a meromorphic first integral outside $H_2 \cap S$. Furthermore $\mathcal{F}|_{H_2}$ admits a rational first integral if and only if $\mathcal{F}$ does.

7. Foliations by curves in higher dimension.

Let M be an n-dimensional complex manifold with a foliation $\mathcal{F}$ whose leaves are curves ($\mathcal{F}$ is locally induced by a holomorphic vector field). In this section we consider the problem of extending a meromorphic function F defined outside a compact subvariety S, whose level surfaces contain the leaves of $\mathcal{F}$. Such a function will still be called a *first integral* for $\mathcal{F}$. We first remark that if S is of codimension two or greater, F extends meromorphically to M as a consequence of Levi's theorem. Therefore, it is enough to consider the case where S is of codimension one. When S is not $\mathcal{F}$-invariant, the extension is automatic and the proof proceeds as that of Proposition 2:

PROPOSITION 7. *Let M, S, F and $\mathcal{F}$ be as above. If S is not $\mathcal{F}$-invariant, then F extends to M as a meromorphic first integral for $\mathcal{F}$.*

For the case where S is $\mathcal{F}$-invariant, a higher dimensional version of Extension Lemma 2 is required:

LEMMA 6. *Let F be a meromorphic first integral for the linear vector field $X(z_1, \ldots, z_n) = \lambda_1 z_1 \partial/\partial z_1 + \cdots + \lambda_n z_n \partial/\partial z_n$, where $\lambda_i \neq 0$ for $i = 1, \ldots, n$, defined outside the hyperplane $\{z_1 = 0\}$. If X admits a finite number of separatrices at* 0 (outside $\{z_1 = 0\}$)*, then F extends to a neighborhood of* 0 *as a meromorphic first integral for X.*

PROOF. We consider the development of F in the Laurent series:

$$F(z_1,\dots,z_n)=\sum_{i_1\in\mathbf{Z},i_2\geq l_2,\dots,i_n\geq l_n} a_{i_1\dots i_n} z_1^{i_1}\dots z_n^{i_n}\,.$$

Since F is a first integral for X outside $\{z_1=0\}$, we have

$$\begin{aligned}0&=dF(z_1,\dots,z_n)X(z_1,\dots,z_n)\\&=\sum_{i_1\in\mathbf{Z},i_2\geq l_2,\dots,i_n\geq l_n}(\lambda_1 i_1+\dots+\lambda_n i_n)a_{i_1\dots i_n} z_1^{i_1}\dots z_n^{i_n}\,.\end{aligned}$$

Whenever $a_{i_1\dots i_n}\neq 0$, we have

$$\lambda_1 i_1+\dots+\lambda_n i_n=0\,,$$

which is equivalent to

$$i_1=-\frac{\lambda_2}{\lambda_1}i_2-\dots-\frac{\lambda_n}{\lambda_1}i_n\,.$$

Restricting the field X to invariant two dimensional planes $z_1\times z_i$, $i=2,\dots,n$, we see that $\lambda_i/\lambda_1\in\mathbf{Q}$ (since there exists a meromorphic first integral outside $z_1=0$). On the other hand, the hypothesis on the finite number of separatrices implies that, in fact, $\lambda_i/\lambda_1\in\mathbf{Q}^+$. This means that i_1 is bounded from below by $l_1=-(\lambda_2/\lambda_1)l_2-\dots-(\lambda_n/\lambda_1)l_n$, which gives the meromorphic extension of F to the hyperplane $\{z_1=0\}$. □

The hypothesis on the number of separatrices is necessary. For instance $F(z_1,z_2,z_3)=\exp(z_2^2/z_1)$ is a first integral for $X(z_1,z_2,z_3)=2z_1\partial/\partial z_1+z_2\partial/\partial z_2+z_3\partial/\partial z_3$, which does not extend meromorphically to $\{z_1=0\}$. In view of the previous lemma, we may state the following:

THEOREM 2. *Let M, S, F and $\mathcal{F}$ be as in the beginning of this section. Assume that S is $\mathcal{F}$-invariant. If $p\in S$ is a linearizable singularity of $\mathcal{F}$, which is a saddle* (only non-zero eigenvalues) *admitting a finite number of separatrices outside S. Then F extends to M as a meromorphic first integral for $\mathcal{F}$.*

PROOF. We apply the previous lemma to extend F to a neighborhood of p, and Levi's theorem to obtain an extension to the whole M. □

8. Closed meromorphic one-forms. In this section we seek conditions for extending a closed meromorphic one-form which defines a foliation $\mathcal{F}$ outside a compact complex curve. We remark that in $\mathbf{C}^2$ closed meromorphic one-forms with simple poles correspond to foliations admitting as a first integral a multiform function of the kind $f_1^{\lambda_1}\dots f_p^{\lambda_p}$, where $f_1,\dots,f_p$ are holomorphic and $\lambda_1,\dots,\lambda_p\in\mathbf{C}$ (see [CM]). We will see that the techniques developed above also apply to this situation.

