Tohoku Math. J.
52 (2000), 629–631

CORRECTION: A NOTE ON THE FACTORIZATION THEOREM OF TORIC BIRATIONAL MAPS AFTER MORELLI AND ITS TOROIDAL EXTENSION

(Tohoku Math. J. 51 (1999), 489–537)

KENJI MATSUKI

(Received July 5, 2000)

This erratum describes:

1. The failure of the algorithm in [AMR] and [Morelli1] for the strong factorization pointed out by Kalle Karu,

2. The statement of a refined weak factorization theorem for toroidal birational morphisms in [AMR], in the form utilized in [AKMR] for the proof of the weak factorization theorem for general birational maps, avoiding the use of the above mentioned algorithm for the strong factorization, and

3. A list of corrections for a few other mistakes in [AMR], mostly pointed out by Laurent Bonavero.

We would like to emphasize that though [AMR] is a joint work with D. Abramovich and S. Rashid, the author of this erratum, Kenji Matsuki, is solely responsible for all the errors above.

1. K. Karu pointed out that the procedure in Proposition 7.8 in [AMR] does not preserve the condition $(*)$, contrary to its assertion, and thus the proof does not work. Moreover, the entire algorithm for the strong factorization in Section 7 of [AMR] based upon Proposition 7.8, which attempted to correct the logic of the line of argument of the original one described in [Morelli1] but is identical to it as an algorithm, does not work, as shown by the following example of a cobordism representing a toric birational map in dimension 3:

Consider a 4-dimensional simplicial cobordism Σ describing three smooth star subdivisions of the 3-dimensional cone $\langle v_1, v_2, v_3\rangle$ at $\langle v_1 + v_2\rangle$, $\langle v_2 + v_3\rangle$ and $\langle v_1 + v_2 + v_3\rangle$, in this order. The four maximal cones in this cobordism are all pointing up, but the new cobordism Σ', obtained by the procedure of Proposition 7.8 in [AMR], contains a cone that is not pointing up. Indeed, each of the four given maximal cones has exactly one positive extremal ray and one extremal ray in the link of its circuit. The algorithm subdivides the cobordism at the barycenters of the 2-dimensional cones generated by the positive and the link extremal rays. After subdividing the two topmost cones, one of the new cones will be a pointing up cone with one positive extremal ray $\langle\rho\rangle$ and three negative extremal rays $\langle\rho_{12}\rangle$, $\langle\rho_{23}\rangle$ and $\langle\rho_3\rangle$ with $n(\pi(\rho_{12})) = v_1 + v_2$, $n(\pi(\rho_{23})) = v_2 + v_3$ and $n(\pi(\rho_3)) = v_3$. The next two subdivisions are at the midrays (barycenters) $\zeta_1 = \mathrm{Mid}(\langle\rho_{12}, \rho_{23}\rangle, l_{r(\langle\rho_{12},\rho_{23}\rangle)})$ and $\zeta_2 = \mathrm{Mid}(\langle\rho_{12}, \rho_3\rangle, l_{r(\langle\rho_{12},\rho_3\rangle)})$. The resulting subdivision Σ' contains a cone $\langle\rho, \rho_{12}, \zeta_1, \zeta_2\rangle$, which is not pointing up as it contains two positive extremal rays ρ, ρ_{12} and two negative extremal rays ζ_1, ζ_2.

2. Weak Factorization Theorem for Toroidal Birational Morphisms: *Every proper and toroidal birational morphism $f : (U_X \subset X) \to (U_Y \subset Y)$ between two nonsingular toroidal embeddings without self-intersections can be factored into a sequence of blowups and blowdowns with centers being smooth irreducible closed strata*

$$\begin{aligned}(U_X \subset X) = (U_{V_0} \subset V_0) &\dashrightarrow (U_{V_1} \subset V_1) \dashrightarrow \cdots \\ &\dashrightarrow (U_{V_{i-1}} \subset V_{i-1}) \dashrightarrow (U_{V_i} \subset V_i) \dashrightarrow \\ &\cdots \dashrightarrow (U_{V_{l-1}} \subset V_{l-1}) \dashrightarrow (U_{V_l} \subset V_l) = (U_Y \subset Y)\end{aligned}$$

where all the toroidal embeddings are toroidal over $(U_Y \subset Y)$ and where there is an index i_0 such that for all $i \leq i_0$ the map $V_i \dashrightarrow X$ is a projective morphism and for all $i \geq i_0$ the morphism $V_i \to Y$ is a projective morphism. In particular, if both X and Y are projective, then all the V_i are projective.

