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THE BRAIDINGS OF MAPPING CLASS GROUPS AND LOOP SPACES

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Abstract. The disjoint union of mapping class groups forms a braided monoidal category. We give an explicit expression of braidings in terms of both their actions on the fundamental group of the surface and the standard Dehn twists. This braided monoidal category gives rise to a double loop space. We prove that the action of little 2-cube operad does not extend to the action of little 3-cube operad by showing that the Browder operation induced by 2-cube operad action is nontrivial. A rather simple expression of Reshetikhin-Turaev representation is given for the sixteenth root of unity in terms of matrices with entries of complex numbers. We show by matrix calculation that this representation is symmetric with respect to the braid structure.

1. Introduction. The braided monoidal category has been playing a key role in the quantum theory and its related topics for about a decade. In the homotopy theoretic point of view, a braided monoidal category gives rise to a double loop space. Precisely speaking, the gorup completion of the nerve of a braided monoidal category has the same homotopy type as a double loop space ([4]). In this paper we deal with the braid structure on the orientable surfaces with one boundary component. We give an explicit description of the braiding of the mapping class groups, and then investigate the loop space structure resulting from it.

Let $\Gamma_{g,1}$ be the mapping class group of the orientable surface of genus g with one boundary component. It was noticed by Miller ([10]) that there is an action of the little 2-cube operad on the disjoint union of $\Gamma_{g,1}$'s extending the product which is induced by a kind of connected sum of surfaces. Although he did not provide any explicit description of the action of the little 2-cube operad, from this he induced the theorem that the group completion of the disjoint union of $B\Gamma_{g,1}$'s has the homotopy type of a double loop space, because, in general, the action of the little n-cube operad gives rise to an n-fold loop space ([8]). On the other hand, since a braided monoidal category gives rise to a double loop space, we may conjecture that there should be a certain structure of a braided monoidal category on the disjoint union of the mapping class groups $\Gamma_{g,1}$'s.

In Section 2, we first show (Theorem 2.4) that the disjoint union of $\Gamma_{g,1}$'s is a braided monoidal category with the product induced by the connected sum. By the connected sum of the surfaces $S_{g,1}$ and $S_{h,1}$, we mean attaching a pair of pants to the surfaces along the boundary circles of the surfaces. Hence the group completion of $\coprod_{g\geq 0} B\Gamma_{g,1}$ has the homotopy type of a double loop space. We provide an explicit algebraic description of the braiding of $\coprod_{g\geq 0} \Gamma_{g,1}$,

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regarding $\Gamma_{g,1}$ as the subgroup of the automorphism group of the fundamental group of the surface $S_{g,1}$ that consists of the automorphisms fixing the fundamental relator.

Let $x_1, y_1, \dots, x_g, y_g$ be the generators of the free group $\pi_1 S_{g,1}$, which are represented by the loops of the surface $S_{g,1}$ that are parallel to the standard Dehn twists $a_1, b_1, \dots, a_g, b_g$, respectively, of Figure 1. We should first determine the orientations of the loops representing $x_1, y_1, \dots, x_g, y_g$, which are compatible with our description of the actions of the Dehn twists on $\{x_1, y_1, \dots, x_g, y_g\}$ and that of the fundamental relator. It is very important to note that the fundamental relator is given by $R = [y_1, x_1][y_2, x_2] \cdots [y_g, x_g]$ rather than $[x_1, y_1] \cdots [x_g, y_g]$. The formula (2.8) gives the explicit description of the $(1, 1)$-braiding of mapping class group of $S_{2,1}$ in term of the standard Dehn twists. There is an action of little 2-cube operad on $\coprod_{g \geq 0} B\Gamma_{g,1}$. We prove (Theorem 2.11) that this action does not extend to the action of 3-cube operad. We prove this by showing that the Browder operation applied to a class in $H_0(\Gamma_{1,1})$ is nontrivial. Here the explicit formula for the braiding plays a key role.

