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REMARKS ON THE ENERGY SCATTERING FOR NONLINEAR KLEIN-GORDON AND SCHRÖDINGER EQUATIONS

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Abstract. We give an improved proof for the result established recently by the present author that the scattering operators are well-defined in the whole energy space for a class of nonlinear Klein-Gordon and Schrödinger equations in any spatial dimension. Using some Sobolev-type inequatilies, we can simplify and somewhat enhance the Morawetz-type estimates and thereby weaken the required repulsivity conditions.

1. Introduction. In this note, we study the scattering theory in the energy space for nonlinear Klein-Gordon equations (NLKG):

$$\Box u + u + f(u) = 0\,, \tag{1.1}$$

and for nonlinear Schrödinger equations (NLS):

$$i\dot{u} - \Delta u + f(u) = 0\,, \tag{1.2}$$

where $u = u(t,x) : \boldsymbol{R}^{1+n} \to \boldsymbol{C}$, $\dot{u} = \partial u/\partial t$, $\Box = \partial_t^2 - \Delta$, $n \in \boldsymbol{N}$ and $f : \boldsymbol{C} \to \boldsymbol{C}$. Our main objective is to prove that the wave operators and the scattering operators for (1.1) and for (1.2) are well-defined and bijective in the whole energy space E (for NLKG, $E = H^1 \oplus L^2$ and for NLS, $E = H^1$). Such results were obtained in [5, 6, 7, 8, 9] for $n \geq 3$ by using the following weak decay estimate called the Morawetz estimate:

$$\iint_{\boldsymbol{R}^{1+n}} \frac{G(u)}{|x|} dxdt \leq CE(u)\,, \tag{1.3}$$

where u is any finite-energy solution, $E(u)$ denotes its energy and $G : \boldsymbol{C} \to \boldsymbol{R}$ is a certain function derived from f (see (1.11)).

However, since the derivation of (1.3) can not work for $n \leq 2$ (actually (1.3) is false in the one-dimensional case), the scattering in the lower dimensional case had been completely open until the author derived some variants of (1.3) in [13] for any spatial dimension. In the simplest case (two-dimensional NLS), the estimates can be written as

$$\iint_{\boldsymbol{R}^{1+n}} \frac{G(u)}{|t| + |x| + 1} dxdt \leq CE(u)\,, \tag{1.4}$$

although the estimates in the other cases were more complicated, especially for the one-dimensional NLKG.

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On the other hand, using some Hardy-Sobolev type inequalities, the author also derived a similar Morawetz-type estimate independent of the nonlinearity f for NLKG in the case where $n \geq 3$ [10]:

$$\iint_{\boldsymbol{R}^{1+n}} \frac{|u|^{2^*}}{|t|+|x|} dxdt \leq CE(u), \tag{1.5}$$

where $2^* := 2n/(n-2)$ is the Sobolev critical exponent, and also similar estimates for Hartree equations with $n \geq 3$ [12].

Although (1.5) does not make sense if $n \leq 2$, we can still derive some Morawetz-type estimates (Lemma 2.6) independent of the nonlinearity by using some Sobolev type inequalities. These estimates are simpler and even stronger than those derived in [13], especially for the one-dimensional NLKG. In this note we will derive those estimates and thereby improve the scattering results in [9, 13] relative to the repulsivity conditions on the nonlinearity.

Now we describe the hypotheses on the nonlinearity f, that is, what the repulsivity conditions are and how we can weaken them. First of all, to let the energy conservation law hold, we need to assume that there exists $F : \boldsymbol{C} \to \boldsymbol{R}$ such that

$$\partial_{\bar{z}} F(z) = f(z), \quad F(0) = 0, \tag{1.6}$$

and in the NLS case, to have the charge (L^2) conservation law, we need

$$f(u) = f(|u|)\frac{u}{|u|}. \tag{1.7}$$

We also need certain assumptions on the smoothness and the growth order of f:

$$f(0) = 0, \quad |f(u) - f(v)| \leq C|u - v|(|u|^{p_1} + |v|^{p_1} + |u|^{p_2} + |v|^{p_2}), \tag{1.8}$$

for some $4/n < p_1 \leq p_2 < 2^* - 2$ ($2^* = \infty$ if $n < 3$) and $C > 0$. Finally, to describe the repulsivity condition, we define

$$V(u) := \frac{F(u)}{|u|^2} : \boldsymbol{C} \to \boldsymbol{R}. \tag{1.9}$$

Then the repulsivity condition we need is

$$\partial_{|z|} V(z) = 2\Re \partial_z V(z)\frac{z}{|z|} \geq 0, \tag{1.10}$$

which simply means that $V(u)$ is non-decreasing with respect to $|u|$. The function G in (1.3) and (1.4) is given by

$$G(z) := \Re \partial_z V(z)|z|^2 z = \Re(\bar{z} f(z)) - F(z). \tag{1.11}$$

In preceding works, some stronger assumptions on V were needed, because (1.3) was essential to derive any decay estimate for u. The most general assumption in the literature seems to be the following (see [9, (4.22)])

$$\partial_{|z|} V(z) \geq C \min(|z|^{-1}, |z|^p), \tag{1.12}$$

for some $p > 0$ and $C > 0$, which requires that V is not flat at $u = 0$ but diverges at $u = \infty$. For $n \geq 3$, it is not so difficult to obtain the scattering result under the weaker condition (1.10) if we use (1.5) instead of (1.3) in the arguments of [9] or [13]. In fact, the author proved the scattering result under (1.10) for NLKG in the Sobolev critical case $p_1 = p_2 = 2^* - 2$ [11], and it can be extended to the case $4/n < p_1 \le p_2 = 2^* - 2$, though in the critical case, we need some minor restrictions on the solution class and the nonlinearity (compare the result in [11] and Theorem 1.1 below).

In order to deal with NLKG and NLS in a unified way, we use the following notation:

$$\boldsymbol{u} := \begin{cases} (u, \sqrt{1-\Delta}^{-1}\dot{u}), & \text{for NLKG}, \\ u, & \text{for NLS}. \end{cases} \tag{1.13}$$

Then, we have the following conserved energy:

$$E(u;t) := \int_{\boldsymbol{R}^n} |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 + F(u)dx = E(u;0), \tag{1.14}$$

if $\boldsymbol{u}(0) \in H^1$.

Now we can state the main result of this note as follows.

THEOREM 1.1. *Let $n \in \boldsymbol{N}$ and $f : \boldsymbol{C} \to \boldsymbol{C}$. Assume* (1.6), (1.8) *and* (1.10). *In the* NLS *case, assume* (1.7) *further. Then, the wave operators for* NLKG *and* NLS *are well-defined homeomorphisms on the energy space. More precisely, for any solution u with finite energy $E(u) < \infty$ of* NLKG *or* NLS, *there exists a unique solution v for the free equation*

$$\Box v + v = 0, \quad \textit{for } \text{NLKG}, \tag{1.15}$$

$$i\dot{v} - \Delta v = 0, \quad \textit{for } \text{NLS} \tag{1.16}$$

satisfying

$$\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_{H^1(\boldsymbol{R}^n)} \to 0, \quad \textit{as } t \to \infty, \tag{1.17}$$

and the correspondence $\boldsymbol{v}(0) \mapsto \boldsymbol{u}(0)$ defines a homeomorphism from H^1 into itself (obviously we have the same result for $t \to -\infty$).

Moreover, we have the following global space-time norm estimate for any finite energy solution u:

$$\begin{aligned} &\|u\|_{L^\rho(\boldsymbol{R};B^{1/2}_{\rho,2}(\boldsymbol{R}^n))} \le C(E(u)), && \textit{for } \text{NLKG } \textit{with } n \le 2, \\ &\|u\|_{L^\rho(\boldsymbol{R};B^{1/2}_{\rho,2}(\boldsymbol{R}^n))} + \|u\|_{L^\zeta(\boldsymbol{R};B^{1/2}_{\zeta,2}(\boldsymbol{R}^n))} \le C(E(u)), && \textit{for } \text{NLKG } \textit{with } n \ge 3, \\ &\|u\|_{L^\rho(\boldsymbol{R};B^{1}_{\rho,2}(\boldsymbol{R}^n))} \le C(E(u)), && \textit{for } \text{NLS}, \end{aligned} \tag{1.18}$$

where $\rho := 2 + 4/n$, $\zeta = 2 + 4/(n-1)$, $B^\sigma_{\rho,2}$ denotes the inhomogeneous Besov space (*see, e.g.,* [2]) *and $C(\cdot)$ is a certain positive valued function dependent only on n, p_1, p_2 and the constant in* (1.8).

