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ON p-ADIC ZETA FUNCTIONS AND Z_p-EXTENSIONS OF CERTAIN TOTALLY REAL NUMBER FIELDS

Dedicated to the memory of Professor Kenkichi Iwasawa

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Abstract. In this article, we describe the order of the Galois-invariant part of the p-Sylow subgroup of the ideal class group in the cyclotomic Z_p-extension of a certain totally real number field k in terms of the residue at 1 of the p-adic zeta function of k, where p denotes an odd prime number. By using this, we obtain an alternative formulation of Greenberg's theorem on the vanishing of the cyclotomic Iwasawa λ- and μ-invariants of k for p. We also give some computational data for totally real cubic fields and $p=3$.

1. Introduction. Let k be a number field and p a prime number. For the cyclotomic Z_p-extension k_∞ of k, let k_n be the n-th layer of k_∞ over k and A_n the p-Sylow subgroup of the ideal class group of k_n. Then there exist integers λ, μ and ν, depending only on k and p, such that $\sharp A_n = p^{\lambda n + \mu p^n + \nu}$ for sufficiently large n (cf. [12]). Here, $\sharp G$ denotes the order of a finite group G. The integers $\lambda = \lambda_p(k)$, $\mu = \mu_p(k)$ and $\nu = \nu_p(k)$ are called the (cyclotomic) Iwasawa invariants of k for p. It is conjectured that, for any totally real number field k and any prime number p, both $\lambda_p(k)$ and $\mu_p(k)$ always vanish, that is, $\sharp A_n$ remains bounded as n tends to infinity (cf. [8], also [13, p. 316]). This is often called Greenberg's conjecture. It is known to be valid for $\mu_p(k)$ if k is an abelian number field (cf. [3]), but not yet for $\lambda_p(k)$ even if k has low degree except when $k = Q$.

Recently, several authors invesitgated Greenberg's conjecture in the case where p is an odd prime and k is a real abelian number field with degree prime to p (cf. [7], [10], [16], [17] and their references). For instance, after Greenberg's conjecture for many real quadratic fields was verified by various methods in several papers, Ichimura and Sumida showed in [9] and [10] that $\lambda_3(Q(\sqrt{m}))=0$ for all positive integers $m < 10{,}000$. Also, Kraft and Schoof determined in [16] the structure of the Iwasawa module associated to A_n in the cyclotomic Z_3-extension for certain real quadratic fields with small conductor. However, in general, it is too difficult to determine the structure or even the order of A_n in the cyclotomic Z_p-extension of totally real number fields.

Throughout this paper from now on we assume that k is totally real and p is an odd prime number. Denote by Γ the Galois group $\mathrm{Gal}(k_\infty/k)$ of k_∞ over k, and let A_n^Γ be the subgroup of A_n consisting of ideal classes which are invariant under the action

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of Γ, namely, A_n^Γ is the Γ-invariant part of A_n. In this paper, we give a formula for the order of A_n^Γ in terms of the p-adic zeta function of k. Let $\zeta_p(s, k)$ be the p-adic zeta function of k, which is continuous on $\boldsymbol{Z}_p - \{1\}$ and has simple pole at $s=1$ if Leopoldt's conjecture is valid for k and p (cf. [2]). Let us put

$$\zeta_p^*(s, k) = \frac{\zeta_p(s, k)}{\zeta_p(s, \boldsymbol{Q})}.$$

Note that if k is a real abelian number field, then $\zeta_p^*(s, k) = \prod_{\chi \neq 1} L_p(s, \chi)$, where the product is over all non-trivial p-adic Dirichlet characters χ of $\mathrm{Gal}(k/\boldsymbol{Q})$ and $L_p(s, \chi)$ is the p-adic L-function associated with χ. Let v_p be the p-adic valuation normalized by $v_p(p)=1$. Then the following is our main result, which was shown by a different method in [20, Proposition 1] for real quadratic fields.

THEOREM 1.1. *Let k be a totally real number field and p an odd prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. Then*

$$\sharp A_n^\Gamma = p^{v_p(\zeta_p^*(1,k))}$$

for every n sufficiently large. Furthermore, the right hand side of the above is given by

$$p^{v_p(\zeta_p^*(1,k))} = \sharp A_0 p^{v_p(R_p(k)) - [k:\boldsymbol{Q}] + 1},$$

where $R_p(k)$ denotes the p-adic regulator of k and $[k:\boldsymbol{Q}]$ the degree of k over $\boldsymbol{Q}$.

