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# THE STRONG RIGIDITY THEOREM FOR NON-ARCHIMEDEAN UNIFORMIZATION

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**Abstract.** In this paper, we present a purely algebraic proof of the strong rigidity for non-Archimedean uniformization, in case the base ring is of characteristic zero. In the last section, we apply this result to Mumford's construction of fake projective planes. In view of recent result on discrete groups by Cartwright, Mantero, Steger and Zappa, we see that there exist at least three fake projective planes.

**Introduction.** Let $K$ be a non-Archimedean local field and $R$ the ring of integers. The *Drinfeld upper half space* $\Omega_K^n$ is a $p$-adic analogue of the complex unit ball introduced by Drinfeld in [6]. As a set, $\Omega_K^n$ is the set of all geometric points of the projective space $\boldsymbol{P}_K^{n-1}$ which do not lie on any $K$-rational hyperplanes. Drinfeld proved that the space $\Omega_K^n$ has a natural structure as a rigid analytic space. It has a natural analytic action of $PGL(n, K)$, and considering the procedure of taking discrete quotients, one gets a good uniformization theory in $p$-adic analysis.

The space $\Omega_K^n$ has two essentially different ways of description. One of them is a rigid analytic subspace of $\boldsymbol{P}_K^{n-1}$. The other one is a formal scheme $\hat{\Omega}=\hat{\Omega}_K^n$ over the discrete valuation ring $R\subset K$. The second description was developed by Kurihara [13] and Mustafin [17] independently, and it is sometimes called *p-adic unit ball* of Kurihara and Mustafin.

In this paper, we will take up the viewpoint of the second one, because it is related rather directly with a visual combinatorial object called the *Bruhat-Tits building*. Interesting applications such as [16] were discovered through this viewpoint. Then the procedure of uniformization is presented as follows (cf. [17]): Let $\Gamma$ be a torsionfree co-compact subgroup of $PGL(n, K)$. Then one can take a quotient $\mathscr{X}_\Gamma=\hat{\Omega}/\Gamma$ in the category of formal schemes over Spf $R$. It is known that the resulting formal scheme $\mathscr{X}_\Gamma$ is algebraizable, i.e., the formal completion of a scheme $X_\Gamma$, which is proper and flat over Spec $R$. Taking the generic fiber, one obtains a nonsingular projective variety $X_{\Gamma,\eta}$ over Spec $K$ as the algebraization of the rigid analytic space $\Omega_K^n/\Gamma$.

Recall the following result of Mustafin on the rigidity of the uniformization by the Drinfeld upper half space.

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THEOREM 0.1 (cf. [17, §4]). *The schemes* $X_{\Gamma_1}$ *and* $X_{\Gamma_2}$ *are isomorphic over* Spec $R$ *if and only if* $\Gamma_1$ *and* $\Gamma_2$ *are conjugate in* $PGL(n, K)$.

We generalize this theorem as follows in case char $K=0$.

THEOREM 0.2. *Assume that the characteristic of* $K$ *is zero. Let* $\Gamma_1$ *and* $\Gamma_2$ *be torsionfree co-compact subgroups of* $PGL(n, K)$. *Then the schemes* $X_{\Gamma_1,\eta}\otimes_K\bar{K}$ *and* $X_{\Gamma_2,\eta}\otimes_K\bar{K}$ *are isomorphic over* Spec $\bar{K}$ *if and only if* $\Gamma_1$ *and* $\Gamma_2$ *are conjugate in* $PGL(n, K)$, *where* $\bar{K}$ *denotes an algebraic closure of* $K$.

REMARK 0.3. In the framework of rigid analytic geometry, Berkovich obtained an equivalent theorem without the assumption on the characteristic of $K$ (cf. [2, Theorem 2]). Since the descriptions of the Drinfeld space are not equal, Berkovich's proof is totally different from ours.

This theorem shows a strong rigidity property of the Drinfeld upper half space, and will have possible applications to $p$-adic analysis, number theory and even algebraic geometry.

We will prove the theorem in a purely algebraic way. In proving it, we will also clarify several interesting algebro-geometric aspects of the Drinfeld upper half space such as the behavior after base extensions, simultaneous crepant resolution of singularities, etc.

A nonsingular complex surface of general type with $p_g=q=0$ and $c_1^2=3c_2=9$ is called a *fake projective plane* (cf. [1, V, Rem. 1.2]). In [16], Mumford constructed a torsionfree co-compact subgroup $\Gamma$ of $PGL(3, \boldsymbol{Q}_2)$ such that $X_{\Gamma,\eta}$ is a fake projective plane for a fixed isomorphism $\overline{\boldsymbol{Q}_2}\simeq\boldsymbol{C}$. Recently, Cartwright, Mantero, Steger and Zappa (cf. [4], [5]) looked at certain discrete subgroups of $PGL$-groups rather systematically, and obtained a complete list of torsionfree co-compact subgroups of $PGL(3, \boldsymbol{Q}_2)$ of some kind. Combining this result with our main theorem, we see that there exist at least two more fake projective planes.