PROPOSITION 8. *Let M be a complex surface and $S \subset M$ a compact complex curve. Let $\mathcal{F}$ be a singular holomorphic foliation in M, which is induced in $M \setminus S$ by a closed meromorphic one-form ω. If S is not $\mathcal{F}$-invariant, then ω extends to a meromorphic closed one-form in M.*

PROOF. The proof is similar to that of Proposition 2. Let p be a regular point in S, also regular for $\mathcal{F}$, where the foliation is transversal to S. Choose U_p a coordinate neighborhood around p and $\Phi = (x, y) : U_p \to \boldsymbol{C}^2$ a coordinate chart such that $P := \Phi(U_p)$ is a polydisc, $\Phi(S \cap U_p) = \{y = 0\}$ and $\mathcal{F}|_{U_p}$ is the foliation with vertical leaves given by $dx = 0$. Let $\tilde{\omega} = \Phi_* \omega|_{U_p \setminus S}$. We have $\tilde{\omega}(x, y) = a(x, y)dx$, $(x, y) \in P \setminus \{y = 0\}$, where $a(x, y)$ is meromorphic in $P \setminus \{y = 0\}$. Since ω is closed, we have that $a(x, y)$ is a function of x only. The extension of ω to S is achieved by noticing that the singular points of S, the tangencies of $\mathcal{F}$ and S, and the singularities of $\mathcal{F}$ in S form a codimension two analytic set. □

The following is a generalization of Lemma 1:

LEMMA 7. *Let $p \in s(\mathcal{F})$ be a simple singularity and S a separatrix for $\mathcal{F}$ at p. Suppose that $\mathcal{F}$ is given in a neighborhood V of $S^* = S \setminus \{p\}$ by a closed meromorphic one-form ω with simple poles. Then the holonomy with respect to S is linearizable.*

PROOF. Let $\gamma : [0, 1] \to S^*$ be a closed path such that $[\gamma] \in H_1(S^*)$ is a generator. Choose Σ a small disk such that $\gamma \times \Sigma$ is contained in V. Suppose first that $S \subset (\omega)_\infty$.

Fix $q \in \gamma$. There exists a neighborhood U of q and a local chart (X, Y) in which $\mathcal{F}$ is given by $dY = 0$ and $S \cap U = \{Y = 0\}$. Since ω is closed and has simple poles, it follows that $\omega = a dY/Y + d\phi$, where $\phi \in \mathcal{O}(U)$ and $a \in \boldsymbol{C}$ is the residue of ω with respect to S^* (hence, independent from q). From $\omega \wedge dY = 0$, we have $d\phi \wedge dY = 0$, so that $\phi = \phi(Y)$. In a new system of coordinates $(x, y) = (X, Y \exp(\phi(Y)))$, $\mathcal{F}$ is given by $dy = 0$, while $\omega = a dy/y$.

It follows that we may cover a neighborhood of $\gamma \times \{0\}$ with a finite number of coordinate charts (x_j, y_j) such that $S \cap U_j = \{y_j = 0\}$, $\mathcal{F}|_{U_j} : dy_j = 0$ and $\omega|_{U_j} = a dy_j/y_j$. Whenever $U_i \cap U_j \neq \emptyset$, we have

$$a\frac{dy_i}{y_i} = a\frac{dy_j}{y_j},$$

so that $y_i = c_{ij} y_j$, where c_{ij} is locally constant in $U_i \cap U_j$. It follows that the holonomy mapping associated to $[\gamma]$ is linear.