In the case where k is a real quadratic field in which p splits, Fukuda and Komatsu defined n_2 as the integer with the property that $\mathfrak{p}^{n_2} \| (\varepsilon^{p-1} - 1)$ in k, ε being the fundamental unit of k and $\mathfrak{p}$ a prime ideal of k lying above p, and showed that

$$\sharp A_n^\Gamma = \sharp A_0 p^{n_2 - 1}$$

for all integers $n \geq n_2 - 1$ (cf. [4, Proposition 1] or [5]). Theorem 1.1 can be regarded as a generalization of this. In deed, If k is a real quadratic field in which p splits, then we see that $n_2 = v_p(R_p(k))$ by Lemma 5.5 in [22]. Also, this theorem can be regarded as an explicit version of a formula for the order of A_n^Γ given by Inatomi [11, Proposition 2]. Further, by Theorem 1.1, we see that $v_p(R_p(k)) \geq [k:\boldsymbol{Q}] - 1$.

REMARK 1.2. The formula $\sharp A_n^\Gamma = \sharp A_0 p^{v_p(R_p(k)) - [k:\boldsymbol{Q}] + 1}$ which is obtained in Theorem 1.1 does not hold in general without the assumption on the decomposition of p in $k/\boldsymbol{Q}$. In fact, if only one prime ideal of k lies over p and if this prime is totally ramified in k_∞/k, then one can see that $\sharp A_n^\Gamma = \sharp A_0$ for all integers $n \geq 0$ (cf. [8, The proof of Theorem 1]). However, $v_p(R_p(k))$ is not always equal to $[k:\boldsymbol{Q}] - 1$. For example, $v_3(R_3(\boldsymbol{Q}(\sqrt{257}))) = 3$ and $v_3(R_3(\boldsymbol{Q}(\sqrt{326}))) = 2$ (cf. [19, Section 4]). Therefore, the formula above does not hold in these cases.

Let O_n be the ring of integers in k_n and O_n' the ring of p-integers in k_n, namely,

$O'_n = O_n[1/p]$. Then an invertible O'_n-submodule in k_n is called a p-ideal of k_n. Let A'_n be the p-Sylow subgroup of the p-ideal class group of k_n. Here, the p-ideal class group $C'_n = I'_n/P'_n$ of k_n is the multiplicative group I'_n of p-ideals of k_n modulo the subgroup P'_n of principal p-ideals of k_n. Then the surjective homomorphism from the multiplicative group I_n of ideals of k_n to I'_n defined by

$$\mathfrak{a} \mapsto \mathfrak{a}O'_n$$

induces a surjective homomorphism from A_n to A'_n. Now we put

$$D_n = \ker(A_n \to A'_n) .$$

Then D_n is the subgroup of A_n consisting of ideal classes represented by products of prime ideals of k_n lying above p. It is clear that $D_n \subset A_n^{\Gamma}$. Let L_∞ be the maximal unramified abelian pro-p-extension of k_∞, and let L_∞^* be the maximal unramified abelian pro-p-extension of k_∞ in which every prime of k_∞ lying above p splits completely. Using Theorem 1.1, we obtain the following alternative formulation of a theorem of Greenberg [8, Theorem 2] (see Theorem 3.1 in Section 3) on the vanishing of the Iwasawa invariants.

THEOREM 1.3. *Let k be a totally real number field and p an odd prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. Then the following six conditions are equivalent*:

(1) $\lambda_p(k) = \mu_p(k) = 0$,
(2) $\sharp D_n = p^{v_p(\zeta_p^*(1,k))}$ *for every n sufficiently large*,
(3) $\sharp D_n = p^{v_p(\zeta_p^*(1,k))}$ *for some $n \geq 0$*,
(4) $\sharp D_n = \sharp A_0 p^{v_p(R_p(k)) - [k:\boldsymbol{Q}] + 1}$ *for every n sufficiently large*,
(5) $\sharp D_n = \sharp A_0 p^{v_p(R_p(k)) - [k:\boldsymbol{Q}] + 1}$ *for some $n \geq 0$*,
(6) $\sharp \mathrm{Gal}(L_\infty/L_\infty^*) = p^{v_p(\zeta_p^*(1,k))}$.

Although Theorem 1.3 seems to be only a little different from a theorem of Greenberg [8, Theorem 2], Theorem 1.3 suggests that the validity of Greenberg's conjecture can be regarded as based on a deep arithmetic relation between an analytic object and an algebraic object. Moreover, using (5) in Theorem 1.3 for $n=0$, we obtain the following which is a partial generalization of a result of Fukuda and Komatsu [5, Theorem 1].

COROLLARY 1.4. *Under the same assumptions as in Theorem 1.3, if $v_p(R_p) = [k:\boldsymbol{Q}] - 1$ and if $A_0 = D_0$, then $\lambda_p(k) = \mu_p(k) = 0$. In particular, if $v_p(\zeta_p^*(1, k)) = 0$, then $\lambda_p(k) = \mu_p(k) = 0$.*

We will prove Theorem 1.1 in Section 2 and Theorem 1.3 in Section 3. Further, using these, we give in Section 4 some computational examples for totally real cubic fields and $p=3$.