REMARK 1.2. The function V might be regarded as the 'nonlinear potential' associated with f, in view of the form of the energy

$$E(u) = \int_{\boldsymbol{R}^n} |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 + V(u)|u|^2 dx, \tag{1.19}$$

compared with the linear case:

$$E(u) = \int_{\mathbf{R}^n} |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 + V(x)|u|^2 dx\,, \tag{1.20}$$

where $V(x)$ is a linear potential. By [1], there exist standing wave solutions for NLKG and NLS if f satisfies (1.7) and $V(z) < V(0)$ for some $z \in \mathbf{C}$, so that the wave operators are not surjective. However, to the best of the author's knowledge, it is open whether the scattering operator exists in the whole energy space when (1.10) does not hold but $V(z) \geq V(0)$ for any $z \in \mathbf{C}$.

The content of this note is organized as follows. In Section 2, we derive certain Morawetz-type estimates independent of the nonlinearity and valid in any spatial dimension, after proving some Sobolev type inequalities. In Sections 3 and 4, we derive a global space-time estimate of the solutions, which is the main step in the proof of Theorem 1.1. In Section 3, we consider the NLS case for any dimension and the NLKG case for one or two dimensions. In Section 4, we will consider the NLKG case for $n \geq 3$, which is more complicated by the inhomogeneity of KG and the existence of the Sobolev critical exponent. Section 5 is devoted to proving the main theorem from the estimate derived in Section 3 and 4, though this step is essentially known. Throughout this note, we will use the notation $C(\cdot, \cdot, \dots)$ to denote any positive continuous function whose explicit form we refrain from writing for simplicity.

2. Morawetz-type estimates. Under the monotonicity condition of the nonlinear potential (1.10), we will derive certain Morawetz-type estimates (Lemma 2.6) which are independent of the nonlinearity and valid in any spatial dimension. The idea is similar to that in [10]. To dominate $|u|$, we use certain quadratic terms involving the derivative of u in certain integral identities of approximate conservation laws for the equations. However, for $n < 3$, the derivative of u alone can not dominate $|u|$ itself through the Sobolev inequality. So we will use a certain Gagliardo-Nirenberg type inequality and L^2 boundedness of the solutions.

First we derive the inequality needed to prove the Morawetz-type estimates.

LEMMA 2.1. *Let $n \in \mathbf{N}$, $p > 2$ and $q := n(p-2)/2$. Let $\chi(x)$ and $\lambda(x)$ be real-valued functions. Then for any complex-valued $u(x) \in H^1(\mathbf{R}^n)$, we have*

$$\int_{\mathbf{R}^n} \chi^2 |u|^p dx \leq C\|u\|_{L^q}^{p-2} \int_{\mathbf{R}^n} \chi^2 |\nabla u + i\lambda u|^2 + |u\nabla\chi|^2 dx\,, \tag{2.1}$$

where C is a positive constant dependent only on n and p.

PROOF. (I) First we prove the inequality in the case where $n \leq 2$. Let $s := (2+q)/2$. Then, by Hölder's inequality, we have

$$\int \chi^2 |u|^p dx \leq \|\chi|u|^s\|_{L^{n/(n-1)}}^2 \|u\|_{L^q}^{p-2s}\,, \tag{2.2}$$

where we used the assumption $n \leq 2$ to have $p \geq 2s$. Then, by the Sobolev inequality, we have

$$\|\chi|u|^s\|_{L^{n/(n-1)}} \leq C\|\nabla(\chi|u|^s)\|_{L^1}\,. \tag{2.3}$$

The right hand side can be rewritten by

$$\nabla(\chi|u|^s) = \Re s(\chi|u|^{s-2}u\overline{\nabla u + i\lambda u}) + |u|^s\nabla\chi \,. \tag{2.4}$$

Thus we obtain

$$\begin{aligned}\|\chi|u|^s\|_{L^{n/(n-1)}} &\le C\||u|^{s-1}\|_{L^2}(\|\chi(\nabla u + i\lambda u)\|_{L^2} + \|u\nabla\chi\|_{L^2}) \\ &\le C\|u\|_{L^q}^{q/2}(\|\chi(\nabla u + i\lambda u)\|_{L^2} + \|u\nabla\chi\|_{L^2}) \,.\end{aligned} \tag{2.5}$$

From this estimate and (2.2), we obtain the desired result.

(II) Next we prove the inequality in the case where $n \ge 2$. By the Sobolev inequality, we have for $n \ge 2$,

$$\|\chi|u|^{p/2}\|_{L^2} \le C\|\nabla(\chi|u|^{p/2})\|_{L^{2n/(n+2)}} \,. \tag{2.6}$$

The right hand side can be rewritten by

$$\nabla(\chi|u|^{p/2}) = \frac{p}{2}\chi\Re(|u|^{p/2-2}u\overline{\nabla u + i\lambda u}) + |u|^{p/2}\nabla\chi \,. \tag{2.7}$$

Then, by Hölder's inequality, we have

$$\begin{aligned}\|\nabla(\chi|u|^{p/2})\|_{L^{2n/(n+2)}} &\le C\||u|^{p/2-1}\|_{L^n}(\|\chi(\nabla u + i\lambda u)\|_{L^2} + \|u\nabla\chi\|_{L^2}) \\ &\le C\|u\|_{L^q}^{p/2-1}(\|\chi(\nabla u + i\lambda u)\|_{L^2} + \|u\nabla\chi\|_{L^2}) \,,\end{aligned} \tag{2.8}$$

From this estimate and (2.6), we obtain the desired result. □

REMARK 2.2. In the above lemma, $2 \le q \le 2^*$ is equivalent to $2 + 4/n \le p \le 2^*$.

Any Morawetz-type estimate is based on some integral identity derived by variation of the lagrangian. We mention a general formula for such identities. First we fix some notation.

DEFINITION 2.3.

$$\langle a, b\rangle := \Re(a\bar{b}) \,, \quad \partial = (\partial_t, \nabla_x) \,, \quad \mathcal{D} = \begin{cases} (-\partial_t, \nabla_x) \,, & \text{for NLKG} \,, \\ (-i/2, \nabla_x) \,, & \text{for NLS} \,, \end{cases} \tag{2.9}$$

$$2\ell(u) = \begin{cases} -|\dot{u}|^2 + |\nabla u|^2 + |u|^2 + F(u) \,, & \text{for NLKG} \,, \\ \langle i\dot{u}, u\rangle + |\nabla u|^2 + F(u) \,, & \text{for NLS} \,, \end{cases} \tag{2.10}$$

$$\mathrm{eq}_L(u) := \begin{cases} \Box u + u \,, & \text{for NLKG} \,, \\ i\dot{u} - \Delta u \,, & \text{for NLS} \,, \end{cases} \quad \mathrm{eq}(u) := \mathrm{eq}_L(u) + f(u) \,, \tag{2.11}$$

where $F(u)$ is as in (1.6).

$\ell(u)$ is the lagrangian density assosiated to the equation $\mathrm{eq}(u) = 0$. The differential operator $\mathcal{D}$ appears from the variation of ℓ:

$$\begin{aligned}\delta_v\ell(u) &:= \lim_{\varepsilon\to 0}\frac{\ell(u+\varepsilon v) - \ell(u)}{\varepsilon} \\ &= \langle \mathrm{eq}(u), v\rangle + \partial\cdot\langle\mathcal{D}u, v\rangle \,.\end{aligned} \tag{2.12}$$

Using this identity, we obtain the following formula.

LEMMA 2.4. *Assume* (1.6). *In the NLS case, assume* (1.7) *further. Let* $u, h, q : \boldsymbol{R}^{1+n} \to \boldsymbol{R}$ *sufficiently smooth. Then we have for* $\alpha = 0, \dots, n$,

$$\begin{aligned}\langle \mathrm{eq}(u), h\mathcal{D}_\alpha u + qu\rangle &= -\partial \cdot \langle \mathcal{D}u, h\mathcal{D}_\alpha u + qu\rangle + \mathcal{D}_\alpha\left(h\ell(u) + \frac{|u|^2}{2}\partial_\alpha q\right) \\ &\quad + \langle \mathcal{D}u, (\partial h)\mathcal{D}_\alpha u\rangle - \frac{|u|^2}{2}\mathcal{D}_\alpha \partial_\alpha q \\ &\quad + (2q - \mathcal{D}_\alpha h)\ell(u) + G(u)q\,,\end{aligned} \tag{2.13}$$

where $G(u)$ *is defined as in* (1.11).