In §1, we give a brief summary of the construction of the formal scheme $\hat{\Omega}$ and the non-Archimedean uniformization basically according to Mustafin [17]. In §2 and §3, we observe the base changes $\hat{\Omega}_{R'}=\hat{\Omega}\otimes_R R'$ and $X_{\Gamma,R'}=X_\Gamma\otimes_R R'$, where $R'$ is the integer ring of a finite extension of $K$. The proof of Theorem 0.2 is given in §4. In the last section, we apply it to the existence of new fake projective planes.

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**1. Non-Archimedean uniformization in general.** In this section, we give a brief summary of the theory of non-Archimedean uniformization since the theory is not so popular. We introduce it basically according to Mustafin [17] but in the dual formulations, since we wish to formulate notation and materials as in Mumford [16] and Ishida [9].

Throughout this paper, we fix the following notation. Let $R$ be a complete discrete valuation ring. We fix a generator $\pi$ of the maximal ideal of $R$. Let $k=R/\pi R$ be the residue field of $R$ and $K$ the fractional field of $R$. We assume that the field $k$ is finite and consists of $q$ elements. We denote by $\eta$ and 0 the generic point and the closed point of $\operatorname{Spec} R$, respectively. Let $n \geq 2$ be a natural number. A matrix $\alpha=(a_{ij}) \in GL(n, K)$ defines a linear automorphism of the vector space $V=\sum_{i=0}^{n-1} KX_i$ with indeterminates $X_0, \ldots, X_{n-1}$ by

$$\alpha\left(\sum_{i=0}^{n-1} c_i X_i\right)=(X_0, \ldots, X_{n-1})\alpha\,{}^t(c_0, \ldots, c_{n-1})$$

$$=\sum_i\left(\sum_j a_{ij} c_j\right)X_i .$$

Hence the induced automorphism $\alpha^\wedge$ of $\boldsymbol{P}(V)=\operatorname{Proj} K[X_0, \ldots, X_{n-1}]$ is given in terms of the homogeneous coordinates $(X_0 : \cdots : X_{n-1})$ by

$$\alpha^\wedge(X_0 : \cdots : X_{n-1})=(X_0 : \cdots : X_{n-1})\alpha .$$

Thus the composite $\beta^\wedge \circ \alpha^\wedge$ is equal to $(\alpha\beta)^\wedge$.

1.1. Let $\tilde{\Delta}_0$ be the set of all free $R$-submodules in $V=\sum_{i=0}^{n-1} KX_i$ of rank $n$. We define an equivalence relation $\sim$ on $\tilde{\Delta}_0$ by

$$M_1 \sim M_2 \Leftrightarrow M_1=\lambda M_2 \qquad \text{for some} \quad \lambda \in K^\times$$

$$\Leftrightarrow M_1=\pi^d M_2 \qquad \text{for some} \quad d \in \boldsymbol{Z} .$$

Define $\Delta_0=\tilde{\Delta}_0/\sim$. We write the equivalence class of $M \in \tilde{\Delta}_0$ by $[M]$. The group $PGL(n, K)$ acts transitively on the set $\Delta_0$ by $\alpha[M]=[\alpha M]$ for $\alpha \in PGL(n, K)$ and $[M] \in \Delta_0$. Let $\Lambda_1, \Lambda_2 \in \Delta_0$. Take a representative $M_1 \in \Lambda_1$. Then, there exists a unique $M_2 \in \Lambda_2$ such that

$$M_1 \supseteq M_2 \not\subseteq \pi M_1 .$$

Similarly, there exists a unique $\pi^e M_1 \in \Lambda_1$ such that

$$M_2 \supseteq \pi^e M_1 \not\subseteq \pi M_2 .$$

It is easy to see that the nonnegative integer $e$ does not depend on the choice of $M_1$. Define $d(\Lambda_1, \Lambda_2)=e$. Then $d: \Delta_0 \times \Delta_0 \to \boldsymbol{Z}_{\geq 0}$ is a metric function on $\Delta_0$. If $d(\Lambda_1, \Lambda_2)=1$, i.e., if $M_1 \supsetneq M_2 \supsetneq \pi M_1$, then $\Lambda_1$ and $\Lambda_2$ are said to be *adjacent*.

Definition 1.2. The *Bruhat-Tits building* attached to $PGL(n, K)$, denoted by $\Delta=\Delta_K^n$, is a simplicial complex defined as follows:

1. The set of vertices of $\Delta$ is $\Delta_0$.
2. A subset $\{\Lambda_0, \ldots, \Lambda_l\} \subset \Delta_0$ forms an $l$-simplex if and only if $\Lambda_i$ and $\Lambda_j$ are

adjacent for any $i$, $j$ with $i \neq j$.