Finally we mention that for real abelian number fields with degree prime to p, we can also give more precise results of our theorems in which each object is replaced by its Ψ-component, where Ψ denotes an irreducible $\boldsymbol{Q}_p$-character of $\mathrm{Gal}(k/\boldsymbol{Q})$ (cf. [21]).

2. Proof of Theorem 1.1. In this section, we prove the main theorem, i.e., Theorem 1.1. We use the same notation as in the preceding section. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. Let M be the maximal abelian pro-p-extension of k which is unramified outside the primes of k lying above p. Then the validity of Leopoldt's conjecture for k and p assures that M/k_∞ is of finite degree (cf. [1, Lemma 8 in Appendix]). First, we show the following lemma.

LEMMA 2.1. *Under the assumptions stated above*, $\#\mathrm{Gal}(M/k_\infty)=p^{v_p(\zeta_p^*(1,k))}$.

PROOF. Let $h(k)$ be the class number and $d(k)$ the absolute value of the discriminant, respectively, of k. We also denote by N the norm map from k to $\boldsymbol{Q}$. Then a result of Coates [1, Lemma 8 in Appendix] says that

$$v_p(\#\mathrm{Gal}(M/k_\infty))=v_p\left(\frac{w(k(\zeta_p))h(k)R_p(k)}{\sqrt{d(k)}}\prod_{\mathfrak{p}\mid p}(1-N(\mathfrak{p})^{-1})\right),$$

where ζ_p denotes a primitive p-th root of unity, $w(k(\zeta_p))$ the number of the roots of unity contained in $k(\zeta_p)$, and the product is over all prime ideals $\mathfrak{p}$ of k lying above p. Since p splits completely in k, it follows that

$$v_p(\#\mathrm{Gal}(M/k_\infty))=v_p(w(k(\zeta_p)))+v_p(h(k))+v_p(R_p(k))-[k:\boldsymbol{Q}]\,.$$

Further, in our case, since k is real and $k\cap\boldsymbol{Q}(\zeta_p)=\boldsymbol{Q}$, we see that $w(k(\zeta_p))=2p$. Hence,

$$v_p(\#\mathrm{Gal}(M/k_\infty))=v_p(h(k))+v_p(R_p(k))-[k:\boldsymbol{Q}]+1\,. \tag{1}$$

On the other hand, Colmez [2, Main theorem] proved that

$$\lim_{s\to1}(s-1)\zeta_p(s,k)=\frac{2^{[k:\boldsymbol{Q}]-1}h(k)R_p(k)}{\sqrt{d(k)}}\prod_{\mathfrak{p}\mid p}(1-N(\mathfrak{p})^{-1})\,,$$

where the product is over all prime ideals $\mathfrak{p}$ of k lying above p. Since p splits completely in k, it follows from the above limit formula that

$$\begin{aligned}\zeta_p^*(1,k)&=\lim_{s\to1}\frac{\zeta_p(s,k)}{\zeta_p(s,\boldsymbol{Q})}=\frac{\lim_{s\to1}(s-1)\zeta_p(s,k)}{\lim_{s\to1}(s-1)\zeta_p(s,\boldsymbol{Q})}\\&=\frac{2^{[k:\boldsymbol{Q}]-1}h(k)R_p(k)\prod_{\mathfrak{p}\mid p}(1-N(\mathfrak{p})^{-1})}{\sqrt{d(k)}(1-p^{-1})}\\&=\frac{2^{[k:\boldsymbol{Q}]-1}h(k)R_p(k)}{\sqrt{d(k)}}(1-p^{-1})^{[k:\boldsymbol{Q}]-1}\,.\end{aligned}$$

Hence, taking the p-adic valuation, we have

$$(2) \qquad v_p(\zeta_p^*(1, k)) = v_p(h(k)) + v_p(R_p(k)) - [k : \boldsymbol{Q}] + 1 .$$

Therefore, combining (1) with (2), we obtain $v_p(\#\mathrm{Gal}(M/k_\infty)) = v_p(\zeta_p^*(1,k))$. ∎

REMARK 2.2. It is easy to see that Lemma 2.1 also holds when the degree of k over $\boldsymbol{Q}$ is prime to p, regardless of the decomposition of p in k (cf. [21]).

Let L_n be the maximal unramified abelian p-extension of k_n and L_n' the maximal abelian extension of k contained in L_n. Then, by genus theory and the fact that k_∞/k is totally ramified at p, one can easily see that

$$\#A_n^\Gamma = [L_n' : k_n] = [L_n' k_\infty : k_\infty] .$$

for all integers $n \geq 0$. Note that if Leopoldt's conjecture is valid for k and p, then $\#A_n^\Gamma$ remains bounded as $n \to \infty$ (cf. [8, Proposition 1]). Next, we show the following lemma, the idea of whose proof was suggested by Manabu Ozaki. The author would like to thank him very much for this discussion.