PROOF. Let $T(\lambda)$ be a one-parameter group of transformations acting on functions defined on $\boldsymbol{R}^{1+n}$. Denote by T' the infinitesimal transformation of T. For any parallel translation group $T(\lambda)$, we have $\ell(T(\lambda)u) = T(\lambda)\ell(u)$, so that we obtain from (2.12),

$$\langle \mathrm{eq}(u), T'u\rangle = T'\ell(u) - \partial \cdot \langle \mathcal{D}u, T'u\rangle\,. \tag{2.14}$$

Let $T(\lambda)u := e^{i\lambda}u$ and assume (1.7). Then we have $T'u = iu$, and $\ell(T(\lambda)u) = \ell(u)$ so that from (2.12) we have

$$\langle \mathrm{eq}(u), iu\rangle = -\partial \cdot \langle \mathcal{D}u, iu\rangle\,. \tag{2.15}$$

Let $T(\lambda)u := e^{\lambda}u$. Then we have $T'u = u$ and, if $V(u)$ did not depend on u, we would have $\ell(T(\lambda)u) = T(2\lambda)\ell(u)$. So we have

$$\delta_u \ell(u) = 2\ell(u) + \delta_u(V(u))\frac{|u|^2}{2} = 2\ell(u) + G(u)\,. \tag{2.16}$$

Then, it follows from (2.12) that

$$\langle \mathrm{eq}(u), u\rangle = 2\ell(u) + G(u) - \partial \cdot \langle \mathcal{D}u, u\rangle\,. \tag{2.17}$$

From (2.14), (2.15) and (2.17), we have

$$\begin{aligned}\langle \mathrm{eq}(u), h\mathcal{D}_\alpha u + qu\rangle &= \mathcal{D}_\alpha(h\ell(u)) - (\mathcal{D}_\alpha h)\ell(u) - h\partial \cdot \langle \mathcal{D}u, \mathcal{D}_\alpha u\rangle \\ &\quad + q(2\ell(u) + G(u)) - q\partial \cdot \langle \mathcal{D}u, u\rangle \\ &= -\partial \cdot \langle \mathcal{D}u, h\mathcal{D}_\alpha u + qu\rangle + \langle \mathcal{D}u, u\partial q\rangle \\ &\quad + \mathcal{D}_\alpha(h\ell(u)) + \langle \mathcal{D}u, (\partial h)\mathcal{D}_\alpha u\rangle \\ &\quad + (2q - \mathcal{D}_\alpha h)\ell(u) + G(u)q\,,\end{aligned} \tag{2.18}$$

where the second term in the right hand side can be rewritten as

$$\langle \mathcal{D}u, u\partial q\rangle = \mathcal{D} \cdot \left(\frac{|u|^2}{2}\partial q\right) - \frac{|u|^2}{2}\mathcal{D} \cdot \partial q\,. \tag{2.19}$$

Hence we obtain the desired result. □

To have some positivity of the right hand side of (2.13), we choose $h = (h_0, \dots, h_n)$ and q such that $(\partial_\beta h_\alpha)_{\alpha,\beta=0,\dots,n}$ is everywhere nonnegative definite and $\Re(2q - \mathcal{D} \cdot h) = 0$. Now we choose $h := (t, x)/|(t, x)|$ and $q := \Re\mathcal{D} \cdot h/2$. Then we obtain the following identities.

Let $M := h \cdot \mathcal{D}u + qu$. For NLKG, we obtain the identity (cf. [10, Lemma 4.2]):

$$
\begin{aligned}
\langle \mathrm{eq}(u), M\rangle = -\partial \cdot \langle \mathcal{D}u, M\rangle + \mathcal{D} \cdot \left(h\ell(u) + \frac{|u|^2}{2}\partial q \right) \\
+ \frac{|t\nabla u + x\dot{u}|^2 + |x|^2|\nabla u|^2 - |x \cdot \nabla u|^2}{|(t,x)|^3} + \frac{|u|^2}{2}\Box q \\
+ G(u)\left\{ \frac{n-1}{2|(t,x)|} + \frac{t^2 - |x|^2}{2|(t,x)|^3} \right\}.
\end{aligned} \tag{2.20}
$$

For NLS, we have the identity (cf. [13, (5.19) (5.20)]):

$$
\begin{aligned}
\langle \mathrm{eq}(u), M\rangle = -\partial \cdot \langle \mathcal{D}u, M\rangle + \nabla \cdot \left(h\ell(u) + \frac{|u|^2}{2}\nabla q \right) \\
+ \frac{|t\nabla u + ixu/2|^2 + |x|^2|\nabla u|^2 - |x \cdot \nabla u|^2}{|(t,x)|^3} - \frac{|u|^2}{2}\Delta q \\
+ G(u)\left\{ \frac{n-1}{2|(t,x)|} + \frac{t^2}{2|(t,x)|^3} \right\}.
\end{aligned} \tag{2.21}
$$

We also have

$$
|h| \le C, \quad |q| \le \frac{C}{|(t,x)|}, \quad |\partial q| \le \frac{C}{|(t,x)|^2}, \quad |\partial^2 q| \le \frac{C}{|(t,x)|^3}. \tag{2.22}
$$

Using Lemma 2.1, now we can derive Morawetz-type estimates from the above identities (2.20) and (2.21).

REMARK 2.5. To derive the original Morawetz estimate (1.3), let $h := (0,x)/|x|$ and $q := \Re\mathcal{D} \cdot h/2$ (cf. [16, (2.27)]). Obviously, M is then too singular at $x = 0$ for $n \le 2$. By replacing h with $(0,x)/\sqrt{1+|x|^2}$, we can avoid the singularity (cf. [9, Lemma 4.3]). However, we can not estimate the term $-|u|^2\Delta q$ for $n \le 2$, which is nonnegative for $n \ge 3$. Remark that there is no nontrivial q satisfying $q \ge 0$ and $-\Delta q \ge 0$ if $n \le 2$.

In the following, we integrate (2.20) over the inside of a hyperboloid. So it is useful to see what we obtain from the divergence theorem on the truncating space-like surface. If $\chi : \boldsymbol{R}^n \to \boldsymbol{R}$, then we formally calculate

$$
\begin{aligned}
\iint_{t>\chi(x)} -\partial \cdot \langle \mathcal{D}u, M\rangle + \mathcal{D} \cdot \left(h\ell(u) + \frac{|u|^2}{2}\partial q \right) dxdt \\
= \int_{t=\chi(x)} (1, \nabla\chi) \cdot \left\{ -\langle \partial u, M\rangle + h\ell(u) + \frac{|u|^2}{2}\partial q \right\} dx.
\end{aligned} \tag{2.23}
$$

Let $v(x) := u(\chi(x), x)$ and $h = (h_0, h^0) \in \boldsymbol{R} \times \boldsymbol{R}^n$. Then we have $\nabla u = \nabla v - \nabla\chi\dot{u}$. Thereby we can rewrite for $t = \chi(x)$,

$$\begin{aligned}&(1, \nabla\chi) \cdot \partial u = (1 - |\nabla\chi|^2)\dot{u} + \nabla\chi \cdot \nabla v,\\ &M = -(1, \nabla\chi) \cdot h\dot{u} + h^0 \cdot \nabla v + qv,\\ &2\ell(u) = (|\nabla\chi|^2 - 1)|\dot{u}|^2 + |\nabla v|^2 - 2\langle\nabla\chi \cdot \nabla v, \dot{u}\rangle + |v|^2 + F(v).\end{aligned} \tag{2.24}$$

Then we have

$$\begin{aligned}(2.23) = \int_{t=\chi(x)} \Bigg[&\frac{(1, \nabla\chi) \cdot h}{2}\{(1 - |\nabla\chi|^2)|\dot{u}|^2 + |\nabla v|^2 + |v|^2 + F(v)\}\\ &- \langle(1 - |\nabla\chi|^2)\dot{u} + \nabla\chi \cdot \nabla v, h^0 \cdot \nabla v + qv\rangle + \frac{|v|^2}{2}(1, \nabla\chi) \cdot \partial q\Bigg]dx.\end{aligned} \tag{2.25}$$

Let $\chi \in C^1(\boldsymbol{R}^{1+n}, \boldsymbol{R})$. Then, taking $M = \dot{u}$ in (2.25), we have for $v(t, x) := u(\chi(t, x), x)$

$$\int (1 - |\nabla\chi|^2)|\dot{v}|^2 + |\nabla v|^2 + |v|^2 + F(v)dx = E(u), \tag{2.26}$$

and in particular, if $|\dot{\chi}|^2 + |\nabla\chi|^2 = 1$, then we have

$$E(v) = E(u). \tag{2.27}$$

LEMMA 2.6. *Assume* (1.6), (1.8) *and* (1.10). *Let* $p \geq 2 + 4/n$ *and assume* $p \leq 2^* = 2n/(n-2)$ *if* $n \geq 3$. *Then, for any finite energy solution* u *for* NLKG *we have*

$$\iint_{|x|<|t|} \frac{|u|^p}{|t|} dxdt \leq CE(u)^{p/2}, \tag{2.28}$$

where C *is a positive constant depending only on* n *and* p. *Assume* (1.7) *in addition. Then, for any finite energy solution* u *for* NLS *we have*

$$\iint_{\boldsymbol{R}^{1+n}} \frac{t^2|u|^p}{|(t, x)|^3} dxdt \leq CE(u)^{p/2}, \tag{2.29}$$

where C *is a positive constant dependent only on* n *and* p.