Suppose now that $S \not\subset (\omega)_\infty$. As above, we produce a covering of $\gamma \times \{0\}$ with a finite number of open sets U_j provided with coordinates (x_j, y_j) such that $\mathcal{F}|_{U_j} : dy_j = 0$. We can thus write $\omega|_{U_j} = a_j(y_j)dy_j$, where $a_j(y_j)$ is holomorphic. Let $A_j(y_j)$ be a primitive of $a_j(y_j)$ such that $A_j(0) = 0$. A_j is a holomorphic first integral for $\mathcal{F}|_{U_j}$. If $U_i \cap U_j \neq \emptyset$, we have $dA_i = \omega|_{U_i \cap U_j} = dA_j$, which gives $A_i = A_j$ in $U_i \cap U_j$. The function $A : U = \bigcup_j U_j \to \boldsymbol{C}$ such that $A|_{U_j} = A_j$ is a holomorphic first integral for $\mathcal{F}|_U$. If h_γ is the holonomy map associated to γ, we have that $A|_\Sigma \circ h_\gamma = A|_\sigma$. Therefore, h_γ is linearizable. □

LEMMA 8 (Extension Lemma I). *Let $p \in s(\mathcal{F})$ be a simple singularity and S a separatrix for $\mathcal{F}$ at p. Suppose that $\mathcal{F}$ is given in a neighborhood V of $S^* = S \setminus \{p\}$ by a closed meromorphic one-form ω with simple poles. Then ω extends to a meromorphic one-form defined in a neighborhood of p.*

PROOF. Lemma 7 and [MM] give that $\mathcal{F}$ is linearizable at p, that is, there are coordinates (x, y) such that the one-form $\eta = xdy - \lambda ydx$, $\lambda \in \boldsymbol{C} \setminus \boldsymbol{Q}^+$, induces the foliation in a neighborhood of $p = (0, 0)$. Suppose that $S = \{y = 0\}$ in this coordinate system. Let us write

$$\begin{aligned}\omega &= a(x, y)dx + b(x, y)dy \\ &= \left(\sum_{j\geq -1, i\in \boldsymbol{Z}} a_{ij}x^i y^j\right) dx + \left(\sum_{j\geq -1, i\in \boldsymbol{Z}} b_{ij}x^i y^j\right) dy\,.\end{aligned}$$

Since ω is closed, we have

$$\sum_{j\geq -1, i\in \boldsymbol{Z}} ib_{i,j}x^{i-1}y^j - \sum_{j\geq -1, i\in \boldsymbol{Z}} ja_{i,j}x^i y^{j-1} = 0\,.$$

Therefore

$$(i+1)b_{i+1,j} = (j+1)a_{i,j+1} \quad \text{for } j \geq -1,\, i \in \boldsymbol{Z}\,. \tag{4}$$

On the other hand, since $\omega \wedge \eta = 0$ in a neighborhood where both forms are defined, we have

$$\sum_{j\geq -1, i\in \boldsymbol{Z}} a_{ij}x^{i+1}y^j + \lambda \sum_{j\geq -1, i\in \boldsymbol{Z}} b_{ij}x^i y^{j+1} = 0\,,$$

which gives

$$a_{i,j+1} = -\lambda b_{i+1,j} \quad \text{for } j \geq -1,\, i \in \boldsymbol{Z}\,. \tag{5}$$

Suppose that some $b_{i_0,j_0} \neq 0$, where $j_0 \neq -1$. From relations (4) and (5) we have

$$\lambda = -\frac{a_{i_0-1,j_0+1}}{b_{i_0,j_0}} = -\frac{i_0}{j_0+1} = -\frac{p}{q},$$

where $p, q \in \boldsymbol{Z}^+$ are such that $(p, q) = 1$. This means that whenever $b_{i,j} \neq 0$ with $j \neq -1$, we have

$$-\frac{i}{j+1} = -\frac{p}{q}\,.$$

That is, there exists $l \in \boldsymbol{Z}$ such that $i = lp$ and $j = -1 + lq$. When $b_{i,-1} \neq 0$, equation (4) implies that $i = 0$. Therefore the set of indices (i, j) such that $b_{i,j}$ is possibly non-zero is of the form

$$\begin{cases} i = lp\,, \\ j = -1 + lq\,, \end{cases} \quad l \geq 0\,.$$

This means that $b(x, y)$ extends meromorphically to a neighborhood of p, possibly having a simple pole in $\{y = 0\}$. From equation (5) we see that

$$a_{i,j} \neq 0 \Rightarrow b_{i+1,j-1} \neq 0$$

$$\Rightarrow \begin{cases} i = -1 + lp\,, \\ j = lq\,, \end{cases} \quad l \geq 0\,.$$

Therefore $a(x, y)$ also extends meromorphically to p. □

In the case of closed forms with poles of higher order we have:

LEMMA 9 (Extension Lemma II). *Let $p \in s(\mathcal{F})$ be a simple singularity and S a separatrix for $\mathcal{F}$ at p. Suppose that $\mathcal{F}$ is given in a neighborhood V of $S^* = S \setminus \{p\}$ by a closed meromorphic one-form ω with a pole of order $k + 1 \geq 2$ in S. Then ω extends to a meromorphic one-form defined in a neighborhood of p.*