Turaev and Reshetikhin introduced an invariant of ribbon graphs which is derived from the theory of quantum groups and is a generalization of Jones polynomials. These invariants were extended to those of 3-manifolds and of mapping class groups (cf. [11], [12], [6]). The definitions are abstract and a little complicated, since they are defined through quantum groups. Wright ([14]) computed the Reshetikhin-Turaev invariant of mapping class groups explicitly in the case $r = 4$, that is, at the sixteenth root of unity. For each $h \in \Gamma_{g,0}$, we can find the corresponding (colored) ribbon graph, whose Reshetikhin-Turaev invariant turns out to be an automorphism of the 1-dimensional summand of $V^{k_1} \otimes V^{k_1 *} \otimes \cdots \otimes V^{k_g} \otimes V^{k_g *}$, which we denote by $V_{r,g}$. We get this ribbon graph using the Heegaard splitting and the surgery theory of 3-manifolds. Wright showed as a result of her calculation that the restriction of this invariant to the Torelli subgroup of $\Gamma_{g,0}$ is equal to the sum of the Birman-Craggs homeomorphisms. The dimension of $V_{4,g}$ equals $2^{g-1}(2^g + 1)$, so the Reshetikhin-Turaev invariant of $h \in \Gamma_{g,0}$, when $r = 4$, is a $2^{g-1}(2^g + 1) \times 2^{g-1}(2^g + 1)$ matrix with entries of complex numbers. Wright proved a very interesting lemma that there is a natural one-to-one correspondence between the basis vectors of $V_{4,g}$ and the $\mathbf{Z}/2$-quadratic forms of Arf invariant zero. In Section 3, we express the Reshetikhin-Turaev representation ρ_4 of mapping class groups in easier words than Wright's description. We describe ρ_4 at genus 2 in terms of 10×10 matrices. A calcultion of 10×10 matrices gives a proof that the Reshetikhin-Turaev representation is symmetric.

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2. Braided monoidal category and loop spaces. Let $S_{g,k}$ be a compact connected orientable surface of genus g with k boundary components. The mapping class group $\Gamma_{g,k}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms of $S_{g,k}$ fixing the boundary of $S_{g,k}$ that consists of k disjoint circles. We also have the following alternative definition $\Gamma_{g,k} = \pi_o \mathrm{Diff}^+(S_{g,k})$. A Dehn twist along a simple closed curve γ is an isotopy class of a homeomorphism h, supported in a tubular neighborhood N of γ, obtained as follows: We regard N as an annulus in the Euclidean plane with its usual orientation. Then h is the identity outside N, and inside N the concentric circles rotate counterclockwise while the rotation's angle increases from 0 to 2π going inwards. The composition of Dehn twists (or homeomorphisms) of $S_{g,k}$ will be written from left to right, that is, the mapping class group $\Gamma_{g,k}$ acts on $\pi_1 S_{g,k}$ on the *right*. We in this paper mainly deal with the case $k = 0$ and $k = 1$. Both $\Gamma_{g,0}$ and $\Gamma_{g,1}$ are generated by the Dehn twists $a_1, \dots, a_g, b_1, \dots, b_g, \omega_1, \dots, \omega_{g-1}$ of Figure 1 (cf. [13]). We call these Dehn twists the standard Dehn twists.

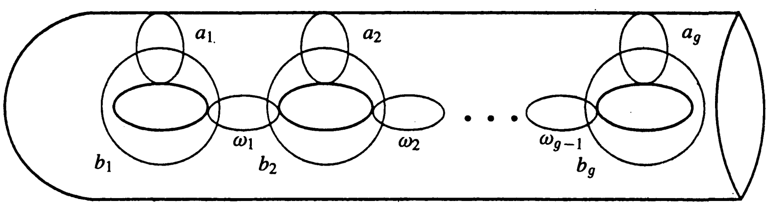

FIGURE 1. Dehn twists.

There exists an obvious surjection $\Gamma_{g,1} \to \Gamma_{g,0}$. Wajnryb ([13]) showed that both $\Gamma_{g,1}$ and $\Gamma_{g,0}$ are generated by $2g+1$ Dehn twists a_1, a_2, $b_1, \dots, b_g$, $\omega_1, \dots, \omega_{g-1}$. We now recall the definition of a strict braided monoidal category. We will deal with strict monoidal categories. This does not lead to a loss of generality because according to MacLane's coherence theorem, any (braided) monoidal category is equivalent to a certain strict (braided) monoidal category (cf. [12], Remark XI.1.4).

DEFINITION 2.1. A (*strict*) *monoidal category* $(\mathcal{C}, \otimes, E)$ is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called *tensor product*) and an object E (called the *unit object*) satisfying

(a) $\otimes$ is strictly associative, and
(b) E is a strict 2-sided unit for $\otimes$.