PROOF. Denote

$$r := |x|, \quad \lambda := |(t, x)| = \sqrt{t^2 + r^2}, \quad \tau := \sqrt{t^2 - r^2}. \tag{2.30}$$

By (1.8), we have the local well-posedness for NLKG and NLS, and by the standard approximation argument, it suffices to prove the estimates, assuming that u is sufficiently smooth.

First we consider the NLKG case. We integrate (2.20) over the region $\{(t, x) | \tau = \sqrt{t^2 - r^2} > 1\}$. Let $v(\tau, x) := u(\sqrt{|x|^2 + \tau^2}, x)$. Then we have $\nabla_x v = \nabla u + x\dot{u}/t$ and $|t_\tau|^2 + |\nabla t|^2 = 1$. Using the divergence theorem, (2.25) and (2.22), we obtain

$$\iint_{\tau>1} \frac{|t\nabla u + x\dot{u}|^2}{\lambda^3} dxdt \leq C \iint_{\tau>1} \frac{|u|^2}{\lambda^3} dxdt + CE(v; 1), \tag{2.31}$$

where the first term in the right hand side is bounded by the L^2 norm, and the last term is bounded by the energy identity (2.27). Thus we obtain

$$\iint_{\tau>1} \frac{|t\nabla u + x\dot{u}|^2}{\lambda^3} dxdt \le CE(u)\,. \tag{2.32}$$

Since $dxdt = \tau/t dxd\tau$, it follows from (2.32) that

$$\iint_{\tau>1} \frac{\tau}{r^2+\tau^2} |\nabla v|^2 dxd\tau \le CE(u)\,. \tag{2.33}$$

Now let $\chi := (r^2+\tau^2)^{-1/2}$. Then we have $|\nabla\chi| \le C/(r^2+\tau^2)$. By Lemma 2.1 and the Sobolev embedding, we obtain

$$\int \frac{\tau}{r^2+\tau^2} |v|^p dx \le C\|v\|_{H^1}^{p-2} \int \frac{\tau}{r^2+\tau^2} |\nabla v|^2 dx + C\tau^{-3}\|v\|_{H^1}^p\,. \tag{2.34}$$

By the energy identity (2.27), we have $\|v\|_{H^1}^2 \le CE(u)$, and hence

$$\iint_{\tau>1} \frac{\tau}{r^2+\tau^2} |v|^p dxd\tau \le CE(u)^{p/2}\,. \tag{2.35}$$

Returning to the original coordinates (t, x), we obtain

$$\iint_{\tau>1} \frac{|u|^p}{|t|} dxdt \le CE(u)^{p/2}\,. \tag{2.36}$$

By the Sobolev embedding and the energy identity (2.26), we have

$$\begin{aligned} \iint_{r<t<r+1} |u|^p dxdt &= \int_0^1 \int_{\mathbf{R}^n} |u(r+s,x)|^p dxds \\ &\le C\int_0^1 \|u(|x|+s,x)\|_{H_x^1(\mathbf{R}^n)}^p ds \le CE(u)^{p/2}\,. \end{aligned} \tag{2.37}$$

So it remains only to prove

$$\iint_{r<|t|<1} \frac{|u|^p}{|t|} dxdt \le CE(u)^{p/2}\,. \tag{2.38}$$

In the case $p = 2^*$ for $n \ge 3$, this follows from [10, Proposition 4.4]. So we may assume $p < 2^*$. By Hardy's inequality, we have

$$\|r^{-1/3}u\|_{L^2} \le C\|u\|_{L^2}^{2/3}\|\nabla u\|_{L^2}^{1/3}\,. \tag{2.39}$$

Interpolating with the Sobolev embedding, we then have

$$\|r^{-\varepsilon}u\|_{L^p} \le C\|u\|_{H^1} \tag{2.40}$$

for some $\varepsilon > 0$ depending on $p < 2^*$. Thus we obtain

$$\begin{aligned} \iint_{r<|t|<1} \frac{|u|^p}{|t|} dxdt &\le \iint_{|t|<1} \frac{|u|^p}{|t|^{1-p\varepsilon} r^{p\varepsilon}} dxdt \\ &\le C\int_{-1}^1 \frac{\|u(t)\|_{H^1}^p}{|t|^{1-p\varepsilon}} dt \le CE(u)^{p/2}\,. \end{aligned} \tag{2.41}$$

Now we turn to the NLS case. We integrate (2.21) over the region $\{(t,x) \mid |t| > 1\}$ to obtain

$$\iint_{|t|>1} \frac{|t\nabla u + ixu/2|^2}{\lambda^3} dxdt \leq CE(u). \tag{2.42}$$

Now let $\chi := |t|\lambda^{-3/2}$. Then we have $|\nabla \chi| \leq C|t|/\lambda^{-5/2} \leq C|t|^{-3/2}$. It follows from Lemma 2.1 that

$$\int_{\boldsymbol{R}^n} \frac{t^2|u|^p}{\lambda^3} dx \leq C\|u\|_{H^1}^{p-2} \int_{\boldsymbol{R}^n} \frac{|t\nabla u + ixu/2|^2}{\lambda^3} dx + C|t|^{-3}\|u\|_{H^1}^p. \tag{2.43}$$

Thus we obtain

$$\iint_{|t|>1} \frac{t^2|u|^p}{\lambda^3} dxdt \leq CE(u)^{p/2}. \tag{2.44}$$

If $p < 2^*$, in the same way as in the NLKG case, we have

$$\iint_{|t|<1} \frac{t^2|u|^p}{\lambda^3} dxdt \leq C\int_{-1}^{1} \frac{dt}{t^{1-p\varepsilon}} \|r^{-\varepsilon}u\|_{L^p}^p dt \leq CE(u)^{p/2}, \tag{2.45}$$

where $\varepsilon > 0$ is as in (2.40). Now we have only to prove in the case where $n \geq 3$

$$\iint_{|t|<1} \frac{t^2|u|^{2^*}}{\lambda^3} dxdt \leq CE(u)^{2^*/2}. \tag{2.46}$$

However, in this case we can prove the following estimate in the same way as in [12]:

$$\iint_{\boldsymbol{R}^{1+n}} \frac{|t|^{1+\nu}|u|^{2^*}}{\lambda^{2+\nu}} dxdt \leq C(\nu)E(u)^{2^*/2} \tag{2.47}$$

for any $\nu > 0$. □

3. Global space-time estimate (Case I). In this section, we derive some global space-time estimates of the solutions, which is the main step in the proof of Theorem 1.1, for NLS with any n and for NLKG with $n \leq 2$. The remaining case is treated in the next section. The arguments are essentially the same as in [13], but in order to control the general nonlinearities we should choose the space-time norms more carefully. In this section, we will repeatedly refer to [13], so for brevity, we denote by $(\cdot,\cdot)_*$ those equations in [13] and similarly by Lemma $\cdot.\cdot_*$ the lemmas therein.

$$\begin{aligned} &(E;I) := L^\infty(I; H^1(\boldsymbol{R}^n)), && (B;I) := L^\infty(I; B^{1-n/2-\sigma}_{\infty,\infty}(\boldsymbol{R}^n)), \\ &(X;I) := L^q(I \times \boldsymbol{R}^n), && (X';I) := L^{q'}(I \times \boldsymbol{R}^n), \\ &(K;I) := L^\rho(I; B^{\sigma_K}_{\rho,2}(\boldsymbol{R}^n)), && (\bar{K};I) := L^{\bar\rho}(I; B^{\sigma_K}_{\bar\rho,2}(\boldsymbol{R}^n)), \\ &(Y;I) := L^q(I; L^\rho(\boldsymbol{R}^n)), && (\tilde{Y};I) := L^{q/p_2}(I; L^{\bar\rho}(\boldsymbol{R}^n)), \end{aligned} \tag{3.1}$$

where $\rho = 2 + 4/n$, $1/\rho + 1/\bar\rho = 1$, $p_2/q = p_1/q' = 1/\bar\rho - 1/\rho = 2/(n+2)$ and

$$\sigma_K = \begin{cases} 1/2, & \text{in the NLKG case}, \\ 1, & \text{in the NLS case}. \end{cases} \tag{3.2}$$

$\sigma > 0$ should be taken so small that we have

$$0 < \frac{\rho}{q}\sigma_K + \left(1 - \frac{\rho}{q}\right)\left(1 - \frac{n}{2} - \sigma\right). \tag{3.3}$$

From $4/n < p_1 \le p_2$, we have $\rho < q' \le q$. I is an interval in $\boldsymbol{R}$, which we will occasionally omit.