LEMMA 2.3. *Assume that p splits completely in k. Then M is an unramified extension over k_∞. In particular, $M = k_\infty L_n'$ for every n sufficiently large.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of k lying above p, $T_\mathfrak{p}$ the inertia group of $\mathfrak{p}$ for the abelian extension M/k, and $k_\mathfrak{p}$ the completion of k at $\mathfrak{p}$. Then $k_\mathfrak{p} \simeq \boldsymbol{Q}_p$ by the assumption on the decomposition of p in $k/\boldsymbol{Q}$. Local class field theory (cf. [18, Theorem 3a]) says that the inertia group $I_\mathfrak{p}^{\mathrm{ab}}$ for the maximal abelian extension of $k_\mathfrak{p}$ is isomorphic to the unit group of $k_\mathfrak{p}$, and hence $I_\mathfrak{p}^{\mathrm{ab}} \simeq \boldsymbol{Z}_p^\times$. Since $T_\mathfrak{p}$ is a pro-p-group, it follows from this that $T_\mathfrak{p}$ is isomorphic to a quotient group of $\boldsymbol{Z}_p$. On the other hand, $\mathfrak{p}$ is totally ramified in the cyclotomic part k_∞/k of M/k, whence $T_\mathfrak{p} \simeq \boldsymbol{Z}_p$. This isomorphism implies that $T_\mathfrak{p} \cap \mathrm{Gal}(M/k_\infty)$ is trivial. Indeed, if $T_\mathfrak{p} \cap \mathrm{Gal}(M/k_\infty)$ were non-trivial, then the rank of $T_\mathfrak{p}$ over $\boldsymbol{Z}_p/p\boldsymbol{Z}_p$ would be at least two because M/k_∞ is the non-cyclotomic part of M/k. Therefore M/k_∞ is an unramified p-extension.

Finally, since a finite unramified extension of k_∞ is obtained by lifting an unramified extension of k_m, for some integer m, to k_∞ (cf. [12, Lemma 6.1]), it immediately follows from the definitions of M and L_n' that $M = k_\infty L_n'$ for sufficiently large n. ∎

Now, Lemmas 2.1 and 2.3, and the fact mentioned before Lemma 2.3 yield

$$\#A_n^\Gamma = \#\mathrm{Gal}(k_\infty L_n'/k_\infty) = \#\mathrm{Gal}(M/k_\infty) = p^{v_p(\zeta_p^*(1,k))}$$

for sufficiently large n. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.3. We continue with the same notation as in the previous sections. In this section, we prove Theorem 1.3. First, we recall the following theorem on the vanishing of the Iwasawa invariants of k with p splitting completely.

THEOREM 3.1 (cf. Theorem 2 in [8]). *Let k be a totally real number field and p a prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p. Then the following two conditions are equivalent*:

(1) $\lambda_p(k)=\mu_p(k)=0$,

(2) $\sharp A_n^{\Gamma}=\sharp D_n$ *for every n sufficiently large.*

Now we prove Theorem 1.3. By the theorem above and Theorem 1.1, it is clearly seen that (1), (2) and (4) in Theorem 1.3 are equivalent one another. Also, since k_∞/k is totally ramified at p, the norm map from D_m to D_n for $m\geq n\geq 0$ is surjective. Thus $\sharp D_n\leq\sharp D_m$ for $m\geq n\geq 0$. Hence, from Theorem 1.1 and the fact that $\sharp D_n\leq\sharp A_n^{\Gamma}$ for all integers $n\geq 0$, it follows that in Theorem 1.3, (2) (resp. (4)) is equivalent to (3) (resp. (5)). Therefore, it suffices to show only that (1) and (5) in Theorem 1.3 are quivalent.

Let L_n^* be the maximal extension of k_n contained in L_n, in which every prime of k_n lying above p splits completely. Then we have $L_\infty=\bigcup_{n\geq 0}L_n$, $L_\infty^*=\bigcup_{n\geq 0}L_n^*$ and

$$\mathrm{Gal}(L_\infty/L_\infty^*)=\mathrm{proj\,lim}\,\mathrm{Gal}(L_n/L_n^*)\,,$$

where the projective limit is taken with respect to the restriction maps. By class field theory, it is easy to see that

$$\mathrm{Gal}(L_\infty/L_\infty^*)\simeq\mathrm{proj\,lim}\,D_n\,,$$

where the projective limit is taken with respect to the norm maps. As already mentioned before, since A_n^{Γ} remains bounded as $n\to\infty$, so does D_n. Hence, it follows that there exists an integer n such that $D_m\simeq D_n$ with respect to the norm maps for all integers $m\geq n$. Therefore $\mathrm{Gal}(L_\infty/L_\infty^*)\simeq D_n$ for sufficiently large n. Consequently, Theorems 1.1 and 3.1 imply that (1) is equivalent to (5) in Theorem 1.3. This completes the proof of Theorem 1.3.