DEFINITION 2.2. A (strict) monoidal category $(\mathcal{C}, \otimes, E)$ is called a (*strict*) *braided monoidal category* if for each pair of objects (A, B) there exists a natural commutativity isomorphism $\beta_{A,B}$: $A \otimes B \to B \otimes A$ satisfying

(a) $\beta_{A,E} = \beta_{E,A} = 1_A$ for each object A, and
(b) the following braid relations hold:

$$\beta_{A\otimes B,C} = (\beta_{A,C} \otimes 1_B) \circ (1_A \otimes \beta_{B,C}) \text{ and } \beta_{A,B\otimes C} = (1_B \otimes \beta_{A,C}) \circ (\beta_{A,B} \otimes 1_C).$$

The isomorphism $\beta_{A,B}$ is called a braiding. The naturality of the braiding means that for any morphisms $f : A \to A'$, $g : B \to B'$, we have

$$(g \otimes f) \circ \beta_{A,B} = \beta_{A',B'} \circ (f \otimes g).$$

Note that the equalities in (b) of Definition 2.2 implies the following Yang-Baxter equality:

$$(1_C \otimes \beta_{A,B})(\beta_{A,C} \otimes 1_B)(1_A \otimes \beta_{B,C}) = (\beta_{B,C} \otimes 1_A)(1_B \otimes \beta_{A,C})(\beta_{A,B} \otimes 1_C).$$

In the homotopy theoretic point of view, a braided monoidal category has the following remarkable property in the coherence problem:

LEMMA 2.3 ([4]). *The group completion of the classifying space of a braided monoidal category is the homotopy type of a double loop space.*

This lemma implies that there is a certain connection between braided monoidal categories and the mapping class groups $\Gamma_{g,1}$, in view of Miller's proposition (Proposition 2.1, [10]) in which he claims that there is an action of the little square operad of disjoint squares in D^2 on the disjoint union of the $\mathrm{Diff}^+(S_{g,1})$'s, extending the F-product that is induced by the connected sum. Here let us describe the F-product more precisely. The F-product

$$\mathrm{Diff}^+(S_{g,1}) \times \mathrm{Diff}^+(S_{h,1}) \to \mathrm{Diff}^+(S_{g+h,1})$$

is obtained by attaching a pair of pants (a sphere with three boundary components) to the surfaces $S_{g,1}$ and $S_{h,1}$ along the fixed boundary circles and extending the identity map on the boundary to the whole pants.

From the standard result of May for the loop spaces ([8]), Miller concluded, in his proposition, that the group completion of $\coprod_{g\geq 0} B\,\mathrm{Diff}^+(S_{g,1})$ (the disjoint union of $B\,\mathrm{Diff}^+(S_{g,1})$'s) is a double loop space. He needed the proposition mainly for using a remarkable aspect of the homology of mapping class groups, which is the following: $\lim_{\to} H_*(B\,\mathrm{Diff}^+(S_{g,1}); \boldsymbol{Q})$ is a commutative, cocommutative, associative, coassociative Hopf algebra. His sketchy proof of the proposition, however, is a little obscure. We, in this paper, are going to take another route that leads us to a similar destination. Instead of finding an action of the little 2-cube operad we find a braid structure in the collection of the mapping class groups which is supposed to give us a double loop space structure. By the disjoint union of $\Gamma_{g,1}$'s we mean the category whose objects are $[g]$, $g \in \boldsymbol{Z}$, $g \geq 0$ and whose morphisms satisfy

$$\mathrm{Hom}([g],[h]) = \begin{cases} \Gamma_{g,1} & \text{if } g = h\,, \\ \emptyset & \text{if } g \neq h\,. \end{cases}$$

We denote this category by $\coprod_{g\geq 0} \Gamma_{g,1}$.

THEOREM 2.4. *The disjoint union of $\Gamma_{g,1}$'s is a braided monoidal category.*

