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ON THE FATOU SET OF AN ENTIRE FUNCTION WITH GAPS

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Abstract. It is shown that every component of the Fatou set of an entire function with certain gaps is bounded.

1. Introduction. Let f be a nonlinear entire function of the complex variable z. Its natural iterates f^n are defined by $f^0(z) = z$, $f^1(z) = f(z)$, $f^{n+1} = f(f^n(z))$, $n = 1, 2, \dots$. The *Fatou set* $\mathcal{F}(f)$ of the function f is the largest open set of the complex plane where the family $\{f^n\}$ forms a normal family. The complement of $\mathcal{F}(f)$ is called the *Julia set* and is denoted by $\mathcal{J}(f)$. Then $\mathcal{F}(f)$ is open and completely invariant under f, and $\mathcal{J}(f)$ is closed, perfect and also completely invariant. For more details of the concepts and properties in the iteration theory, we refer to Beardon's [7], Carleson and Gamelin's [10], McMullen's [19] books as well as Milnor's [18] lecture notes for rational functions and the survey articles of Baker [6] and Eremenko and Lyubich [11] for rational and entire functions and Bergweiler [8] for transcendental meromorphic functions.

If f is a polynomial of degree at least two, then $\mathcal{F}(f)$ contains the component $D = \{z; f^n(z) \to \infty\}$, which is unbounded and completely invariant. If f is transcendental entire, it is obvious from Picard's theorem together with the invariance of $\mathcal{J}(f)$ that $\mathcal{J}(f)$ is unbounded, so that $\mathcal{F}(f)$ no longer contains a neighbourhood of ∞.

Baker [5] raised the question of whether every component of $\mathcal{F}(f)$ must be bounded if f is of sufficiently small growth. The appropriate growth condition would be of order $< 1/2$, since Baker [5] showed that for any sufficiently large positive a, the function $f_0(z) = z^{-1/2}\sin z^{1/2} + z + a$ is of order $\rho = 1/2$ and has an unbounded component D of $\mathcal{F}(f)$ containing a segment $[x_0, \infty)$ of the positive real axis. Moreover, Baker proved that the growth of a transcendental entire function f must exceed order $1/2$, minimal type, if $\mathcal{F}(f)$ has an unbounded invariant component.

Baker [5], Stallard [23], Anderson and Hinkkanen [3] obtained a few results in the positive direction to this problem and proved that $\mathcal{F}(f)$ has no unbounded components if f is of order less than $1/2$, and satisfies certain different growth conditions. The key point in their proofs is that for any entire function f of order less than $1/2$ with those growth conditions, the iterates f^k have a certain property of self-sustaining spread on any compact subset of the

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components of the Fatou set of f. The case that $\mathcal{F}(f)$ contains a simply connected wandering component remains open.

In the present paper, we shall consider the iteration of entire functions with gaps, of finite or infinite order of growth, and show that the Fatou sets of such functions have no unbounded components.

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2. Statement of main results. We shall use the standard notation for the *maximum modulus* $M(r, f)$, *minimum modulus* $L(r, f)$, *order* ρ and *lower order* μ of a function f, namely,

$$M(r, f) = \max\{|f(z)|; |z| = r\},$$

$$L(r, f) = \min\{|f(z)|; |z| = r\},$$

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

and

$$\mu = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function with gaps, i.e., such that many of the a_n are zero, in a certain sense. Then the function has the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}. \tag{2.1}$$