Now, our goal estimate is $\|u\|_{(K;\boldsymbol{R})} \le C(E(u))$. First we derive several basic estimates as in [13, Sect. 3]. Since $\rho < q' \le q$, we have by the Hölder and the Sobolev inqualities,

$$\|u\|_{(X')} \le C\|u\|_{(K)}^{1-\alpha}\|u\|_{(X)}^{\alpha}, \tag{3.4}$$

where $0 < \alpha \le 1$ is defined by $(1-\alpha)/\rho + \alpha/q = 1/q'$. By the Sobolev embedding, we also have

$$\|u\|_{(B)} \le C\|u\|_{(E)}, \quad \|\varphi_j * u\|_{(B)} \le C2^{-\sigma j}\|u\|_{(E)}, \tag{3.5}$$

where $\{\varphi_j\}_{j=0}^{\infty}$ is the Paley-Littlewood partition of $\delta(x)$ as in $(2.5)_*$. By (3.3), we have by the interpolation inequalities and the Sobolev embedding,

$$\|u\|_{(X)} \le C\|u\|_{(K)}^{\rho/q}\|u\|_{(B)}^{1-\rho/q}, \tag{3.6}$$

$$\|u\|_{(X)} \le C\|u\|_{(Y)}^{\beta}\{\|u\|_{(K)}^{\rho/q}\|u\|_{(E)}^{1-\rho/q}\}^{1-\beta}, \tag{3.7}$$

where $0 < \beta < 1$ is defined by $\beta\{\sigma_k\rho/q + 2(1-\rho/q)/(n+2)\} = n/\rho - n/q$. By the assumption (1.8), [9, Lemma 3.1] and Hölder's inequality, we have

$$\|f(u)\|_{(\tilde{K})} \le C\|u\|_{(K)}(\|u\|_{(X)}^{p_2} + \|u\|_{(X')}^{p_1}), \tag{3.8}$$

and

$$\|f(u) - f(v)\|_{(\tilde{Y})} \le C\|u - v\|_{(Y)}(\|u\|_{(X)}^{p_2} + \|v\|_{(X)}^{p_2} + \|u\|_{(X')}^{p_1} + \|v\|_{(X')}^{p_1}). \tag{3.9}$$

In this section, we use the Strichartz estimates for the following norms: Let $T > 0$ and $I = (0, T)$. Then we have

$$\|\boldsymbol{u}\|_{(E;I)} + \|\boldsymbol{u}\|_{(K;I)} \le C\|\boldsymbol{u}(0)\|_{H^1} + C\|\mathrm{eq}_L(u)\|_{(\tilde{K};I)}. \tag{3.10}$$

The next lemma is a substitute for Lemma 3.1_*, which is sufficient for the rest of the arguments in [13].

LEMMA 3.1. *Assume* (1.8). *Let u be a solution of* NLKG *or* NLS *on an interval $I = (S, T)$ satisfying $\|\boldsymbol{u}\|_{(E;I)} \le E < \infty$ and $\|u\|_{(X;I)} = \eta < \infty$. In the* NLKG *case, assume $n < 3$. Then there exists $0 < \eta_0 = \eta_0(E)$ such that if $\eta \le \eta_0(E)$, we have*

$$\|\boldsymbol{u}\|_{(K;I)} + \|u\|_{(X';I)} \le C(E). \tag{3.11}$$

PROOF. Let v be the free solution with the same initial data as u at $t = S$. By (3.10), (3.8) and (3.4), we have

$$\begin{aligned} \|u - v\|_{(K;I)} &\le C\|f(u)\|_{(\bar{K};I)} \\ &\le C\|u\|_{(K)}(\|u\|_{(X)}^{p_2} + \|u\|_{(X')}^{p_1}) \\ &\le C(\|u\|_{(K)}\|u\|_{(X)}^{p_2} + \|u\|_{(K)}^{(1-\alpha)p_1+1}\|u\|_{(X)}^{\alpha p_1}) . \end{aligned} \tag{3.12}$$

So, taking $\eta_0(E)$ sufficiently small, we obtain an estimate $\|u\|_{(K;I)} \le C(E)$. Then the desired estimate follows from (3.10) and (3.4). □

Now, assume (1.6), (1.8) and (1.10). Assume (1.7) further in the NLS case. Then we can prove Lemma 4.1$_*$, if we replace (4.5)$_*$ with the following estimate:

$$\eta = \|u\|_{(X;I)} \le C(E)\|u\|_{(Y;I)}^{\beta} \le C(E)|I|^{\beta/q} , \tag{3.13}$$

which follows from (3.7), Hölder's inequality and the Sobolev embedding. By Lemma 2.6, we can prove Lemma 5.3$_*$ with $p = 1 + 4/n$. Lemma 6.2$_*$ is obviously valid. Thus we can prove Lemma 6.1$_*$, without the log-weight. Then Lemma 7.1$_*$ follows, but here we should add the estimate

$$\|\omega^{-1}\dot{u}\|_{(K;(S,T))} \le \nu^2 \tag{3.14}$$

to (7.1)$_*$ in the NLKG case, where $\omega = \sqrt{1-\Delta}$. However, it is obvious that this estimate can be derived with very little modification, if we use the modified Lemma 3.1. Then we may replace the assumption (8.1)$_*$ with

$$\|u\|_{(X;(S,T))}^2 + \|\boldsymbol{u}\|_{(K;(S,T))}^2 \le \nu^2 \le \int_{|x-c|<R} e_N(u; S)dx , \tag{3.15}$$

where $e_N(u) := |\nabla \boldsymbol{u}|^2 + |\boldsymbol{u}|^2 + F(u)$ denotes the energy density. Here we need a small modification, since in [13] we used $p > 2$ in the estimates for (8.19)$_*$ and (8.20)$_*$ (remark that $p - 1 = p_1 = p_2$ in [13]). In the NLKG case, we can use the assumption $\|\omega^{-1}\dot{u}\|_{(K;I)} \le \nu$ to have $\|\omega^{-1}\dot{w}\|_{(K;I)} \le C\nu$. Then we obtain

$$E(w;T) \le E(w;S) + C(E)(\nu^{p_1+2} + \nu^{p_2+2}) , \tag{3.16}$$

instead of (8.19)$_*$. So it suffices that $p_1, p_2 > 0$. In the NLS case, we have (3.16), which indeed follows from the arguments in [13]. Thus we have Lemma 8.1$_*$ if (8.1)$_*$ is replaced with (3.15). $B^{-3}_{\infty,2}$ in (8.16)$_*$ and (8.17)$_*$ should be replaced with $B^{-(n+3)/2}_{\infty,2}$.

Now we have only to check the arguments in the proof of Lemma 9.1$_*$, where we need a substantial modification. We need the following additional lemma.

LEMMA 3.2. *Assume* (1.8). *Let $n < 3$ in the* NLKG *case. Then there exist certain positive continuous functions C_j for $j = 0, 1$ satisfying $C_1(0, E) = 0$ and the following properties*: *Let u be a solution to* NLKG *or* NLS *on an interval I with $\|\boldsymbol{u}\|_{(E;I)} \le E < \infty$ and $\|u\|_{(Y;I)} \le \eta < \infty$. If $\eta \le C_0(E)$, we have*

$$\|u\|_{(X;I)} + \|u\|_{(X';I)} \le C_1(\eta, E) . \tag{3.17}$$

PROOF. Let $I=(S,T)$. Let v be the solution of $\mathrm{eq}_L(v)=0$ with $\boldsymbol{v}(S)=\boldsymbol{u}(S)$. Then, by (3.10), (3.8), (3.4) and (3.7) we have

$$\begin{aligned}\|u\|_{(K;I)} &\le CE + C\|f(u)\|_{(\bar{K})} \\ &\le CE + C\|u\|_{(K)}\|u\|^{p_2}_{(X)} + C\|u\|_{(K)}\|u\|^{p_1}_{(X')} \\ &\le CE + C(E)(\|u\|^a_{(K)}\|u\|^b_{(Y)} + \|u\|^c_{(K)}\|u\|^d_{(Y)}),\end{aligned} \tag{3.18}$$

where $a=1+p_2(1-\beta)\rho/q>1$, $b=p_2\beta>0$, $c=1+p_1(1-\alpha)+p_1\alpha(1-\beta)\rho/q>1$ and $d=p_1\alpha\beta>0$. So, if we take $C_0(E)$ sufficiently small, we have $\|u\|_{(K;I)}\le C(E)$. Then, from (3.4) and (3.7), we obtain the desired result. □

Now we will prove a lemma which corresponds to Lemma 9.1_*.