PROOF. Let $x_1, y_1, \dots, x_g, y_g$ be the generators of the fundamental group of $S_{g,1}$ which are represented by the Dehn twists $a_1, b_1, \dots, a_g, b_g$, respectively (Figure 1). The mapping class group $\Gamma_{g,1}$ can be identified with the subgroup of the automorphism group of the free group on $x_1, y_1, \dots, x_g, y_g$ that consists of the automorphisms fixing the fundamental relator $R = [y_1, x_1][y_2, x_2] \cdots [y_g, x_g]$. The binary operation on $\coprod_{g\geq 0} \Gamma_{g,1}$ induced by the F-product can be identified with the operation taking the free product of the automorphisms. The (r, s)-braiding $\beta_{r,s}$ on the free group on $x_1, y_1, \dots, x_{r+s}, y_{r+s}$ is given by:

$$\begin{gathered} x_1 \mapsto Sx_{s+1}S^{-1}\,, \quad y_1 \mapsto Sy_{s+1}S^{-1}, \dots\,, \\ x_r \mapsto Sx_{s+r}S^{-1}\,, \quad y_r \mapsto Sy_{s+r}S^{-1}\,, \\ x_{r+1} \mapsto x_1\,, \quad y_{r+1} \mapsto y_1, \dots\,, \end{gathered}$$

$$x_{r+s} \mapsto x_s\,, \quad y_{r+s} \mapsto y_s\,,$$

where $S = [y_1, x_1][y_2, x_2] \cdots [y_s, x_s]$. Note that both the $(r, 0)$-braiding and the $(0, s)$-braiding are equal to the identity.

The (r, s)-braiding fixes the fundamental relator R:

$$\begin{aligned} R &= [y_1, x_1] \cdots [y_r, x_r][y_{r+1}, x_{r+1}] \cdots [y_{r+s}, x_{r+s}] \\ &\mapsto S[y_{s+1}, x_{s+1}] \cdots [y_{s+r}, x_{s+r}]S^{-1}[y_1, x_1] \cdots [y_s, x_s] = R\,. \end{aligned}$$

It is easy to see that the (r, s)-braiding satisfies the equalities in (b) of Definition 2.2, that is, we have

$$\beta_{r\otimes s,t} = (\beta_{r,t} \otimes 1_s)(1_r \otimes \beta_{s,t}) \quad \text{and} \quad \beta_{r,s\otimes t} = (1_s \otimes \beta_{r,t})(\beta_{r,s} \otimes 1_t)\,.$$

□

Note that in the proof of Theorem 2.4 we have chosen the fundamental relator R as $[y_1, x_1] \cdots [y_g, x_g]$ rather than $[x_1, y_1] \cdots [x_g, y_g]$.

Theorem 2.4 explains the pseudo double loop space structure on the union of the classifying spaces of the mapping class groups observed by Miller. From Lemma 2.3 and Theorem 2.4 we get the following:

THEOREM 2.5. *The group completion of $\coprod_{g\geq 0} B\Gamma_{g,1}$ has the homotopy type of a double loop space.*

We should take the group completion of the resulting topological monoid in order to have reasonable connected components. Here as the group completion of $\coprod_{g\geq 0} B\Gamma_{g,1}$ we may take $\Omega B(\coprod_{g\geq 0} B\Gamma_{g,1})$. See [9] for more details.

The Braid Structure. We now express the braid structure of the collection of mapping class groups more explicitly so that we could extract more applications from it and could handle the braided monoidal structure more easily. First let us express explicitly the $(1, 1)$-braiding on genus 2 surface. $\Gamma_{2,1}$ is generated by the Dehn twists a_1, b_1, a_2, b_2, ω_1. Let x_1, y_1, x_2, y_2 be generators of $\pi_1 S_{2,1}$ which are represented by the loops parallel to a_1, b_1, a_2, b_2, respectively. Here it is very important to determine the *orientations* of the loops which represent the generators x_1, y_1, x_2, y_2. It is given as in Figure 2.

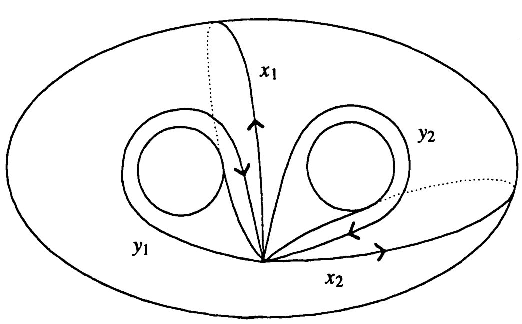

FIGURE 2.

Here we may think that the base point lies at the boundary of the surface. It is important to note that the fundamental relator, which is represented by the loop along the boundary, is $[y_1, x_1][y_2, x_2]$ rather than $[x_1, y_1][x_2, y_2]$. Let z_1 be the loop on $S_{g,0}$ which represents the Dehn twist ω_1. We now choose the orientation of z_1 as in Figure 3. Then z_1 is expressed in terms of the generators x_1, y_1, x_2, y_2 as in Figure 3.