We say that $f(z)$ has *Fabry gaps* if

$$\frac{n_k}{k} \to \infty \tag{2.2}$$

as $k \to \infty$, and $f(z)$ has *Fejér gaps* if

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty. \tag{2.3}$$

Since early this century, a vast number of papers has appeared, concerning the Picard, Borel and asymptotic values, Julia lines, maximum and minimum modulus, and value distribution of entire functions with different gap conditions. For example, Fejér [12] proved in 1908 that an entire function with gaps (2.3) takes every finite complex value and then Biernacki [9] proved that the function has no Picard values, i.e., $f(z)$ takes every finite complex value infinitely often; Pólya [21] obtained that for an entire function of infinite order with Fabry gaps (2.2), every line is a Julia line of $f(z)$, i.e., $f(z)$ assumes in every angle every value infinitely often, with at most one exception, and Anderson and Clunie [2] then extended it to functions of finite positive order; Erdös and Macintyre, Kövari obtained many results about the Borel and asymptotic values of entire functions with different gaps (see, for instance, [15], [16], [17]); Fuchs [13] proved a conjecture of Pólya which implies that an entire function of finite order with Febry gaps has no finite asymptotic and Nevanlinna values, and Sons [22], Hayman [14] solved, among other things, the case of entire function of finite lower order with Febry gaps; Anderson and Binmore [1] first, for the case of finite lower order, and Murai [20] then, for the general case, proved that an entire function with Fejér gaps has no finite Nevanlinna deficient values, etc.

We shall discuss the iteration of entire functins with Febry or other gaps and prove that every component of the Fatou set is bounded, by using the properties of the entire functions with such gaps. Our main results are the following theorems.

THEOREM 1. *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

be an entire function with $0 < \mu \leq \rho < \infty$. *If* $f(z)$ *has Fabry gaps, then every component of* $\mathcal{F}(f)$ *is bounded.*

For each entire function and any given number $T > 1$, it follows from the Hadamard three-circles theorem that

$$\liminf_{r\to\infty} \frac{\log M(r^T, f)}{\log M(r, f)} \geq T\,. \tag{2.4}$$

We shall obtain the same conclusion for functions of arbitrary order of growth, including infinite order, provided that a stronger Fejér gap condition and the strict inequality (2.4) hold.

THEOREM 2. *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

be an entire function, satisfying

$$\liminf_{r\to\infty} \frac{\log M(r^T, f)}{\log M(r, f)} > T \tag{2.5}$$

for some number $T > 1$. *If*

$$n_k > k \log k (\log\log k)^{\alpha} \tag{2.6}$$

as $k \to \infty$, *for some* $\alpha > 2$, *then every component of* $\mathcal{F}(f)$ *is bounded.*

3. Lemmas and proof of theorems. To prove our theorems, we need the following results. The first result was obtained by Baker in [5], using Schottky's theorem.

LEMMA 1. *If, in a domain D, the analytic functions g of the family G omit the values* 0, 1, *and if E is a compact subset of D on which the functions all satisfy* $|g(z)| \geq 1$, *then there exist constants B, C depending only on E and D, such that for any* z, z' *in E and any g in G we have*

$$|g(z')| < B|g(z)|^C\,.$$

Before stating next two lemmas, we recall briefly the definitions of logarithmic densities for a measurable set E on the positive real axis. Let $E(a, b)$ denote the part of E in the interval (a, b). Then the *upper logarithmic density* and *lower logarithmic density* are defined respectively by

$$\overline{\log\text{dens}}\, E = \limsup_{r\to\infty} \frac{1}{\log r} \int_{E(1,r)} \frac{dt}{t}\,; \tag{3.1}$$

$$\underline{\log\text{dens}}\, E = \liminf_{r\to\infty} \frac{1}{\log r} \int_{E(1,r)} \frac{dt}{t}\,. \tag{3.2}$$

If the upper and lower logarithmic density are equal, their common value is called the *logarithmic density* of E.

We shall use the following theorem of Fuchs [13] for the entire functions with Fabry gaps.

LEMMA 2. *Let $f(z)$ be an entire function of finite order with Fabry gaps* (2.2). *Then for given $\varepsilon > 0$, we have*

$$\log L(r, f) > (1 - \varepsilon) \log M(r, f) \tag{3.3}$$

holds for all r outside a set of logarithmic density 0.

Hayman [14] obtained the same conclusion for functions of arbitrary growth, including infinite order, with the stronger gap condition (2.6).

LEMMA 3. *Let $f(z)$ be an entire function with gaps* (2.6). *Then for given $\varepsilon > 0$,* (3.3) *holds for all r outside a set of logarithmic density* 0.

We now prove an useful criterion for $\mathcal{F}(f)$ with no unbounded components.