LEMMA 3.3. *Assume* (1.8). *Let u and w be a global solution to* NLKG *or* NLS. *In the* NLKG *case, assume $n<3$. Let v be the solution to* $\mathrm{eq}_L(v)=0$ *with* $\boldsymbol{v}(0)=\boldsymbol{u}(0)-\boldsymbol{w}(0)$. *Assume $\|\boldsymbol{u}\|_{(E;\boldsymbol{R})}, \|\boldsymbol{w}\|_{(E;\boldsymbol{R})}\le E<\infty$. Then, for any $L<\infty$, there exists $\kappa=\kappa(E,L)>0$ such that if $\|w\|_{(X;(0,\infty))}<L$ and $\|v\|_{(X;(0,\infty))}<\kappa$, we have $\|u\|_{(X;(0,\infty))}<C(E,L)$.*

PROOF. We denote by $D_j(\eta,E)$ certain positive continuous functions which are increasing with respect to η and satisfy $D_j(0,E)=0$ for any $E>0$. Let $\eta\in(0,\eta_0)$, $0=T_0<T_1<\cdots<T_N<T_{N+1}=\infty$, $N^{1/q}\eta\le L$ and

$$\|w\|_{(X;I_j)}\le\eta, \tag{3.19}$$

where $I_j:=(T_j,T_{j+1})$. By Lemma 3.1 and $(3.5)_*$, we have

$$\|w\|_{(Y;I_j)}\le D_0(\eta,E), \quad \|v\|_{(Y;I_j)}\le D_0(\kappa,E). \tag{3.20}$$

Let $u=w+v+g$. Then we have $\boldsymbol{g}(0)=0$ and the integral equation

$$g(t)=g_j(t)+\int_{T_j}^t U(t-s)(f(w)-f(u))ds, \tag{3.21}$$

where U is as in $(2.8)_*$, and g_j is the solution of $\mathrm{eq}_L(g_j)=0$ with $\boldsymbol{g}_j(T_j)=\boldsymbol{g}(T_j)$. By $(3.7)_*$ and (3.9), we have for any $I=(T_j,T)$,

$$\|g\|_{(Y;I)}\le\|g_j\|_{(Y;I)}+C_2(\|w\|^{p_2}_{(X;I)}+\|u\|^{p_2}_{(X;I)}+\|w\|^{p_1}_{(X';I)}+\|u\|^{p_1}_{(X';I)})\|v+g\|_{(Y;I)}, \tag{3.22}$$

where C_2 is a positive constant depending only on n, p_1, p_2 and the constant in (1.8). Let $P_j:=\|g_j\|_{(Y;(T_j,\infty))}$. Then, from $(9.6)_*$, $(3.7)_*$ and (3.9), we have

$$P_{j+1}\le P_j+C_2(\|w\|^{p_2}_{(X;I_j)}+\|u\|^{p_2}_{(X;I_j)}+\|w\|^{p_1}_{(X';I_j)}+\|u\|^{p_1}_{(X';I_j)})\|v+g\|_{(Y;I_j)}. \tag{3.23}$$

Now let $B:=2^N D_0(\kappa,E)$. We will prove that $\|g\|_{(Y;I_j)}\le B$ if κ is sufficiently small. There exists $0<\eta_1(E)<\min(\eta_0(E),C_0(E))$ such that if $\eta\le\eta_1$, we have

$$C_2(2C_1(\eta,E)^{p_2}+2C_1(\eta,E)^{p_1})<1/2, \tag{3.24}$$

where C_1 is given in Lemma 3.2. Then, there exists $0<\eta_2(E)<\eta_1(E)$ such that if $\eta\le\eta_2$, then we have

$$D_0(\eta,E)<\eta_1(E)/4. \tag{3.25}$$

Moreover, there exists $\kappa_1(E, L) > 0$ such that if $\eta > \eta_2/2$ and $\kappa \le \kappa_1$, then we have

$$D_0(\kappa, E) < 2^{-N}\eta_1/4, \tag{3.26}$$

so that $B < \eta_1/4$. Now we fix $\eta := \eta_2(E)$ and $\kappa = \kappa_1(E, L)$. Then we prove that

$$P_j \le (2^j - 1)D_0(\kappa, E) \tag{3.27}$$

by induction on j. For $j = 0$, it is trivial. Assume (3.27) for $j - 1$, and suppose that there exists some $T \in I_j$ such that $\|g\|_{(Y;(T_j,T)))} = (2^j - 1)D_0(\kappa, E)$. Then, by (3.20), (3.26), (3.25), Lemma 3.2, (3.24) and (3.22), we have

$$\|g\|_{(Y;I)} < 2\|g_j\|_{(Y;I)} + \|v\|_{(Y;I)} \le (2^j - 1)D_0(\kappa, E), \tag{3.28}$$

where $I = (T_j, T)$, which is a contradiction. Thus we obtain $\|g\|_{(Y;(T_j,T)))} < (2^j - 1)D_0(\kappa, E)$ for any $T \in I_j$. In particular, we have $\|g\|_{(Y;I_j)} \le (2^j - 1)D_0(\kappa, E)$, so that from (3.23) we have (3.27) for this j. Thus the induction is completed and we obtain $\|u\|_{(Y;I_j)} \le D_0(\eta_2, E) + D_0(\kappa_1, E) + B < \eta_1 < C_0$. Then, from Lemma 3.2, we obtain $\|u\|_{(X;(0,\infty))} \le NC_1(C_0(E), E)) \le C(E, L)$. □

Now we can prove the global estimate $\|u\|_{(X;\boldsymbol{R})} < C(E(u))$ in the same way as in Proposition 10.1$_*$. Then, partitioning the time axis into intervals and applying Lemma 3.1 on each interval, we obtain the desired

$$\|u\|_{(K;\boldsymbol{R})} < C(E(u)). \tag{3.29}$$

4. Global space-time estimates (Case II). In this section, we derive the global space-time estimates for NLKG with $n \ge 3$, which was not considered in the previous section. Again, we denote by $(\cdot,\cdot)_*$ those equations in [13] and similarly by Lemma $\cdot.\cdot_*$ the lemmas therein. In this section we need more norms than in the previous section. We will assume (1.8) with

$$\frac{4}{n} < p_1 < \frac{4(n+1)}{(n+2)(n-1)} < \frac{4}{n-1} < p_2 < \frac{4}{n-2}, \tag{4.1}$$

where we need the upper bound of p_1 and the lower bound of p_2 only for some technical reasons. We will use the following norms in this section:

$$\begin{aligned}
&(E; I) := L^\infty(I; H^1(\boldsymbol{R}^n)), && (B; I) := L^\infty(I; B^{1-n/2-\sigma}_{\infty,\infty}(\boldsymbol{R}^n)),\\
&(M; I) := L^\mu(I \times \boldsymbol{R}^n), && (X'; I) := L^{q'}(I \times \boldsymbol{R}^n),\\
&(K; I) := L^\rho(I; B^{\sigma_K}_{\rho,2}(\boldsymbol{R}^n)), && (\bar{K}; I) := L^{\bar\rho}(I; B^{\sigma_K}_{\bar\rho,2}(\boldsymbol{R}^n)),\\
&(S; I) := L^\zeta(I; B^{\sigma_K}_{\zeta,2}(\boldsymbol{R}^n)), && (\bar{S}; I) = L^{\bar\zeta}(I; B^{\sigma_K}_{\bar\zeta,2}(\boldsymbol{R}^n)),\\
&(V; I) := L^\nu(I \times \boldsymbol{R}^n), && (\tilde{V}; I) := L^{\bar\rho}(I \times \boldsymbol{R}^n),\\
&(W; I) := L^\zeta(I \times \boldsymbol{R}^n), && (\bar{W}; I) := L^{\bar\zeta}(I \times \boldsymbol{R}^n),
\end{aligned} \tag{4.2}$$

where $\rho = 2 + 4/n$, $\zeta = 2 + 4/(n-1)$, $1/\rho + 1/\bar\rho = 1/\zeta + 1/\bar\zeta = 1$, $p_1/q' = 1/\bar\rho - 1/\rho = 2/(n+2)$, $p_2/\mu = 1/\bar\zeta - 1/\zeta = 2/(n+1)$, $(p_1+1)/\nu = 1/\bar\rho$, $\sigma_K = 1/2$ and $\sigma > 0$ should be taken such that

$$0 < \frac{\zeta}{\mu}\sigma_K + \left(1 - \frac{\zeta}{\mu}\right)\left(1 - \frac{n}{2} - \sigma\right). \tag{4.3}$$

Remark that we have $\rho < \nu < q' < \zeta < \mu < 2(n+1)/(n-2)$ from (4.1).