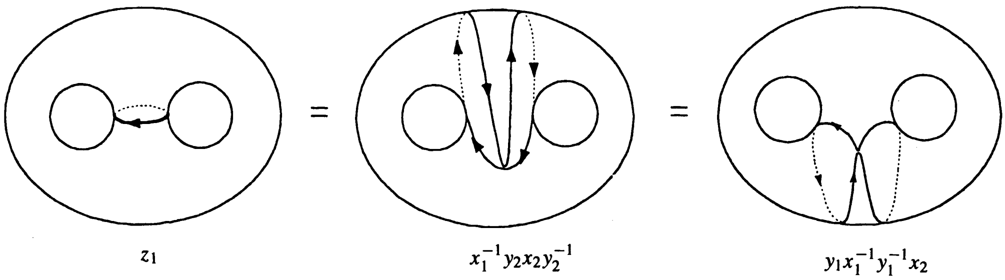

FIGURE 3.

In $\pi_1 S_{2,1}$, we have $z_1 = x_1^{-1}y_2x_2y_2^{-1}$. For an arbitrary genus g ($g \geq 1$), the orientations of the generators of $\pi_1 S_{g,1}$, which are represented by the loops parallel to the Dehn twists a_1, $b_1, \ldots, a_g, b_g$, are given in the same way as in Figure 2, and the fundamental relator should be $[y_1, x_1]\cdots[y_g, x_g]$ rather than $[x_1, y_1]\cdots[x_g, y_g]$ in our formation.

We regard the Dehn twists a_1, b_1, a_2, b_2, ω_1 as automorphisms of $\pi_1 S_{g,1}$, which is the free group on the generators $\{x_1, y_1, x_2, y_2\}$. Then the Dehn twists act as follows:

$$\begin{aligned}
&a_1 : y_1 \mapsto y_1x_1^{-1}, \quad b_1 : x_1 \mapsto x_1y_1, \\
&a_2 : y_2 \mapsto y_2x_2^{-1}, \quad b_2 : x_2 \mapsto x_2y_2, \\
&\omega_1 : x_1 \mapsto z_1^{-1}y_2x_2y_2^{-1}, \quad y_1 \mapsto y_1z_1, \quad y_2 \mapsto z_1^{-1}y_2,
\end{aligned}$$

where $z_1 = x_1^{-1}y_2x_2y_2^{-1}$. These automorphisms fix the generators that do not appear in the list.

We now find the explicit expression of the braiding, which is an element of $\Gamma_{2,1}$, and acts on $\pi_1 S_{g,1} = F_{\{x_1,y_1,x_2,y_2\}}$ as follows:

$$\begin{aligned}
&x_1 \mapsto [y_1, x_1]x_2[x_1, y_1], \quad y_1 \mapsto [y_1, x_1]y_2[x_1, y_1], \\
&x_2 \mapsto x_1, \quad y_2 \mapsto y_1.
\end{aligned} \tag{2.6}$$

The braiding is as follows:

LEMMA 2.7. *The* $(1, 1)$*-braiding* $\beta_{1,1}$ *for the monoidal structure in genus* 2 *is given by*

$$\beta_{1,1} = (a_1b_1a_1)^4(a_2b_2(a_1b_1a_1)^{-1}\omega_1a_1b_1a_1^2b_1)^{-3}. \tag{2.8}$$

We can easily confirm that the formula (2.8) satisfies (2.6). Here the Dehn twists act on the *right* on $\pi_1 S_{g,1}$.

The braid group of all braidings in the mapping class group of genus g is generated by

$$\beta_i = (a_i b_i a_i)^4 (a_{i+1} b_{i+1} (a_i b_i a_i)^{-1} \omega_i a_i b_i a_i^2 b_i)^{-3}$$

for $i = 1, 2, \dots, g-1$. We can obtain the following formula for the (r, s)-braiding in terms of the braiding generators:

$$\beta_{r,s} = (\beta_s \beta_{s-1} \cdots \beta_1)(\beta_{s+1} \beta_s \cdots \beta_2) \cdots (\beta_{r+s-1} \cdots \beta_r)$$

or alternatively as

$$(\beta_s \beta_{s+1} \cdots \beta_{r+s-1})(\beta_{s-1} \beta_s \cdots \beta_{r+s-2}) \cdots (\beta_1 \beta_2 \cdots \beta_r).$$