THEOREM 3. *Let f be an entire function. Suppose that there exist $t > 0$ and $m > 1$ such that*

$$L(r, f) > M(r, f)^t \tag{3.4}$$

for all r outside a set E of upper logarithmic density less than $1 - 1/m$. If for some positive number R_1, and $R_{n+1} = M(R_n, f)$ $(n = 1, 2, \ldots)$, there exists a number $T > 1$ such that

$$M(R_n^T, f)^t \geq M(R_n, f)^{mT}\,, \tag{3.5}$$

for all sufficiently large n, then the Faton set $\mathcal{F}(f)$ has no unbounded components.

PROOF. We suppose on the contrary that $\mathcal{F}(f)$ has an unbounded component D. Without loss of generality we may assume that 0, 1 belong to $\mathcal{J}(f)$. Hence each function f^k omits the value 0, 1 in D.

It follows from (3.5) that there exists $N_0 \in \boldsymbol{N}$ such that

$$M(R_n^T, f)^t > M(R_n, f)^{mT} \tag{3.6}$$

for all $n \geq N_0$.

On the other hand, it follows from (3.4) that for any given sufficiently large R, say $R \geq R_0$,

$$L(r, f) > M(r, f)^t \tag{3.7}$$

holds for some r in the range $R \leq r \leq R^m$. Indeed, suppose there exists a sequence $R_j \to \infty$ as $j \to \infty$, such that

$$L(r, f) \leq M(r, f)^t$$

in the interval $R_j \leq r \leq R_j^m$ for all $j = 1, 2, \ldots$. Then we have

$$\begin{aligned}\overline{\log \operatorname{dens}} E &\geq \limsup_{j\to\infty} \frac{1}{\log(R_j)^m} \int_{E(1,(R_j)^m)} \frac{dt}{t} \\ &\geq \limsup_{j\to\infty} \frac{1}{m \log R_j} \int_{R_j}^{R_j^m} \frac{dt}{t} = 1 - \frac{1}{m},\end{aligned}$$

which is a contradiction.

Hence we can choose $N_1 (\geq N_0)$ such that for each $n \geq N_1$, there exists ρ_n satisfying

$$R_n{}^T \leq \rho_n \leq (R_n)^{mT} \tag{3.8}$$

with

$$L(\rho_n, f) > M(R_n{}^T, f)^t\,. \tag{3.9}$$

Hence (3.6) and (3.9) give

$$L(\rho_n, f) > (R_{n+1})^{mT}\,. \tag{3.10}$$

In the following discussion, we use an argument of Beker in [5] (see also [23]). Since D is unbounded and connected, there must exist $N_2 \geq N_1$ such that D meets the circles $\gamma_n = \{z; |z| = R_n\}$, $(\gamma_n)' = \{z; |z| = (R_n)^{mT}\}$ and $(\gamma_n)'' = \{z; |z| = \rho_n\}$ for all $n \geq N_2$.

We choose a value $N \in \mathbf{N}$ such that $N \geq N_2$, and note that D must contain a path Γ joining a point $w_N \in \gamma_N$ to a point $(w_{N+1})' \in (\gamma_{N+1})'$. It is clear that Γ must contain a point $(w_{N+1})'' \in (\gamma_{N+1})''$. Now, by (3.8) and (3.10), $f(D)$ must be an unbounded component of $\mathcal{F}(f)$ containing the path $f(\Gamma)$. Note that $M(R_N, f) = R_{N+1}$ and so $|f(w_N)| \leq R_{N+1}$. Also, $L(\rho_{N+1}, f) > (R_{N+2})^{mT}$ and so $|f((w_{N+1})'')| > (R_{N+2})^{mT}$. Hence $f(\Gamma)$ must contain an arc joining a point $w_{N+1} \in \gamma_{N+1}$ to a point $(w_{N+2})' \in (\gamma_{N+2})'$.

We repeat the process inductively to find that $f^k(D)$ is an unbounded component of $\mathcal{F}(f)$ containing an arc of $f^k(\Gamma)$ which joins a point $w_{N+k} \in \gamma_{N+k}$ to a point $(w_{N+k+1})' \in (\gamma_{N+k+1})'$.