4. Some examples. In the simplest case where k is a real quadratic field with small discriminant and p is a small odd prime number which splits in k, the order of A_n^{Γ}, i.e., the value $v_p(\zeta_p^*(1, k))$, was already computed in [4], [5] and [7] by calculating the integer n_2 mentioned in Section 1. Also, we gave in [21] some computational data for $p=5$, 7 and cyclic cubic fields with small discriminant in which p splits completely.

In this section, for $p=3$ and totally real cubic fields k in which p splits completely, we calculate the order of A_n^{Γ} for sufficiently large n by using Theorem 1.1. We also give some examples of k with $\lambda_3(k)=\mu_3(k)=0$ by applying Theorem 1.3 or Theorem 3.1. Our computation has been carried out by means of the excellent number theoretic calculation packages "KASH 1.7" which is available by ftp at ftp://ftp.math.tu-berlin.de/pub/algebra/Kant/ and "UBASIC86 Ver.8.8" which is available at ftp://rkmath.rikkyo.ac.jp/. Note that most of the previous effective methods to verify Greenberg's conjecture have been developed in the case where p is an odd prime number and k is a real abelian number field such that the exponent of $\mathrm{Gal}(k/\boldsymbol{Q})$ divides $p-1$ (cf. [7],

[10], [16], [17] and their references), though Greenberg's conjecture for an odd prime number p and a real cyclic number field of degree p was studied barely in [6], [8] and [15].

EXAMPLE 4.1. Let k be the cyclic cubic field defined by $f(x)=x^3-x^2-30x-27$. Then the discriminant of k is 8281 (the conductor of k is $91=7\cdot 13$) and $p=3$ splits completely in k. Note that $\mu_3(k)=0$ by Ferrero and Washington [3] (or by Iwasawa [14]). Let θ be a root of $f(x)=0$ and θ' one of its conjugates. By using KASH 1.7, we see that a system of fundamental units of k is $\{1+\theta, (3+5\theta+\theta^2)/3\}$. Put $\varepsilon_1=1+\theta$ and $\varepsilon_2=(3+5\theta+\theta^2)/3$. Further, put $\varepsilon_1'=1+\theta'$ and $\varepsilon_2'=(3+5\theta'+\theta'^2)/3$, which are conjugates of ε_1 and ε_2 respectively. Since we may take the following values as θ and θ' (other pairs are possible and we obtain the same conclusion for any other pair):

$$\theta\equiv 5735755845 \pmod{3^{21}},$$
$$\theta'\equiv 10181147757 \pmod{3^{21}},$$

we obtain

$$\varepsilon_1\equiv 2248971445 \pmod{3^{20}},$$
$$\varepsilon_2\equiv 2980924492 \pmod{3^{20}},$$
$$\varepsilon_1'\equiv 3207578956 \pmod{3^{20}},$$
$$\varepsilon_2'\equiv 679402073 \pmod{3^{20}}.$$

Taking the 3-adic logarithms of these, we get

$$\log_3\varepsilon_1\equiv 10385055 \pmod{3^{16}},$$
$$\log_3\varepsilon_2\equiv 34739103 \pmod{3^{16}},$$
$$\log_3\varepsilon_1'\equiv 8307618 \pmod{3^{16}},$$
$$\log_3\varepsilon_2'\equiv 18692673 \pmod{3^{16}}.$$

Hence it follows that $R_3(k)\equiv 7534224 \pmod{3^{16}}$, so that

$$R_3(k)\equiv 3^2 \pmod{3^3}.$$

Thus, $v_3(R_3(k))=2$. Again, by using KASH 1.7, we see that $h(k)=3$, $A_0\simeq \boldsymbol{Z}/3\boldsymbol{Z}$ and the primes of k lying over $p=3$ are non-principal in k. Hence $\#A_0=\#D_0=3$. Therefore, by Theorem 1.1, $\#A_n^{\Gamma}=3$ for sufficiently large n. Since $\#D_n\le\#D_m\le\#A_m^{\Gamma}$ for $m>n\ge 0$, we have $\#D_n=3$ for sufficiently large n. Hence it follows from Theorem 1.3 or 3.1 that $\lambda_3(k)$ $(=\mu_3(k))=0$.