REMARK 2.9. It has been believed by some people that the (r, r)-braiding squared equals a Dehn twist around the fixed boundary of the surface $S_{2r,1}$ (cf. [7]). However, we can easily see that our braiding $\beta_{r,r}$ does not satisfy it. We can see, for example, that $\beta_{1,1}^2$ (or $\beta_{1,1}^{-2}$) acts on the fundamental group of $S_{2,1}$ in a different way from the Dehn twist around the boundary, which acts as follows:

$$x_1 \mapsto R^{-1} x_1 R, \quad y_1 \mapsto R^{-1} y_1 R,$$

$$x_2 \mapsto R^{-1} x_2 R, \quad y_2 \mapsto R^{-1} y_2 R,$$

where $R = [y_1, x_1][y_2, x_2]$.

REMARK 2.10. The braid structure gives rise to the double loop space structure, so it is supposed to be related to the Browder operation of the homology of mapping class groups. Let $D : B_{2g} \to \Gamma_{g,1}$ be the obvious map given by

$$D(\sigma_i) = \begin{cases} b_{\frac{i+1}{2}} & \text{if } i \text{ is odd}, \\ \omega_{\frac{i}{2}} & \text{if } i \text{ is even}. \end{cases}$$

Cohen in [3] dealt with this map D and the following commutative diagram:

$$\begin{array}{ccc} B_p \int B_{2g} & \xrightarrow{\theta'} & B_{2pg} \\ {\scriptstyle B_p \int D}\big\downarrow & & \big\downarrow{\scriptstyle D} \\ B_p \int \Gamma_{g,1} & \xrightarrow{\theta} & \Gamma_{pg,1} \end{array}$$

where θ is the group analogue of the Browder operation. The braiding structure (2.8) plays a key role in the explicit formula for $\theta : B_2 \int \Gamma_{1,1} \to \Gamma_{2,1}$, which should be given as follows:

$$(\sigma_1; 1, 1) \overset{\theta}{\longmapsto} (a_1 b_1 a_1)^4 (a_2 b_2 (a_1 b_1 a_a)^{-1} \omega_1 a_1 b_1 a_1^2 b_1)^{-3}.$$

The group completion of $\coprod_{g \geq 0} B\Gamma_{g,1}$ has the double loop space structure. There exists the Browder operation of homology induced by the action of little 2-cube operad. In the following theorem we prove that the double loop space structure of the group completion of $\coprod_{g \geq 0} B\Gamma_{g,1}$ does not support the action of little 3-cube operad on $\coprod_{g \geq 0} B\Gamma_{g,1}$. We get this by showing that the Browder operation induced by the little 2-cube operad action is nontrivial.

THEOREM 2.11. *Let $X = \coprod_{g\geq 0} B\Gamma_{g,1}$. The action of little 2-cube operad on X does not extend to the action of little 3-cube operad.*

PROOF. Consider the action map $\phi : \mathcal{C}_2(2) \times X^2 \to X$ of the little 2-cube operad $\mathcal{C}_2$. If the action extends to the action of little 3-cube operad $\mathcal{C}_3$, we have the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{C}_2(2) \times X^2 \arrow[d] \arrow[dr, "\phi"] & \\
\mathcal{C}_3(2) \times X^2 \arrow[r] & X
\end{tikzcd}
$$

Recall that $\mathcal{C}_n(2)$ has the same homotopy type as S^{n-1} (cf. [8]). By taking the homology we get:

$$
\begin{tikzcd}
H_*(S^1 \times X^2) \arrow[d] \arrow[dr, "\theta_*"] & \\
H_*(S^2 \times X^2) \arrow[r] & H_*(X)
\end{tikzcd}
$$

The map $\theta_* : H_i(X) \otimes H_j(X) \to H_{i+j+1}(X)$, which is induced by the action map ϕ, is called the Browder operation. From the commutativity of this diagram we have that θ_* should be trivial. Hence for the proof of the theorem it suffices to show that the Browder operation on the homology of X is nontrivial. By restricting the map ϕ to each connected component, we get the map

$$S^1 \times B\Gamma_{g,1} \times B\Gamma_{g,1} \to B\Gamma_{2g,1}\,.$$

This map is, in the group level, denoted by the map

$$\theta : B_2 \int \Gamma_{g,1} \to \Gamma_{2g,1}$$

as described in Remark 2.10. In order to show that θ_* is nonzero it suffices to show that