Thus, on Γ, the function f^k takes a value of modulus at least R_{N+k}. Since $R_{N+k} \to \infty$ as $k \to \infty$ and D is a component of $\mathcal{F}(f)$, we conclude that $f^k \to \infty$ as $k \to \infty$, locally uniformly in D. It follows that there exists $K \in \mathbf{N}$ such that, for all $k > K$ and all $z \in \Gamma$, we have $|f(z)| > 1$.

Hence the family $\{f^k\}_{k>K}$ satisfies the conditions of Lemma 1 on Γ and so there exist constants B and C such that

$$|f^k(z')| < B|f^k(z)|^C \tag{3.11}$$

for all z, z' in Γ, $k > K$.

We know that, for any $k > K$, we can choose z_k, $(z_k)' \in \Gamma$ such that $f^k(z_k) = w_{N+k} \in \gamma_{N+k}$ and $f^k((z_k)') = (w_{N+k+1})' \in (\gamma_{N+k+1})'$. So it follows from (3.11) that

$$M(R_{N+k}, f) = R_{N+k+1} < (R_{N+k+1})^{mT} < B(R_{N+k})^C$$

for each $k > K$. This contradicts the fact that f is a transcendental function, since $R_{N+k} \to \infty$ as $k \to \infty$. The proof of Theorem 3 is complete.

PROOF OF THEOREM 1. We only need to show that f satisfies the conditions in Theorem 3.

(i) Let $t = 1 - \varepsilon$ and $m > 1$. Then from Lemma 2 we have

$$L(r, f) > M(r, f)^t$$

for all r outside a set E of upper logarithmic density less than $1 - 1/m$.

(ii) On the other hand, for any given $R_1 > 1$, we let $T > \rho/\mu$ and $R_{n+1} = M(R_n, f)$, $n = 1, 2, \ldots$. Then (3.5) holds. Otherwise, there exists a sub-sequence $\{R_{n_j}\}_{j \geq 1}$ of $\{R_n\}$, $R_{n_j} \to \infty$ as $j \to \infty$, such that

$$M(R_{n_j}^T, f)^t < M(R_{n_j}, f)^{mT}$$

for $j \geq 1$. Hence

$$\frac{\log t + \log\log M(R_{n_j}^T, f)}{\log R_{n_j}} \leq \frac{\log mT + \log\log M(R_{n_j}, f)}{\log R_{n_j}}.$$

By letting $j \to \infty$, we then have

$$T\mu \leq \rho,$$

which contradicts the assumption.

Therefore it follows from Theorem 3 that $\mathcal{F}(f)$ has no unbounded components and the proof is complete.

PROOF OF THEOREM 2. The proof of Theorem 2 is similar.

(i) Let $t = 1 - \varepsilon$ and $m > 1$. Then from Lemma 3 we have

$$L(r, f) > M(r, f)^t$$

for all r outside a set E of upper logarithmic density less than $1 - 1/m$.

(ii) It follows from (2.5) that there exist R_0 and $T_1 > T$ such that

$$\log M(r^T, f) > T_1 \log M(r, f)$$

for all $r \geq R_0$. Hence we have

$$\log M(r^{T^i}, f) > {T_1}^i \log M(r, f)$$

for $r \geq R_0$ and $i = 1, 2, \ldots$. Since $T_1 > T$, we can choose a i_0 such that

$${T_1}^{i_0} > \frac{T^{i_0} m}{t}.$$

For a given R_1, we define $R_{n+1} = M(R_n, f)$, $n = 1, 2, \ldots$. Then by letting $T_2 = T^{i_0}$, we have

$$M(R_n^{T_2}, f)^t \geq M(R_n, f)^{mT_2}$$

for all sufficiently large n.

Therefore Theorem 2 follows from Theorem 3 and the proof is complete.

CONCLUDING REMARK. It remains open whether or not Theorems 1 and 2 are sharp, although the proofs given do not seem to extend to more general cases. We propose the following problem with the Fejér gap condition for further study.

PROBLEM. Let $f(z)$ be an entire function with Fejér gaps, i.e.,

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty .$$

Is every component of $\mathcal{F}(f)$ bounded?

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