1.3. It is easy to see that a subset $\{\Lambda_0, \ldots, \Lambda_l\} \subset \Delta_0$ forms an $l$-simplex if and only if, changing indices if necessary, there exist representatives $M_i \in \Lambda_i$ for $i=0, \ldots, l$ such that

$$M_0 \supset M_1 \supset \cdots \supset M_l \supset \pi M_0$$

(cf. [17, Lemma 1.1]). Hence, there exists a one-to-one correspondence between the set of $l$-simplices having a fixed vertex $\Lambda_0 = [M_0]$ and the set of flags of length $l$ in the vector space $M_0/\pi M_0$ by considering quotients $M_i/\pi M_0$. In particular, $l$ is at most $n-1$. Moreover, we have the following basic properties of the Bruhat-Tits building $\Delta$ attached to $PGL(n, K)$ (cf. [3] and [17]):

1. $\Delta$ is an $(n-1)$-dimensional locally finite simplicial complex.
2. $\Delta$ is a *chamber complex*, i.e., any simplex is a face of some chamber, where we mean by a *chamber* a simplex of dimension $n-1$.
3. $\Delta$ is *labelable* by $\boldsymbol{Z}/n\boldsymbol{Z}$, i.e., there exists a map $\tau: \Delta_0 \to \boldsymbol{Z}/n\boldsymbol{Z}$, called a *labeling*, in such a way that the vertices of each chamber are mapped bijectively onto $\boldsymbol{Z}/n\boldsymbol{Z}$ ($\tau(\Lambda)$ is called the *type* of the vertex $\Lambda$ with respect to the labeling $\tau$).
4. The group $PGL(n, K)$ acts on $\Delta$ by $\alpha\{\Lambda_0, \ldots, \Lambda_l\} = \{\alpha\Lambda_0, \ldots, \alpha\Lambda_l\}$.

A subgroup $\Gamma \subset PGL(n, K)$ is discrete, if and only if the stabilizer in $\Gamma$ of each $\Lambda \in \Delta$ is a finite subgroup. $\Gamma$ is said to be *co-compact*, if it is discrete and $\Delta$ has only finitely many $\Gamma$-orbits. It is known that a co-compact subgroup $\Gamma$ acts on $\Delta$ freely, if and only if $\Gamma$ is torsionfree.

EXAMPLE 1.4. In case $n=2$, the Bruhat-Tits building $\Delta$ attached to $PGL(2, K)$ is a *tree* such that each vertex is an end of $q+1$ edges (cf. [14]).

EXAMPLE 1.5. We consider the case $n=3$. Let $\Lambda$ be a vertex of $\Delta$. Then $\Lambda$ is contained in $2(q^2+q+1)$ edges of $\Delta$, and this set of edges has a natural one-to-one correspondence with the set of $k$-rational points and $k$-rational lines of $\boldsymbol{P}_k^2$. Let $B^2$ be the algebraic surface obtained by blowing up $\boldsymbol{P}_k^2$ along all $k$-rational points. Then the dual graph of the configuration of exceptional curves and proper transforms of the lines is a one-dimensional simplicial complex which is isomorphic to the link of $\Lambda$ in $\Delta$. The dual graph of case $q=2$ is in Figure 1 (cf. [16, §1]), where $\tilde{p}_i$ $(i=0, \ldots, 6)$ are the exceptional curves and $\tilde{l}_i$ $(i=0, \ldots, 6)$ are the proper transforms of the lines.

1.6. To each $\Lambda = [M] \in \Delta_0$, we associate the scheme $\boldsymbol{P}(\Lambda) = \mathrm{Proj}(\mathrm{Sym}_R M)$ over $\mathrm{Spec}\, R$. This definition is independent of the choice of $M$. The generic fiber of $\boldsymbol{P}(\Lambda)$ is equal to the projective space $\boldsymbol{P}(V)$ for every $\Lambda$. Hence all these integral $R$-schemes are canonically birational.

A subset $S$ of $\Delta_0$ is said to be *convex* if $M_1, M_2 \in \tilde{\Delta}_0$ and $[M_1], [M_2] \in S$ imply $[M_1 + M_2] \in S$. Then we denote by $\Delta(S)$ the subcomplex of $\Delta$ consisting of all simplices in $\Delta$ whose vertices are in $S$. For a subset $T$ of $\Delta_0$, the *convex hull* of $T$ is the smallest convex set which contains $T$. It is equal to the intersection of all convex sets which contain $T$.

Let $S \subset \Delta$ be a nonempty convex subset. In [17] the $R$-scheme $\boldsymbol{P}(\Delta(S))$ is constructed as the limit of "joins" of $\boldsymbol{P}(\Lambda)$ for $\Lambda \in S$. $\boldsymbol{P}(\Delta(S))$ is an integral scheme locally of finite type over $R$ with the generic fiber $\boldsymbol{P}(V)$.

The following characterization of $\boldsymbol{P}(\Delta(S))$ is convenient. Let $D_0$ be the rational function field $K(X_1/X_0, \ldots, X_{n-1}/X_0)$, i.e., the rational function field of $\boldsymbol{P}(V)$. Then a local ring $(A, P)$ in $D_0$ with $\pi \in P$ has a center in $\boldsymbol{P}(\Delta(S))$ if and only if there exists a simplex in $\Delta(S)$ represented by $\{M_0 \supset M_1 \supset \cdots \supset M_l \supset \pi M_0\}$ satisfying the following condition: For each $i=0, \ldots, l$, there exists nonzero $x_i \in M_i$ such that $x_i^{-1} M_i \subset A$, and $M_{i+1}$ is the largest among $R$-submodules $M \subset M_i$ with $[M] \in S$ and $x_i^{-1} M \subset P$, where we set $M_{l+1} := \pi M_0$.