$$\tilde{\theta}_* : H_0(B\Gamma_{1,1}) \otimes H_0(B\Gamma_{1,1}) \to H_1(B\Gamma_{2,1})$$

is nonzero. The image of the map $\tilde{\theta}_*$ equals that of the homology homeomorphism $\alpha : H_1(S^1) \to H_1(B\Gamma_{2,1})$ induced by the map $S^1 \to B\Gamma_{2,1}$, which is the restriction of the map $S^1 \times B\Gamma_{1,1} \times B\Gamma_{1,1} \to B\Gamma_{2,1}$. The map α sends the generator of $H_1(S^1)$ to the abelianization class of

$$(a_1b_1a_1)^4(a_2b_2(a_1b_1a_1)^{-1}\omega_1a_1b_1a_1^2b_1)^{-3}\,,$$

which is nonzero. Note that the isomorphism $H_1(\ ) \cong (\ )_{ab}$ is natural. □

3. Reshetikhin-Turaev Representation. Reshetikhin and Turaev ([11], [12]) defined the invariants of ribbon groups, of 3-manifolds and of maping class groups shortly after Witten's 3-dimensional interpretation of the Jones polynomial of a link. Wright ([14]) described the Reshetikhin-Turaev invariant of mapping class groups more explicitly in the case $r = 4$.

For each $h \in \Gamma_{g,0}$ we need the corresponding ribbon graph, whose Reshetikhin-Turaev invariant turns out to be a $2^{g-1}(2^g+1) \times 2^{g-1}(2^g+1)$ matrix with entries of complex numbers. More precisely, the Reshetikhin-Turaev invariant is the projective representation $\rho_{r,g}$: $\Gamma_{g,0} \to \mathrm{End}(V_{r,g})$, where $V_{r,g}$ is the 1-dimensional summand of $V^{k_1} \otimes V^{k_1 *} \otimes \cdots \otimes V^{k_g} \otimes V^{k_g *}$ and $\dim(V_{4,g}) = 2^{g-1}(2^g+1)$. Let $N(g) = 2^{g-1}(2^g+1)$. By a $\mathbf{Z}/2$-quadratic form of Arf invariant zero we mean a $2 \times g$ matrix

$$\alpha = \begin{pmatrix} m_1 & m_2 & \cdots & m_g \\ l_1 & l_2 & \cdots & l_g \end{pmatrix}$$

in $M_{2\times g}(\mathbf{Z}/2)$ satisfying $\sum_{i=1}^{g} m_i l_i = 0$. It is easy to see that the number of $\mathbf{Z}/2$-quadratic forms of Arf invariant zero equals $N(g)$. We denote $m(\alpha) = \sum_{i=1}^{g} m_i$. Wright proved the following beautiful lemma.

LEMMA 3.1 ([14], §4.1, Lemma 3). *There is a natural one-to-one correspondence between basis vectors v_i of $V_{4,g}$ and $\mathbf{Z}/2$-quadratic forms α_i of Arf invariant zero.*

Reshetikhin-Turaev's definition of the quantum invariant is quite abstract and complicated, since it comes through a quantum group theory. Wright in [14] described $\rho_{4,g}$ in an explicit form. In the following theorem we express her description in easier words.

THEOREM 3.2. *Let g be a natural number. Let $\alpha_1, \ldots, \alpha_{2^{g-1}(2^g+1)}$ be $\mathbf{Z}/2$-quadratic forms of Arf invariant zero. Regard α_i's as the basis vectors of $V_{4,g}$. Let t be the sixteenth root of unity. Then the images under $\rho_4 = \rho_{4,g}$ of the generators a_k, b_k, ω_k of $\Gamma_{g,0}$ (see Figure* 1), *up to a scalar multiple, are as follows*:

(a) *$\rho_4(a_k)$ is a $N(g) \times N(g)$ matrix that* (i) *interchanges α_i and α_j if the k-th column of α_i equals $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the k-th column of α_j equals $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and all other columns are the same, and* (ii) *maps all other basis elements α_l to $\bar{t}^3 \alpha_l$.*

(b) *$\rho_4(b_k)$ is a $N(g) \times N(g)$ matrix that* (i) *interchanges α_i and α_j if the k-th column of α_i equals $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and that of α_j equals $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and all other columns are the same, and* (ii) *maps all other basis elements α_l to $\bar{t}^3 \alpha_l$.*

(c) *$\rho_4(\omega_k)$ is a $N(g) \times N(g)$ matrix that* (i) *interchanges α_i and $-\alpha_j$ if $\alpha_i^{k,k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\alpha_j^{k,k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,* (ii) *interchanges α_i and α_j if $m(\alpha_i) = m(\alpha_j) = 0$ and $l_s(\alpha_i) = l_s(\alpha_j) + 1$ for all s, and* (iii) *maps all other basis elements α_l to $\bar{t}^3 \alpha_l$.*

We now express the map ρ_4 in terms of 10×10 matrices in the case $g = 2$. Such an explicit expression would provide us with a good vision of the Reshetikhin-Turaev invariant of mapping class groups and its related topics.