Now our goal estimate is $\|\boldsymbol{u}\|_{(K;\boldsymbol{R})} + \|\boldsymbol{u}\|_{(S;\boldsymbol{R})} < C(E(u))$, which will be derived from $\|u\|_{(M;\boldsymbol{R})} < C(E(u))$. The outline of the proof is the same as in the first case, but we have to use a little more complicated version of the Strichartz estimate:

$$\begin{aligned} &\|\boldsymbol{u}\|_{(E;I)\cap(K;I)\cap(S;I)} \le C\|\boldsymbol{u}(0)\|_{H^1} + C\|\mathrm{eq}_L(u)\|_{(\bar{K};I)+(\bar{S};I)}, \\ &\|u\|_{(V;I)\cap(W;I)} \le C\|\boldsymbol{u}(0)\|_{H^1} + C\|\mathrm{eq}_L(u)\|_{(\bar{V};I)+(\bar{W};I)}, \end{aligned} \tag{4.4}$$

where the second estimate follows from the first one. We will use the following nonlinear estimates (see [9, Lemma 3.1]):

$$\|f(u)\|_{(\bar{K})+(\bar{S})} \le C(\|u\|_{(K)}\|u\|_{(X')}^{p_1} + \|u\|_{(S)}\|u\|_{(M)}^{p_2}), \tag{4.5}$$

$$\|f(u) - f(v)\|_{(\bar{V})+(\bar{W})} \le C(\|u-v\|_{(V)}\|u\|_{(V)}^{p_1} + \|u-v\|_{(W)}\|u\|_{(M)}^{p_2}). \tag{4.6}$$

We have the following interpolation inequalities:

$$\|u\|_{(W)} \le C\|u\|_{(K)}^{1-\alpha}\|u\|_{(M)}^{\alpha}, \tag{4.7}$$

$$\|u\|_{(V)} \le C\|u\|_{(K)}^{1-\beta}\|u\|_{(M)}^{\beta}, \tag{4.8}$$

$$\|u\|_{(X')} \le C\|u\|_{(K)}^{1-\gamma}\|u\|_{(M)}^{\gamma}, \tag{4.9}$$

$$\|u\|_{(M)} \le C\{\|u\|_{(W)}^{1-\delta}\|u\|_{(S)}^{\delta}\}^{\zeta/\mu}\|u\|_{(B)}^{1-\zeta/\mu}, \tag{4.10}$$

where $0 < \alpha, \beta, \gamma, \delta < 1$ should be taken such that $(1-\alpha)/\rho + \alpha/\mu = 1/\zeta$, $(1-\beta)/\rho + \beta/\mu = 1/\nu$, $(1-\gamma)/\rho + \gamma/\mu = 1/q'$ and

$$0 < \delta\frac{\zeta}{\mu}\sigma_K + \left(1 - \frac{\zeta}{\mu}\right)\left(1 - \frac{n}{2} - \sigma\right). \tag{4.11}$$

Then the next lemma is a substitute for Lemma 3.1_*.

LEMMA 4.1. *Let $n \ge 3$ and assume* (1.8). *Let u be a solution of* NLKG *on an interval $I = (S,T)$ satisfying $\|\boldsymbol{u}\|_{(E;I)} \le E < \infty$ and $\|u\|_{(M;I)} = \eta < \infty$. Then there exists $0 < \eta_5 = \eta_5(E)$ such that if $\eta \le \eta_5(E)$, we have*

$$\|\boldsymbol{u}\|_{(K;I)} + \|\boldsymbol{u}\|_{(S;I)} + \|u\|_{(X';I)} \le C(E). \tag{4.12}$$

PROOF. Let v be the free solution with the same initial data as u at $t = S$. By (4.4), (4.5) and (4.9), we have

$$\begin{aligned} \|u - v\|_{(S;I)\cap(K;I)} &\le C\|f(u)\|_{(\bar{S};I)+(\bar{K};I)} \\ &\le C\|u\|_{(S)\cap(K)}(\|u\|_{(M)}^{p_2} + \|u\|_{(X')}^{p_1}) \\ &\le C(\|u\|_{(S)\cap(K)}\|u\|_{(M)}^{p_2} + \|u\|_{(S)\cap(K)}^{(1-\gamma)p_1+1}\|u\|_{(M)}^{\gamma p_1}). \end{aligned} \tag{4.13}$$

So, taking $\eta_5(E)$ sufficiently small, we obtain an estimate $\|u\|_{(S;I)\cap(K;I)} \le C(E)$. Then the desired estimate follows from (4.4) and (4.9). □

The next lemma is a substitute for Lemma 4.1_*.

LEMMA 4.2. *Let $n \geq 3$ and assume* (1.8). *Let u be a solution of* NLKG *on an interval I satisfying $\|\boldsymbol{u}\|_{(E;I)} \leq E < \infty$ and $\|u\|_{(M;I)} = \eta \in (0, \eta_5(E))$, where η_5 is as given in Lemma 4.1. Then, there exist a subinterval $J \subset I$, $R > 0$ and $c \in \boldsymbol{R}^n$ satisfying $|J| \geq C(E, \eta)$, $R \leq C(E, \eta)$ and*

$$\int_{|x-c|<R} |u(t)|^s dx \geq C(E, \eta, s) \tag{4.14}$$

for any $t \in J$ and any $s \geq 1$.

PROOF. This lemma can be proved almost in the same way as Lemma 4.1_*. Instead of $(4.2)_*$, it follows from Lemma 4.1 and (4.10) that

$$\eta = \|u\|_{(M)} \leq C\|u\|_{(S)}^{\zeta/\mu}\|u\|_{(B)}^{1-\zeta/\mu} \leq C(E)\|u\|_{(B)}^{1-\zeta/\mu}, \tag{4.15}$$

and, by the Sobolev embedding, that

$$\eta = \|u\|_{(M)} \leq C(E)\|u\|_{(W)}^{(1-\delta)\zeta/\mu} \leq C(E)|I|^{(1-\delta)/\mu}, \tag{4.16}$$

which is a substitute for $(4.5)_*$ to have $|I| \geq C(E, \eta)$. Then, the rest of the proof is just the same as for Lemma 4.1_*. □

Now assume (1.6), (1.8) and (1.10). By Lemma 2.6, we can prove Lemma 5.3_* with $p = 1 + 4/n$. By Lemmas 4.2 and 2.6, Lemma 6.1_* can be proved without the log-weight, if we replace the (X)-norm with the (M)-norm. Then, in the same way as in [13], we can prove the following substitute for Lemma 7.1_*:

LEMMA 4.3. *Let $n \geq 3$ and assume* (1.6), (1.8) *and* (1.10). *Let u be a global solution of* NLKG *with $E(u) = E < \infty$. Let $\nu, \varepsilon > 0$ and $M < \infty$. Then there exists $\nu_1 = \nu_1(E) > 0$, $N = N(E, \nu, M, \varepsilon) < \infty$ with the following properties: If $\nu \leq \nu_1$ and $\|u\|_{(M;I)} > N$ on some interval I, then there exist $(S, T) \subset I$, $c \in \boldsymbol{R}^n$ and $R \in (1, \infty)$ such that $|T - S| > MR$ and that for $t = S$ or $t = T$ we have*

$$\begin{gathered} \|\boldsymbol{u}\|^2_{(K;(S,T))\cap(S;(S,T))\cap(M;(S,T))} \leq \nu^2 \leq \int_{|x-c|<R} e_N(u; t)dx, \\ \left\| \frac{u(t)}{\langle x - c\rangle} \right\|_{L^2} < \varepsilon, \end{gathered} \tag{4.17}$$

where e_N denotes the energy density.

Thus we may replace the assumption $(8.1)_*$ with (4.17). Then, we can prove the corresponding modified version of Lemma 8.1_*, where we use the modification to avoid the explicit use of $p > 2$ in [13], as in the previous section. Now we have only to prove the following substitute of Lemma 9.1_* in the same way as in the previous section.