The closed fiber $\boldsymbol{P}(\Delta(S))_0$ is a reduced normal crossing divisor with the dual graph isomorphic to $\Delta(S)$. The formal scheme $\hat{\Omega}(\Delta(S))$ is defined as a formal completion of an integral $R$-scheme $\boldsymbol{P}(\Delta(S))$ along the closed fiber. $\hat{\Omega} = \hat{\Omega}(\Delta)$ is the Drinfeld upper half space defined as a formal scheme.

1.7. Let $Y_0, \ldots, Y_{n-1}$ be a basis of $V$. Then the set of vertices $S = \{[\sum_{i=0}^{n-1} R\pi^{\alpha_i} Y_i] \mid \alpha_i \in \boldsymbol{Z}\}$ is convex in $\Delta_0$. The subcomplex $A(Y_0, \ldots, Y_{n-1}) := \Delta(S)$ is isomorphic to the triangulation of $\boldsymbol{R}^{n-1}$ by the Weyl chambers of type $\mathrm{A}_{n-1}$. This subcomplex is called an *apartment*.

1.8. Each irreducible component of the closed fiber of $\boldsymbol{P}(\Delta)$ is isomorphic to the $(n-1)$-dimensional smooth $k$-scheme $B^{n-1}$ which is defined as follows (cf. Example 1.5): For each integer $0 \le i \le n-2$, let $\Sigma_i$ be the set of $i$-dimensional $k$-rational linear subspaces of $\boldsymbol{P}_k^{n-1}$. Set $P_0 := \boldsymbol{P}_k^{n-1}$. $P_1$ is defined to be the blow-up of $P_0$ at all the points belonging to $\Sigma_0$. For $1 < i \le n-2$, $P_i$ is defined to be the blow-up of $P_{i-1}$ at the union of proper transforms of the elements of $\Sigma_{i-1}$. Then we set $B^{n-1} = P_{n-2}$.

1.9. Let $S$ be a nonempty convex subset of $\Delta_0$. If a subgroup $\Gamma \subset PGL(n, K)$ stabilizes $S$, then $\Gamma$ acts on $\boldsymbol{P}(\Delta(S))$ as well as on $\hat{\Omega}(\Delta(S))$. Furthermore, the action of $\Gamma$ on $\hat{\Omega}(\Delta(S))$ is free if and only if that on $\Delta(S)$ is free.

Assume that the action is free. Then we can take the quotient $\hat{\Omega}(\Delta(S))/\Gamma$ of the formal scheme, but the base becomes an algebraic space in general. In case $S = \Delta_0$, we denote the quotient $\hat{\Omega}(\Delta)/\Gamma$ by $\mathscr{X}_\Gamma$. In this case, the base of $\mathscr{X}_\Gamma$ is a scheme since the relative canonical sheaf is ample on each irreducible component.

As for the algebraizability of $\mathscr{X}_\Gamma$. Kurihara and Mustafin showed the following.

THEOREM 1.10 (cf. [13, §2], [17, Thm. 4.1]). *Let $\Gamma$ be a torsionfree co-compact subgroup of $PGL(n, K)$. Then, the formal scheme $\mathscr{X}_\Gamma$ is algebraizable, i.e., $\mathscr{X}_\Gamma$ is the completion of a projective scheme $X_\Gamma$ over* Spec $R$ *along its closed fiber. Moreover, the algebraization $X_\Gamma$ has the following properties*:

1. *The closed fiber $X_{\Gamma,0}$ is a reduced algebraic $k$-scheme with only normal crossing singularities. The normalization of each irreducible component is isomorphic*

*to* $B^{n-1}$. *The dual graph of* $X_{\Gamma,0}$ *is isomorphic to* $\Delta/\Gamma$.

2. *The relative canonical sheaf* $K_{X_\Gamma/R}$ *of* $X_\Gamma$ *over* $\operatorname{Spec} R$ *is invertible and relatively ample. In particular, the canonical invertible sheaf of the generic fiber* $X_{\Gamma,\eta}$ *is ample.*

In case $n=3$, Mumford proved that if $q=2$ and $\Gamma$ acts transitively on $\Delta$, then $X_{\Gamma,\eta}$ is a fake projective plane (cf. [16, §1]).

**2. Base change and desingularization.** We use the following notation in the following three sections. $R$ is a complete discrete valuation ring with the quotient field $K$ and the residue field $k$ as in Section 1. Let $K'$ be a finite extension of $K$, and $R'$ the integral closure of $R$ in $K'$. The residue field of $R'$ is denoted by $k'$. We assume that the fields $k$ and $k'$ are finite. We have $[K':K]:=ef$, where $e$ is the ramification index and $f:=[k':k]$.

$R$ is excellent since it is complete. Hence, $R'$ is also a complete discrete valuation ring, and is a free $R$-module of rank $ef$.