EXAMPLE 3.3. Let $g = 2$. Then $N(2) = 10$. Let $\alpha_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\alpha_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\alpha_5 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $\alpha_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\alpha_7 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\alpha_8 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$\alpha_9 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\alpha_{10} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be the $\mathbf{Z}/2$-quadratic forms of Arf invariant zero which are regarded as basis vectors of $V_{4,2}$. Then we have $\rho_4(a_1) : \alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_5, \alpha_3 \leftrightarrow \alpha_6$. Thus $\rho_4(a_1)$ is the following matrix:

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{t}^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}_3
\end{pmatrix}$$

Similarly, we have

$$\begin{aligned}
&\rho_4(a_2) : \alpha_1 \leftrightarrow \alpha_2, \alpha_4 \leftrightarrow \alpha_5, \alpha_7 \leftrightarrow \alpha_8\,, \\
&\rho_4(b_1) : \alpha_1 \leftrightarrow \alpha_7, \alpha_2 \leftrightarrow \alpha_8, \alpha_3 \leftrightarrow \alpha_9\,, \\
&\rho_4(b_2) : \alpha_1 \leftrightarrow \alpha_3, \alpha_4 \leftrightarrow \alpha_6, \alpha_7 \leftrightarrow \alpha_9\,, \\
&\rho_4(\omega_1) : \alpha_9 \leftrightarrow -\alpha_{10}, \alpha_2 \leftrightarrow \alpha_4, \alpha_1 \leftrightarrow \alpha_5\,.
\end{aligned}$$

Let $M_{N(g)}(\mathbf{C})$ be the ring of $N(g) \times N(g)$ matrices of complex numbers. Theorem 3.2 describes the map $\rho_{4,g} : \Gamma_{g,1} \to M_{N(g)}(\mathbf{C})$ up to scalar for $g \geq 1$. Let $M(\mathbf{C})$ be the disjoint union of the monoids $M_{N(g)}(\mathbf{C})$ for $g \geq 1$. We may regard $M(\mathbf{C})$ as a category as usual, that is, objects are the integers $N(g)$, $g \geq 1$ and

$$\mathrm{Hom}(N(g), N(h)) = \begin{cases} M_{N(g)}(\mathbf{C}) & \text{if } g = h\,, \\ \emptyset & \text{if } g \neq h\,. \end{cases}$$

Let $\coprod_{g\geq 1} \Gamma_{g,1}$ denote the category which is the disjoint union of mapping class groups $\Gamma_{g,1}$ for $g \geq 1$.

THEOREM 3.4. *For $r = 4$, the Reshetikhin-Turaev representation is symmetric, that is, the obvious functor $\rho_4 : \coprod_{g\geq 1} \Gamma_{g,1} \to M(\mathbf{C})$ has the property that for each braiding $\beta_{r,s}$, $\rho_4(\beta_{r,s})^2$ is equivalent to the identity matrix up to scalar.*

PROOF. Let $\mathcal{B}_1$ be the image, under $\rho_{4,2} : \Gamma_{2,1} \to \mathrm{Aut}(V_{4,2})$, of

$$\beta_1 = (a_1 b_1 a_1)^4 (a_2 b_2 (a_1 b_1 a_1)^{-1} \omega_1 a_1 b_1 a_1^2 b_1)^{-3}\,.$$

Then we have

$$\mathcal{B}_1 = \begin{pmatrix} \bar{t}^{-6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{t}^{-6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}^{-6} & 0 & 0 & 0 \\ 0 & \bar{t}^{-6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{t}^{-6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}^{-6} & 0 & 0 \\ 0 & 0 & \bar{t}^{-6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{t}^{-6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}^{-6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{t}^{2} \end{pmatrix}$$

Hence $\mathcal{B}_1^2$ equals the identity matrix up to scalar. $\square$

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