LEMMA 4.4. *Let $n \geq 3$ and assume* (1.8). *Let u, w be global solutions of* NLKG, *and v be a global solution of the free equation satisfying $\boldsymbol{u}(0) = \boldsymbol{v}(0) + \boldsymbol{w}(0)$ and $\|\boldsymbol{u}\|_{(E;\boldsymbol{R})}$, $\|\boldsymbol{w}\|_{(E;\boldsymbol{R})} \leq E < \infty$. Then, for any $L < \infty$, there exists $\kappa = \kappa(E, L) > 0$ such that if $\|w\|_{(M;(0,\infty))} < L$ and $\|v\|_{(M;(0,\infty))} < \kappa$, we have $\|u\|_{(M;(0,\infty))} < C(E, L)$.*

To prove this lemma, we need the following estimate, which corresponds to Lemma 3.2 in the previous section.

LEMMA 4.5. *Assume* (1.8) *and* $n \geq 3$. *Then, there exist certain positive continuous functions* C_j *for* $j = 0, 1$ *satisfying* $C_1(0, E) = 0$ *and the following properties. Let* u *be a solution to* NLKG *on an interval* I *with* $\|\boldsymbol{u}\|_{(E;I)} \leq E < \infty$ *and* $\|u\|_{(W;I)} \leq \eta < \infty$. *If* $\eta \leq C_0(E)$, *then we have*

$$\|u\|_{(M;I)} + \|u\|_{(V;I)} \leq C_1(\eta, E)\,. \tag{4.18}$$

PROOF. Let $I = (S, T)$. Let v be the solution of $\mathrm{eq}_L(v) = 0$ with the same initial data as u at $t = S$. Then, by (4.4), (4.5), (4.9) and (4.10), we have

$$\begin{aligned}\|u\|_{(S;I)\cap(K;I)} &\leq CE + C\|f(u)\|_{(\bar{S};I)+(\bar{K};I)} \\ &\leq CE + C\|u\|_{(S)}\|u\|_{(M)}^{p_2} + C\|u\|_{(K)}\|u\|_{(X')}^{p_1} \\ &\leq CE + C(E)(\|u\|_{(S)}^a\|u\|_{(W)}^b + \|u\|_{(S)\cap(K)}^c\|u\|_{(W)}^d)\,,\end{aligned} \tag{4.19}$$

where $a = 1 + p_2\delta\zeta/\mu > 1$, $b = p_2(1-\delta)\zeta/\mu > 0$, $c = 1 + p_1(1-\gamma) + p_1\gamma\delta\zeta/\mu > 1$ and $d = p_1\gamma(1-\delta)\zeta/\mu > 0$. So, if we take $C_0(E)$ sufficiently small, we have $\|u\|_{(S;I)\cap(K;I)} \leq C(E)$. Then, from (4.8) and (4.10), we obtain the desired result. □

Now we can prove Lemma 4.4 in a way similar to that for Lemma 3.3. We partition $(0, \infty)$ into intervals I_j such that

$$\|w\|_{(M;I_j)} \leq \eta\,, \tag{4.20}$$

instead of (3.19). Then we have the estimate for the number of the intervals $N^{1/\mu}\eta \leq L$. By Lemma 4.1, (4.8) and (4.7), we have

$$\|w\|_{(V;I_j)\cap(W;I_j)} \leq D_0(\eta, E)\,, \quad \|v\|_{(V;I_j)\cap(W;I_j)} \leq D_0(\kappa, E)\,, \tag{4.21}$$

instead of (3.20). By (4.4) and (4.6), we have

$$\begin{aligned}\|g\|_{(V;I)\cap(W;I)} \leq \|g_j\|_{(V;I)\cap(W;I)} &+ C_2(\|w\|_{(M;I)}^{p_2} + \|u\|_{(M;I)}^{p_2} \\ &+\|w\|_{(V;I)}^{p_1} + \|u\|_{(V;I)}^{p_1})\|v + g\|_{(V;I)\cap(W;I)}\,,\end{aligned} \tag{4.22}$$

instead of (3.22), and a similar estimate instead of (3.23). Then, in the same way as for Lemma 3.3, we obtain $\|u\|_{(V;I_j)\cap(W;I_j)} < C_0(E)$ and by Lemma 4.1, $\|u\|_{(M;(0,\infty))} \leq NC_1 \leq C(E, L)$.

Now we can prove the global estimate $\|u\|_{(M;\boldsymbol{R})} < C(E(u))$ in the same way as in Proposition 10.1_*. Then, partitioning the time axis into intervals and applying Lemma 3.1 on each interval, we obtain the desired estimate

$$\|\boldsymbol{u}\|_{(K;\boldsymbol{R})} + \|\boldsymbol{u}\|_{(S;\boldsymbol{R})} < C(E(u))\,. \tag{4.23}$$

5. Scattering result. In this section we prove the scattering results in Theorem 1.1 from the global space-time estimates derived in the previous sections. First, we show that the wave operators $v(0) \mapsto u(0)$ is well-defined in the whole H^1. In this step, we need only the global estimate for the free solutions. We have to solve the integral equation

$$\boldsymbol{u}(t) = \boldsymbol{v}(t) + \int_\infty^t K(t-s) f(u(s)) ds\,, \tag{5.1}$$

locally near $t = \infty$, where

$$K(t) := \begin{cases} \omega^{-1}(\sin t\omega, \cos t\omega)\,, & \text{for NLKG}\,, \\ e^{-it\Delta}\,, & \text{for NLS}\,. \end{cases} \tag{5.2}$$

Let $u_1 = v$, and define u_j inductively by the linear equation

$$\boldsymbol{u}_j(t) = \boldsymbol{v}(t) + \int_\infty^t K(t-s) f(u_{j-1}(s)) ds\,. \tag{5.3}$$

For NLS with any n and NLKG with $n \leq 2$, by $(3.7)_*$ and (3.9) we have

$$\begin{aligned} \|u_{j+1} - u_j\|_{(Y;(T,\infty))} &\leq C\|u_j - u_{j-1}\|_{(Y;(T,\infty))} \\ &\times (\|u_j\|^{p_2}_{(X;(T,\infty))} + \|u_{j-1}\|^{p_2}_{(X;(T,\infty))} + \|u_j\|^{p_1}_{(X';(T,\infty))} + \|u_{j-1}\|^{p_1}_{(X';(T,\infty))})\,. \end{aligned} \tag{5.4}$$

By Lemma 3.2, if $\|v\|_{(Y,(T,\infty))}$ is sufficiently small, then we can show, by the standard argument, that u_j converges in $(Y; (T, \infty))$ and is bounded in $(X; (T, \infty)) \cap (K; (T, \infty))$. Then the limit function is the unique solution to (5.1) satisfying $\|u\|_{(K;(T,\infty))} + \|u\|_{(X;(T,\infty))} < \infty$ (the uniqueness follows from an estimate silimar to (5.4)). Now we can extend the local solution to a global one by the standard and well-known argument. By (3.10) and (3.8), we have

$$\|\boldsymbol{u} - \boldsymbol{v}\|_{(E;(T,\infty))} \leq C\|u\|_{(K;(T,\infty))}(\|u\|^{p_2}_{(X;(T,\infty))} + \|u\|^{p_1}_{(X';(T,\infty))})\,, \tag{5.5}$$

which tends to 0 as $T \to \infty$. Thus we obtain the wave operators defined on H^1.

Next we show that the wave operators are surjective (it is trivial that they are injective). Now we have to prove that any finite energy solution u of NLKG or NLS approaches to some free solution. For NLS with any n and NLKG with $n \leq 2$, by (3.10) and (3.8), we have for $T < t$

$$\left\| \int_T^t K(-s) f(u(s)) ds \right\|_{H^1} \leq C\|u\|_{(K;(T,\infty))}(\|u\|^{p_2}_{(X;(T,\infty))} + \|u\|^{p_1}_{(X';(T,\infty))})\,, \tag{5.6}$$

which tends to 0 as $T \to \infty$, since we have the finiteness of the norms. So, there exists the limit in H^1 of

$$\boldsymbol{a} := \int_0^\infty K(-t) f(u(t)) dt\,. \tag{5.7}$$

Now, let v be the free solution with the initial data $\boldsymbol{v}(0) = \boldsymbol{u}(0) + \boldsymbol{a}$. Then we have (5.1) and (5.5) for these u and v, so that we obtain the surjectivity of the wave operators. For NLKG with $n \geq 3$, we just replace (Y) with $(V) \cap (W)$, (X) with (M), (X') in (5.4) with (V) and (K) with $(K) \cap (S)$ in the above argument. Using the uniform bound for the global space-time norms, we can show the weak continuity of the wave operators and their inverses. Then, the strong continuity follows from the weak continuity and the energy conservation law (see [11, Corollary 6.3] for more detail). Thus we finish the proof of Theorem 1.1.

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