Let $n\geq 2$ be an integer. We denote the Bruhat-Tits buildings attached to $PGL(n, K)$ and $PGL(n, K')$ by $\Delta$ and $\Delta'$, respectively. The sets of their vertices are denoted by $\Delta_0$ and $\Delta'_0$. We regard $\Delta_0$ as a subset of $\Delta'_0$ by the correspondence $[M]\mapsto[M\otimes_R R']$. For a convex subset $S$ of $\Delta'_0$, we denote by $\Delta'(S)$ the subcomplex of $\Delta'$ generated by $S$.

For each simplex $\sigma\in\Delta$ of dimension $n-1$, let $\sigma_0^\dagger$ be the convex hull of the set of vertices of $\sigma$ in $\Delta'_0$, and $\sigma^\dagger$ the subcomplex $\Delta'(\sigma_0^\dagger)$ of $\Delta'$. $\sigma^\dagger$ is a subdivision of $\sigma$ by Weyl chambers of type $\mathrm{A}_{n-1}$. Let $\Delta_0^\dagger$ be the convex hull of $\Delta_0$ in $\Delta'_0$, and $\Delta^\dagger$ the subcomplex $\Delta'(\Delta_0^\dagger)$.

LEMMA 2.1. *The simplicial complex* $\Delta^\dagger$ *is equal to the union of* $\sigma^\dagger$ *for* $\sigma\in\Delta_{n-1}$. *In particular,* $\Delta^\dagger$ *is a subdivision of* $\Delta$.

PROOF. The inclusion $\bigcup_\sigma\sigma^\dagger\subset\Delta^\dagger$ is clear. It is known that any two simplices $\sigma_1$, $\sigma_2$ in $\Delta_{n-1}$ are contained in a common apartment of $\Delta$ (cf. [17, Lem. 1.2]). Hence if $[N_1]\in(\sigma_1)_0^\dagger$ and $[N_2]\in(\sigma_2)_0^\dagger$, then $[N_1+N_2]$ is in $(\sigma_3)_0^\dagger$ for a simplex $\sigma_3$ of the apartment. Hence $\Delta_0^\dagger$ is the union of $\sigma_0^\dagger$ for $\sigma\in\Delta_{n-1}$. Let $\tau$ be a simplex of $\Delta^\dagger$ of dimension $d\geq 1$. We prove that $\tau$ is in $\sigma^\dagger$ for some $\sigma$ by induction on $d$. Let $\Lambda$ be a vertex of $\tau$, and $\tau'$ the complementary $(d-1)$-dimensional face. Then there exist $\sigma_1$, $\sigma_2$ in $\Delta_{n-1}$ with $\Lambda\in(\sigma_1)_0^\dagger$ and $\tau'\in(\sigma_2)^\dagger$. $\sigma_1$ and $\sigma_2$ are contained in a common apartment $A$ of $\Delta$. Since $\bigcup_{\sigma\in A}\sigma^\dagger$ is an apartment of $\Delta'$, $\tau$ is a simplex of this apartment. Hence $\tau$ is contained in $\sigma^\dagger$ for a simplex $\sigma$ of the apartment $A$. q.e.d.

Let $\Gamma\subset PGL(n, K)$ be a torsionfree co-compact subgroup. Set $\hat{\Omega}=\hat{\Omega}(\Delta)$ and let $\hat{\Omega}^\dagger$ be the formal scheme $\hat{\Omega}(\Delta^\dagger)$ over $\operatorname{Spf} R'$ (cf. 1.6). The quotient formal schemes of $\hat{\Omega}$ and $\hat{\Omega}^\dagger$ with respect to $\Gamma$ are denoted by $\mathscr{X}_\Gamma$ and $\mathscr{X}'_\Gamma$, respectively. By Theorem 1.10, the formal scheme $\mathscr{X}_\Gamma$ is algebraized to a regular scheme $X_\Gamma$ over $\operatorname{Spec} R$. Set $\hat{\Omega}_{R'}=\hat{\Omega}\otimes_R R'$, $\mathscr{X}_{\Gamma,R'}=\mathscr{X}_\Gamma\otimes_R R'$ and $X_{\Gamma,R'}=X_\Gamma\otimes_R R'$. Here note that $R'$ is a flat finite $R$-algebra.

The goal of this section is to prove the following propositions:

PROPOSITION 2.2. *The formal scheme $\mathscr{X}'_\Gamma$ is algebraized to a regular scheme* (*which we denote by $X'_\Gamma$*).