A proof of this theorem is given in the revised version (dated May 2000) of [AKMW] via the process of π-desingularization described in [AMR] (cf. [Morelli1,2]). We remark that this statement without the projectivity claim follows from our original argument on Page 535 of [AMR]. Indeed, one only needs to take a simplicial, π-nonsingular and collapsible cobordism Σ (in $(N_\Delta)^+_Q$ between Δ and Δ') which maps, by definition, to Δ. This can be obtained in the construction by requiring that the simplicial and collapsible cobordism (in N^+_Q between $\widehat{(\Delta^T_B)}$ and $\widehat{(\tilde{\Delta}^T_B)}$) should map to $\widehat{(\Delta^T_B)}$, a requirement which is easily satisfied by first taking the intersection of $\pi^{-1}(\widehat{(\Delta^T_B)})$ with any cobordism and then further star subdividing if necessary.

3. List of corrections for some other mistakes

∘ Example 4.1: We claimed that the presented cobordism Σ in [AMR] was π-nonsingular but it is actually π-singular. An example of a π-nonsingular and non-collabsible cobordism, communicated to us by Dan Abramovich, can be given as follows: We take vectors $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, -1) \in N_Q = \boldsymbol{Z}^2 \otimes \boldsymbol{Q}$ and let Δ be the fan spanned by these vectors and hence corresponding to $\boldsymbol{P}^2$ as a toric variety. If we take the simplicial fan Σ in $N^+_Q = N_Q \oplus \boldsymbol{Q}$ whose maximal cones consist of

$$\begin{aligned}\sigma_{11'2} &= \langle (v_1, 0), (v_1, 1), (v_2, 0) \rangle \\ \sigma_{22'3} &= \langle (v_2, 0), (v_2, 1), (v_3, 0) \rangle \\ \sigma_{33'1} &= \langle (v_3, 0), (v_3, 1), (v_1, 0) \rangle \\ \sigma_{1'22'} &= \langle (v_1, 1), (v_2, 0), (v_2, 1) \rangle \\ \sigma_{2'33'} &= \langle (v_2, 1), (v_3, 0), (v_3, 1) \rangle \\ \sigma_{3'11'} &= \langle (v_3, 1), (v_1, 0), (v_1, 1) \rangle ,\end{aligned}$$

then Σ is a simplicial π-nonsingular cobordism between Δ and Δ itself, but is not collapsible as we have a directed cycle of the three circuits

$$\sigma_{11'} = \langle (v_1, 0), (v_1, 1)\rangle \to \sigma_{22'} = \langle (v_2, 0), (v_2, 1)\rangle \to \sigma_{33'} = \langle (v_3, 0), (v_3, 1)\rangle \to \sigma_{11'} .$$

∘ Definition 5.3: We have to allow the equalities in the last line of the definition "... in the set $\{\rho_i; r_i \leq 0\}$ or in the set $\{\rho_i; r_i \geq 0\}$, exclusively."

∘ The definitions of N_Δ and $(N_\Delta)_{\boldsymbol{Q}}$ on Page 533 in [AMR] should be corrected as follows:

"Given a conical complex Δ, we consider the space $(N^S \cap \sigma^S) \oplus \boldsymbol{Z}$, for each $N^S = N_{\sigma S}$ associated to the cone $\sigma^S \in \Delta$, which can be glued together naturally via the gluing of $N^S \cap \sigma^S$ to form the integral structure N_Δ. By considering the space $(N^S \otimes \boldsymbol{Q} \cap \sigma^S) \oplus (\boldsymbol{Z} \otimes \boldsymbol{Q}) = (N^S \cap \sigma_S)_{\boldsymbol{Q}} \oplus \boldsymbol{Q}$ and gluing them together, we obtain the space

$$(N_\Delta)^+_{\boldsymbol{Q}} = (N_\Delta)_{\boldsymbol{Q}} \oplus \boldsymbol{Q} ."$$

We remark that a cobordism Σ in $(N_\Delta)^+_{\boldsymbol{Q}}$ between Δ' and Δ for a refinement map $f_\Delta : \Delta' \to \Delta$ of the conical complexes maps to Δ by definition.

REFERENCES

[AMR] D. ABRAMOVICH, K. MATSUKI AND S. RASHID, A note on the factorization theorem of toric birational maps after Morelli and its toroidal extension, Tohoku Math. J. 51 (1999), 489–537.

[AKMW] D. ABRAMOVICH, K. KARU, K. MATSUKI AND J. WŁODARCZYK, Torification and factorization of birational maps, math.AG/9904135.

[Morelli1] R. MORELLI, The birational geometry of toric varieties, J. Alg. Geom. 5 (1996), 751–782.

[Morelli2] R. MORELLI, Correction to "The birational geometry of toric varieties", homepage at the Univ. of Utah (1997), 767–770.