EXAMPLE 4.2. Let k be a totally real cubic field defined by $f(x)=x^3-7x-3$ which is unique up to isomorphism. Then the discriminant of k is 1129 and $p=3$ splits completely in k. (This k is the totally real cubic field with smallest discriminant where $p=3$ splits completely.) Let θ be a root of $f(x)=0$ and θ' one of its conjugates. By using KASH 1.7, we see that a system of fundamental units of k is $\{1+3\theta+\theta^2, 1+2\theta-\theta^2\}$ and $h(k)=1$, so $A_0=D_0=\{1\}$. Put $\varepsilon_1=1+3\theta+\theta^2$ and $\varepsilon_2=1+2\theta-\theta^2$. Further, put $\varepsilon_1'=1+3\theta'+\theta'^2$ and $\varepsilon_2'=1+2\theta'-\theta'^2$. We may take the following values as θ and θ':

$$\theta \equiv 6155444868 \pmod{3^{21}},$$

$$\theta' \equiv 3577471696 \pmod{3^{21}}.$$

After a computation similar to that in Example 4.1, we see that

$$R_3(k) \equiv 2 \cdot 3^2 \pmod{3^3},$$

whence $v_3(R_3(k))=2$. In particular, Leopoldt's conjecture is valid in this case. Now, by Theorem 1.1, we obtain $\sharp A_n^\Gamma=1$ for all integers $n\geq 0$, which implies that $\sharp D_n=1$ for all integers $n\geq 0$. Hence it follows from Theorem 1.3 or 3.1 that $\lambda_3(k)=\mu_3(k)=0$.

EXAMPLE 4.3. Let k be a totally real cubic field defined by $f(x)=x^3-40x-84$ which is unique up to isomorphism. Then the discriminant of k is $16372=2^2\cdot 4093$ and $p=3$ splits completely in k. Let θ be a root of $f(x)=0$ and θ' one of its conjugates. By using KASH 1.7, we see that a system of fundamental units of k is $\{31+16\theta+2\theta^2, 527+324\theta+45\theta^2\}$ and $h(k)=1$, so $A_0=D_0=\{1\}$. Put $\varepsilon_1=31+16\theta+2\theta^2$ and $\varepsilon_2=527+324\theta+45\theta^2$. Further, put $\varepsilon_1'=31+16\theta'+2\theta'^2$ and $\varepsilon_2'=527+324\theta'+45\theta'^2$. We may take the following values as θ and θ':

$$\theta \equiv 8256609024 \pmod{3^{21}},$$

$$\theta' \equiv 2636878198 \pmod{3^{21}}.$$

After a computation similar to that in Example 4.1,

$$R_3(k) \equiv 3^7 \pmod{3^8},$$

and hence $v_3(R_3(k))=7$. In particular, Leopoldt's conjecture is valid in this case. Now, by Theorem 1.1, we obtain $\sharp A_n^\Gamma=3^5=243$ for sufficiently large n. However, we cannot determine the order of D_n only by these data concerning the base field. Hence we do not know whether Greenberg's conjecture is valid or not in this case.

Finally, for $p=3$, we give some computational data of totally real cubic fields k with p splitting completely and with discriminant less than 100,000. In this calculation, we use the polynomials generating totally real cubic fields in a table made by Olivier, which is available at ftp://megrez.math.u-bordeaux.fr/pub/numberfields/. There exist exactly 347 such cubic fields up to isomorphism. Since we can see that $R_3(k)\neq 0$ for all of these, Leopoldt's conjecture is valid when $p=3$. We find that there exist exactly 226 cubic fields which satisfy $A_n^\Gamma=\{1\}$ (in these cases, $\lambda_3(k)=\mu_3(k)=0$). Table 1 gives some data for the 121 remaining cubic fields. In this table, $f(x)$ is a polynomial generating k, $\sharp A_\infty^\Gamma$ and $\sharp D_\infty$ are the order of A_n^Γ and D_n, respectively, for sufficiently large n, and the

TABLE 1. All k's with $d(k)<100{,}000$ satisfying $A_\infty^\Gamma \neq \{1\}$ and $p=3$ splits completely.