PROPOSITION 2.3. *There exists a natural $\Gamma$-equivariant morphism $\tilde{\rho} : \hat{\Omega}^\dagger \to \hat{\Omega}_{R'}$ which makes the following diagram commute*:

$$\begin{array}{ccc} \hat{\Omega}^\dagger & \xrightarrow{\tilde{\rho}} & \hat{\Omega}_{R'} \\ \downarrow & & \downarrow \\ \operatorname{Spf} R' & = & \operatorname{Spf} R' . \end{array} \tag{1}$$

*The morphism $\tilde{\rho}$ descends to a resolution of singularities $\rho : X'_\Gamma \to X_{\Gamma,R'}$ which is isomorphic on the generic fiber. Moreover, the morphism $\rho$ has no discrepancy, i.e., $\omega_{X'_\Gamma/R'} = \rho^* \omega_{X_{\Gamma,R'}/R'}$, where $\omega_{X'_\Gamma/R'}$ and $\omega_{X_{\Gamma,R'}/R'}$ are the relative dualizing sheaves of the $R'$-schemes $X'_\Gamma$ and $X_{\Gamma,R'}$, respectively.*

In fact, if $e > 1$ then $X_{\Gamma,R'}$ is no longer regular. Let us fix a generator $\pi$ (resp. $\zeta$) of the maximal ideal of $R$ (resp. $R'$). Clearly, there exists a unit element $u \in (R')^\times$ such that $\pi = u\zeta^e$. Since the scheme $X_\Gamma$ is etale locally defined by an equation

$$z_0 \cdots z_{n-1} = \pi ,$$

the scheme $X_{\Gamma,R'}$ is etale locally defined by

$$z_0 \cdots z_{n-1} = u\zeta^e ,$$

i.e., it has singularities along the double locus of the closed fiber when $e > 1$. Note that these singularities are locally hypersurfaces in smooth varieties over $R'$. In particular, $X_{\Gamma,R'}$ has the relative dualizing invertible sheaf $\omega_{X_{\Gamma,R'}/R'}$ (cf. [8, Chap. III, §1]). Furthermore, since the locus of the singularity is of codimension two, $X_{\Gamma,R'}$ is normal by Serre's criterion.

2.4. The $R'$-scheme $\boldsymbol{P}(\Delta^\dagger)$ dominates $\boldsymbol{P}(\Lambda)$ for all $\Lambda \in \Delta_0$. By the criterion in 1.6, we see that the local rings of $\boldsymbol{P}(\Delta^\dagger)$ has centers in $\boldsymbol{P}(\Delta)$, i.e., there exists a natural morphism $\boldsymbol{P}(\Delta^\dagger) \to \boldsymbol{P}(\Delta)$. By taking formal completions of these schemes, we get the $\Gamma$-equivariant commutative diagram (1). The induced morphism $\tilde{\rho} : \hat{\Omega}^\dagger \to \hat{\Omega}_{R'}$ is locally described as follows: Let $\{Z_0, \ldots, Z_{n-1}\}$ be a basis of the $K$-linear space $V$. For each $0 \le j \le n-1$, let $M_j$ be the $R$-submodule of $V$ generated by

$$\{Z_0, \ldots, Z_{n-j-1}, \pi Z_{n-j}, \ldots, \pi Z_{n-1}\} .$$

Then $\{[M_0], \ldots, [M_{n-1}]\}$ form a chamber in $\Delta$, and this chamber corresponds to a $k$-valued point in the closed fiber of $\hat{\Omega}_{R'}$ which is an $n$-ple intersection of local components. Set $z_1 = Z_1/Z_0$, $z_2 = Z_2/Z_1, \ldots, z_{n-1} = Z_{n-1}/Z_{n-2}$, $z_n = u\zeta^e Z_0/Z_{n-1}$. Then the singularity of $\hat{\Omega}_{R'}$ at this point is defined by

$$A = R'[z_1, \ldots, z_n]/(z_1 \cdots z_n - u\zeta^e) . \tag{2}$$

The restriction of the morphism $\tilde{\rho}: \hat{\Omega}^{\dagger} \to \hat{\Omega}_{R'}$ at this point is described by the theory of toroidal embeddings. We can show that there exists an ideal $I$ of $A$ such that the restriction of $\tilde{\rho}$ is equal to the blow-up of Spf $A$ along this ideal. Let $U$ be a sufficiently small open neighborhood of this point. Set $U' = (\tilde{\rho})^{-1}(U)$. Then $U'_0$ consists of $(e+1)(e+2)\cdots(e+n-1)/(n-1)!$ components and each component corresponds to an $R'$-module

$$R'\zeta^{a_0}Z_0 + R'\zeta^{a_1}Z_1 + \cdots + R'\zeta^{a_{n-1}}Z_{n-1} \tag{3}$$

with integers $0 = a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq e\ (= a_n)$. For $\boldsymbol{a} = (a_1, \ldots, a_n)$, we denote by $D(\boldsymbol{a})$ the associated exceptional divisor. Set

$$C(\boldsymbol{a}) = \sum_{i=1}^{n} (a_i - a_{i-1})(e - a_i + a_{i-1})$$

and $D = \sum_{\boldsymbol{a}} C(\boldsymbol{a})D(\boldsymbol{a})$. Then $D$ is an effective divisor with support in the exceptional set of $\tilde{\rho}$. We can check that the ideal sheaf $\mathcal{O}_{U'}(-D)$ is relatively ample, by using the theory of toric varieties over a discrete valuation ring (cf. [12, IV, §3]). The combinatorial part of this singularity is equal to that of [12, III, Expl. 2.3]. We review it briefly as follows.