PROOF. If the holonomy of S at p is linearizable, then the proof goes as that of Lemma 8. We therefore suppose that the holonomy is not linearizable. We first remark (see [LSc]) that since S is a pole of order $k + 1 \geq 2$ of the closed form ω, the holonomy group of S is conjugated to a subgroup of $G_{k,\lambda}$ for some λ in $\boldsymbol{C}$, where

$$G_{k,\lambda} = \{R_\theta \circ g_{z,k,\lambda};\, z \in \boldsymbol{C},\, \lambda^k = 1\}\,,$$

and

$$g_{z,k,\lambda} = \exp\left(z \frac{x^{k+1}}{1 + \lambda x^k} \frac{\partial}{\partial x}\right).$$

It follows from formal calculations that p must be a resonance. We then have at p the following Martinet-Ramis normal form ([MaR, p. 597]): There are formal coordinates at p such that $\mathcal{F}$ is given in a unique way by a form of the model

$$\omega_{p/q,k,\lambda} = p(1 + (\lambda - 1)(x^p y^q)^k) y dx + q(1 + \lambda(x^p y^q)^k) x dy\,,$$

where $(p, q) = 1$. The holonomy maps at $\{y = 0\}$ and $\{x = 0\}$ are given respectively by

$$\exp(-2\pi i p/q) \circ g_{2\pi i, qk, \lambda q/p}$$

and

$$\exp(-2\pi i q/p) \circ g_{2\pi i, pk, (\lambda - 1)p/q}\,.$$

Since each germ of diffeomorphism in $(\boldsymbol{C}, 0)$ tangent to the identity is formally conjugated to a unique model $g_{z,k,\lambda}$ ([MaR, p. 580]), we see that the holonomy of S at p is analytically normalizable, that is, the coordinates in question are holomorphic. Therefore the Martinet-Ramis normal form is in fact holomorphic.

On the other hand, $\omega_{p/q,k,\lambda}$ has $h(x,y) = pqxy(x^p y^q)^k$ as an integrating factor. That is, $\bar{\omega}_{p/q,k,\lambda} = h(x,y)^{-1}\omega_{p/q,k,\lambda}$ is closed. Therefore, there exists a meromorphic function g defined in V such that $\omega = g\bar{\omega}_{p/q,k,\lambda}$. If g were non-constant, it would be a first integral for $\mathcal{F}$ in V, since ω and $\bar{\omega}_{p/q,k,\lambda}$ are closed. Then the holonomy of S at p would be linearizable, which is not the case. Therefore, g is constant and ω extends to a neighborhood of p as $g\tilde{\omega}_{p/q,k,\lambda}$. This completes the proof. □

We also have:

LEMMA 10. *Let M be a complex surface and S a compact connected complex curve. Suppose that ω is a meromorphic one form defined in $M \setminus S$. If ω extends as a meromorphic one form to $(M \setminus S) \cup V_p$, where V_p is a neighborhood of a point $p \in S$, then it extends meromorphically to M.*

PROOF. The proof is similar to that of Lemma 3, noticing that a meromorphic one-form defined in a Hartogs' domain extends to its holomorphic closure. □

The proofs of theorems A′, B′ and C′, stated below, proceed as those of their counterparts, Theorems A, B and C.

THEOREM A′. *Let $\mathcal{F}$ be a singular holomorphic foliation in a complex surface M induced by a closed meromorphic one-form in $M \setminus S$, where S is a compact, smooth, connected complex curve. If some singularity of $\mathcal{F}$ in S is a non-dicritical generalized curve, then ω extends to a closed meromorphic one-form in M.*

THEOREM B′. *Let $\mathcal{F}$ be a singular holomorphic foliation in a complex surface M induced by a closed meromorphic one-form in $M \setminus S$, where S is a compact, smooth, connected complex curve with negative self-intersection number. If all singularities of $\mathcal{F}$ in S are generalized curves, then ω extends to a closed meromorphic one-form defined in M.*

THEOREM C′. *Let $\mathcal{F}$ be a singular holomorphic foliation in a complex surface M induced by a closed meromorphic one-form ω outside a compact, smooth, connected complex curve S with self-intersection number $n \geq 0$. Suppose that the singularities of $\mathcal{F}$ in S are generalized curves. If there are at least $n+1$ ordinary dicritical singularities in S, then ω extends to a closed meromorphic one-form defined in M.*

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