$d(k)$	$f(x)$	Gal	$h(k)$	$\#A_0$	$\#D_0$	$v_3(R_3(k))$	$\#A_\infty^\Gamma$	$\#D_\infty$
5329	$x^3-x^2-24x+27$	C3	1	1	1	3	3	*
6601	$x^3-13x-9$	S3	1	1	1	4	9	*
6901	$x^3-x^2-25x-2$	S3	1	1	1	3	3	*
7753	$x^3-19x-27$	S3	1	1	1	5	27	*
8281	$x^3-x^2-30x-27$	C3	3	3	3	2	3	3
13189	$x^3-22x-33$	S3	1	1	1	4	9	*
13537	$x^3-x^2-32x+33$	S3	1	1	1	3	3	*
13549	$x^3-x^2-31x+4$	S3	1	1	1	3	3	*
14197	$x^3-16x-9$	S3	2	1	1	3	3	*
14653	$x^3-25x-12$	S3	1	1	1	5	27	*
15412	$x^3-16x-6$	S3	1	1	1	5	27	*
15529	$x^3-19x-21$	S3	2	1	1	3	3	*
15700	$x^3-x^2-33x+27$	S3	1	1	1	3	3	*
16372	$x^3-40x-84$	S3	1	1	1	7	243	*
17581	$x^3-28x-51$	S3	1	1	1	6	81	*
17689	$x^3-x^2-44x-69$	C3	3	3	3	2	3	3
17929	$x^3-x^2-34x+7$	S3	1	1	1	4	9	*
19348	$x^3-x^2-35x+21$	S3	1	1	1	3	3	*
20353	$x^3-43x-105$	S3	1	1	1	3	3	*
22228	$x^3-x^2-39x-27$	S3	1	1	1	3	3	*
22996	$x^3-x^2-37x+19$	S3	1	1	1	3	3	*
25465	$x^3-37x-81$	S3	1	1	1	3	3	*
25645	$x^3-x^2-41x-30$	S3	1	1	1	3	3	*
27193	$x^3-19x-3$	S3	1	1	1	3	3	*
27925	$x^3-x^2-43x-38$	S3	1	1	1	3	3	*
28936	$x^3-x^2-42x+54$	S3	1	1	1	3	3	*
30904	$x^3-x^2-46x+82$	S3	1	1	1	3	3	*
31069	$x^3-x^2-41x+24$	S3	1	1	1	3	3	*
35101	$x^3-37x-48$	S3	1	1	1	4	9	*
35416	$x^3-34x-24$	S3	1	1	1	3	3	*
36469	$x^3-61x-168$	S3	1	1	1	3	3	*
36961	$x^3-x^2-44x+39$	S3	1	1	1	3	3	*
37300	$x^3-40x-90$	S3	3	3	3	5	81	*
38344	$x^3-37x-78$	S3	1	1	1	3	3	*
38917	$x^3-49x-108$	S3	1	1	1	3	3	*
39505	$x^3-x^2-46x+55$	S3	1	1	1	7	243	*
39700	$x^3-40x-60$	S3	1	1	1	3	3	*
40156	$x^3-x^2-52x+106$	S3	1	1	1	3	3	*
40180	$x^3-28x-42$	S3	3	3	3	2	3	3
41332	$x^3-x^2-53x+111$	S3	3	3	3	3	9	*
41944	$x^3-49x-126$	S3	3	3	3	4	27	*
42817	$x^3-25x-27$	S3	3	3	3	2	3	3
43444	$x^3-x^2-57x+135$	S3	1	1	1	3	3	*

TABLE 1. (continued)

$d(k)$	$f(x)$	Gal	$h(k)$	$\sharp A_0$	$\sharp D_0$	$v_3(R_3(k))$	$\sharp A_\infty^\Gamma$	$\sharp D_\infty$
44617	$x^3-x^2-48x-27$	S3	1	1	1	3	3	*
45541	$x^3-x^2-55x+118$	S3	1	1	1	4	9	*
47089	$x^3-x^2-72x+225$	C3	3	3	3	2	3	3
47860	$x^3-x^2-51x+81$	S3	3	3	3	2	3	3
48481	$x^3-x^2-50x+69$	S3	1	1	1	3	3	*
49681	$x^3-37x-12$	S3	1	1	1	3	3	*
49825	$x^3-x^2-48x+27$	S3	1	1	1	3	3	*
50104	$x^3-43x-66$	S3	7	1	1	5	27	*
50737	$x^3-37x-75$	S3	2	1	1	3	3	*
53176	$x^3-x^2-62x-114$	S3	1	1	1	3	3	*
53401	$x^3-43x-99$	S3	2	1	1	6	81	*
53752	$x^3-25x-18$	S3	1	1	1	3	3	*
54292	$x^3-x^2-51x-27$	S3	3	3	3	2	3	3
55672	$x^3-73x-78$	S3	1	1	1	4	9	*
56665	$x^3-x^2-50x+15$	S3	1	1	1	3	3	*
57985	$x^3-x^2-60x+135$	S3	1	1	1	4	9	*
59212	$x^3-55x-126$	S3	1	1	1	3	3	*
60037	$x^3-x^2-57x+108$	S3	1	1	1	3	3	*
60337	$x^3-x^2-58x-77$	S3	1	1	1	4	9	*
61009	$x^3-x^2-82x+64$	C3	3	3	3	2	3	3
61528	$x^3-25x-6$	S3	1	1	1	4	9	*
62041	$x^3-61x-177$	S3	1	1	1	3	3	*
62113	$x^3-49x-123$	S3	1	1	1	3	3	*
62572	$x^3-x^2-52x-2$	S3	1	1	1	4	9	*
63028	$x^3-40x-12$	S3	3	3	3	2	3	3
63508	$x^3-28x-30$	S3	1	1	1	4	9	*
64924	$x^3-67x-78$	S3	1	1	1	3	3	*
65908	$x^3-x^2-59x-75$	S3	3	3	3	2	3	3
67081	$x^3-x^2-86x-48$	C3	3	3	3	2	3	3
67384	$x^3-x^2-54x-18$	S3	1	1	1	3	3	*
67741	$x^3-x^2-57x-54$	S3	2	1	1	4	9	*
69061	$x^3-x^2-55x+64$	S3	1	1	1	3	3	*
69196	$x^3-43x-96$	S3	1	1	1	4	9	*
70021	$x^3-49x-84$	S3	3	3	3	2	3	3
70984	$x^3-x^2-66x+162$	S3	1	1	1	3	3	*
72169	$x^3-79x-174$	S3	1	1	1	4	9	*
72817	$x^3-x^2-62x-87$	S3	1	1	1	3	3	*
73177	$x^3-x^2-56x-27$	S3	3	3	3	2	3	3
73432	$x^3-43x-30$	S3	1	1	1	4	9	*
73441	$x^3-x^2-90x-261$	C3	1	1	1	4	9	*
73768	$x^3-x^2-58x-50$	S3	1	1	1	3	3	*
74089	$x^3-73x-216$	S3	1	1	1	3	3	*
74137	$x^3-x^2-72x-153$	S3	3	3	3	3	9	*