We set $N = \boldsymbol{Z}^n$. Then the ring $A$ is defined by the cone

$$\sigma = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n ; x_1, \ldots, x_n \geq 0\}$$

in $N_{\boldsymbol{R}} = \boldsymbol{R}^n$ and the lattice $N'$, where $N'$ is the sublattice of $N$ defined by

$$N' = \{(c_1, \ldots, c_n) \in \boldsymbol{Z}^n : c_1 + \cdots + c_n \equiv 0 \pmod{e}\} .$$

For integers $i, j$ with $1 \leq i \leq n$ and $0 < j < e$, the hyperplane

$$H_{i,j} = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n ; ex_i = j(x_1 + \cdots + x_n)\}$$

intersects the interior of $\sigma$. Let $\Sigma$ be the fan obtained by dividing $\sigma$ by all these hyperplanes. Then $\Sigma$ is a nonsingular fan and the resolution of the singularity $\tilde{\rho}$ corresponds to the morphism of toric varieties associated to this subdivision.

The projectivity of the resolution is equivalent to the existence of a real-valued continuous function $h$ on $\sigma$ with the following properties (cf. [18, Chap. 2]).

(1) $h$ is linear on each cone $\tau \in \Sigma$.

(2) $h(x) + h(y) \leq h(x+y)$ for $x, y \in \sigma$, and the equality holds if and only if $x$ and $y$ are in a common cone of $\Sigma$.

(3) $h$ is zero on the one-dimensional faces of $\sigma$.

This function $h$ is called a strictly convex $\Sigma$-linear support function.

Let $q_e(x)$ be the function on $[0, e]$ defined by $q_e(x) = (e-2j-1)x + j(j+1)$ if $j \leq x \leq j+1$ for an integer $j$. It is easy to see that $q_e$ is well-defined and is upper convex. Note that $q_e(j) = j(e-j)$ for an integer $j$. An example of $h$ is defined by

$$h(x_1, \dots, x_n) = q_e(x_1) + \cdots + q_e(x_n)$$

for $x=(x_1, \dots, x_n)$ with $x_1 + \cdots + x_n = e$. Actually, $h_i$ defined by $h_i(x_1, \dots, x_n) = q_e(x_i)$ is a strictly convex $\Sigma_i$-linear support function for the fan $\Sigma_i$ obtained by dividing $\sigma$ by the hyperplanes $H_{i,1}, \dots, H_{i,e-1}$ for each $i$. Hence their sum is strictly convex for $\Sigma$.

Let $S = \{(c_1, \dots, c_n) \in \boldsymbol{Z}^n \cap \sigma\,;\, c_1 + \cdots + c_n = e\}$. The set of one-dimensional cones in $\Sigma$ is equal to $\{\boldsymbol{R}_0 c\,;\, c \in S\}$. Let $D_c$ be the associated prime divisor. Then the divisor

$$\sum_{c \in S} (-h(c)) D_c$$

associated to $h$ [18, Chap. 2] is relatively ample. This divisor has support in the exceptional set of the resolution by the condition (3) of $h$. The restriction of this divisor to $U'$ is $-D$ which we described above. Hence $-D$ is relatively ample.

We take a sufficiently large integer $d$, so that the sheaf $\mathcal{O}_{U'}(-dD)$ is relatively very ample. Set $\mathcal{I}_U = \tilde{\rho}_* \mathcal{O}_{U'}(-dD) \subset \mathcal{O}_U$. Then $\tilde{\rho}$ restricted to $U'$ is the blow-up of $U$ by the ideal $\mathcal{I}_U$. The ideal $\mathcal{I}_U$ is a restriction of the ideal $I \subset A$ defined as follows: For a vector $\boldsymbol{b} = (b_1, \dots, b_n) \in (\boldsymbol{Z}_{\geq 0})^n$, we define

$$\langle \boldsymbol{b}, \boldsymbol{a} \rangle = \sum_{i=1}^{n} (a_i - a_{i-1}) b_i\,.$$

Then $I$ is defined by

$$I = \langle z^{\boldsymbol{b}} \zeta^c\,;\, \boldsymbol{b} \in (\boldsymbol{Z}_{\geq 0})^n,\ c \geq 0,\ \langle \boldsymbol{b}, \boldsymbol{a} \rangle + c \geq dC(\boldsymbol{a}) \text{ for every } \boldsymbol{a} \rangle\,,$$

where $z^{\boldsymbol{b}} = z_1^{b_1} \cdots z_n^{b_n} \in A$. Note that $I$ is invariant even if we replace $z_i$ by $u_i z_i$ for unit elements $u_i$, since it is generated by monomials. By the symmetry of the definition of $C(\boldsymbol{a})$'s, it is also invariant by permutations of indices. Hence $\mathcal{I}_U$'s are patched together to a global ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\hat{\Omega}_{R'}}$ and $\tilde{\rho}$ is the blow-up along this ideal.