TABLE 1. (continued)

$d(k)$	$f(x)$	Gal	$h(k)$	$\#A_0$	$\#D_0$	$v_3(R_3(k))$	$\#A_\infty^\Gamma$	$\#D_\infty$
75085	$x^3-x^2-55x+10$	S3	1	1	1	4	9	*
75313	$x^3-x^2-58x+85$	S3	3	3	3	2	3	3
75724	$x^3-x^2-56x+54$	S3	1	1	1	6	81	*
75901	$x^3-28x-21$	S3	3	3	3	2	3	3
77320	$x^3-43x-18$	S3	1	1	1	4	9	*
78853	$x^3-40x-81$	S3	1	1	1	5	27	*
78973	$x^3-79x-162$	S3	1	1	1	4	9	*
79060	$x^3-28x-18$	S3	1	1	1	3	3	*
80101	$x^3-x^2-57x+54$	S3	1	1	1	3	3	*
80116	$x^3-x^2-79x-191$	S3	1	1	1	4	9	*
80692	$x^3-x^2-87x+297$	S3	1	1	1	3	3	*
81769	$x^3-x^2-64x-89$	S3	1	1	1	3	3	*
83077	$x^3-82x-69$	S3	1	1	1	4	9	*
84172	$x^3-31x-36$	S3	1	1	1	3	3	*
84616	$x^3-x^2-82x-206$	S3	1	1	1	3	3	*
85300	$x^3-x^2-63x+117$	S3	1	1	1	5	27	*
86485	$x^3-x^2-65x-90$	S3	3	3	3	2	3	3
86989	$x^3-34x-51$	S3	1	1	1	3	3	*
87013	$x^3-61x-144$	S3	2	1	1	4	9	*
87349	$x^3-x^2-81x+252$	S3	1	1	1	4	9	*
88084	$x^3-76x-228$	S3	3	3	3	3	9	*
90601	$x^3-x^2-100x+379$	C3	3	3	3	2	3	3
90988	$x^3-x^2-84x+270$	S3	1	1	1	3	3	*
91732	$x^3-40x-78$	S3	1	1	1	3	3	*
92185	$x^3-85x-75$	S3	1	1	1	3	3	*
92488	$x^3-x^2-74x+198$	S3	1	1	1	5	27	*
94168	$x^3-70x-192$	S3	1	1	1	4	9	*
94249	$x^3-x^2-102x+216$	C3	1	1	1	3	3	*
94345	$x^3-x^2-76x+211$	S3	1	1	1	3	3	*
94636	$x^3-x^2-80x-186$	S3	3	3	3	2	3	3
95992	$x^3-91x-234$	S3	6	3	3	2	3	3
96724	$x^3-x^2-71x-123$	S3	1	1	1	3	3	*
97645	$x^3-x^2-61x-20$	S3	1	1	1	3	3	*
98809	$x^3-x^2-62x+75$	S3	1	1	1	3	3	*
99208	$x^3-x^2-90x+306$	S3	1	1	1	3	3	*

others are the same as before. In the column labelled "Gal", S3 means that it is a non-Galois extension over $\boldsymbol{Q}$ (i.e., the Galois group of its Galois closure is the symmetric group of degree 3), and C3 means that it is a Galois extension over $\boldsymbol{Q}$ (i.e., it is a cyclic extension of degree 3). For 19 cubic fields in the table, we can determine the order of D_n and show that Greenberg's conj re holds, only by these data of the base field. On the other hand, the asterisks in the column labelled "$\#D_\infty$" mean that we cannot determine the order of D_n for $n \geq 1$. Therefore we do not know whether Greenberg's conjecture in these cases is valid or not merely from our calculation here. However, concerning μ-invariants, it follows from a theorem of Iwasawa [14, Theorem 3] that $\mu_3(k)=0$ for not only any Galois cubic field but also any non-Galois cubic field, because a non-Galois cubic field is a subfield of a Galois extension of degree 3 over a quadratic field.

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