LEMMA 2.5. *We have* $(\tilde{\rho})^* \omega_{\hat{\Omega}_{R'}/R'} = \omega_{\hat{\Omega}^\dagger/R'}$, *where* $\omega_{\hat{\Omega}_{R'}/R'}$ *and* $\omega_{\hat{\Omega}^\dagger/R'}$ *are the relative dualizing sheaves of* $\hat{\Omega}_{R'} \to \operatorname{Spf} R'$ *and* $\hat{\Omega}^\dagger \to \operatorname{Spf} R'$, *respectively.*

PROOF. By the local expression (2), the dualizing sheaf $\omega_{\hat{\Omega}_{R'}/R'}$ is locally generated by

$$\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_{n-1}}{z_{n-1}}\,. \tag{4}$$

Each component of the exceptional divisor of the resolution $\tilde{\rho} : \hat{\Omega}^\dagger \to \hat{\Omega}_{R'}$ corresponds to an $R'$-submodule as (3). Hence $\hat{\Omega}^\dagger$ is etale locally isomorphic to a Zariski open subset of the affine formal scheme

$$\begin{aligned} &\operatorname{Spf} R'[\zeta^{a_1} Z_1/Z_0,\ \zeta^{a_2 - a_1} Z_2/Z_1, \dots, \zeta^{a_{n-1} - a_{n-2}} Z_{n-1}/Z_{n-2},\ u\zeta^{e - a_{n-1}} Z_0/Z_{n-1}] \\ &\quad = \operatorname{Spf} R'[\zeta^{a_1} z_1,\ \zeta^{a_2 - a_1} z_2, \dots, \zeta^{a_{n-1} - a_{n-2}} z_{n-1},\ \zeta^{-a_{n-1}} z_n]\,, \end{aligned}$$

and the local generator of the dualizing sheaf $\omega_{\hat{\Omega}^\dagger/R'}$ is again given by the local section (4) since $\zeta$ in the differentials is a nonzero constant. q.e.d.

2.6. Since $\omega_{\hat{\Omega}_{R'}/R'}$ is ample on each component of the closed fiber (cf. [17, §4]) and $(\tilde{\rho})^*\omega_{\hat{\Omega}_{R'}/R'}=\omega_{\hat{\Omega}^\dagger/R'}$, the invertible sheaf $\omega_{\hat{\Omega}^\dagger/R'}^{\otimes m}\otimes(\tilde{\rho}^{-1}\mathscr{I})\mathcal{O}_{\hat{\Omega}^\dagger}$ is ample on each component of the closed fiber of $\hat{\Omega}^\dagger$ for $m$ sufficiently large. By the invariance, this invertible sheaf descends to the formal scheme $\mathscr{X}'_\Gamma$. Since it is ample on each component of the closed fiber of $\mathscr{X}'_\Gamma$, this proves Proposition 2.2 by the theory of Grothendieck [EGA3, Thm. 5.1.4, Thm. 5.4.5].

By [EGA3, Thm. 5.4.1], we obtain the following commutative diagram

$$\begin{array}{ccccc} X'_\Gamma & \xrightarrow{\rho} & X_{\Gamma,R'} & \longrightarrow & X_\Gamma \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R. \end{array}$$

By construction, the $R'$-morphism $\rho$ is a desingularization of $X_\Gamma\otimes_R R'$. Since $\rho$ is the descent of $\tilde{\rho}$, it is a blow-up along a closed subscheme whose support is contained in the closed fiber. In particular, we have an isomorphism $X'_{\Gamma,\eta}\xrightarrow{\sim}X_{\Gamma,\eta}\otimes_K K'$ over $\operatorname{Spec} K'$. By Lemma 2.5, we have Proposition 2.3.

**3. Automorphisms of formal schemes.** The goal of this section is to prove the following assertion for the formal scheme $\hat{\Omega}_{R'}=\hat{\Omega}\otimes_R R'$:

PROPOSITION 3.1. *Assume* $\operatorname{char} K=0$. *Then the natural homomorphism*

$$PGL(n, K)\to \operatorname{Aut}_{R'}(\hat{\Omega}_{R'})$$

*is an isomorphism.*

In case $R=R'$, i.e., $\hat{\Omega}=\hat{\Omega}_{R'}$, this was proved by Mustafin (cf. [17, Prop. 4.2], [10]).

3.2. For the proof of this proposition, we need to show some lemmas. Let $B=B^{n-1}$ and $\Sigma_i$ $(0\le i\le n-2)$ be as in 1.8. Let $E\subset B$ be the exceptional divisor of the projection $p\colon B\to \boldsymbol{P}_k^{n-1}$ and $A=A^{n-1}\subset B$ the union of $E$ and the proper transforms of elements of $\Sigma_{n-2}$. By the construction of $B$, the morphism $p$ defines a one-to-one correspondence between the set of irreducible components of $A$ and the union $\Sigma_0\cup\cdots\cup\Sigma_{n-2}$. The intersection $D_1\cap\cdots\cap D_s$ of the irreducible components of $A$ is an irreducible subvariety of codimension $s$ if $\{p(D_1),\dots,p(D_s)\}$ with a suitable order is a flag in $\boldsymbol{P}_k^{n-1}$, and is empty otherwise. Set $B_{k'}=B\otimes_k k'$ and $A_{k'}=A\otimes_k k'$.

LEMMA 3.3. *The natural homomorphism*

$$PGL(n, k)\to \operatorname{Aut}_{k'}(B_{k'}, A_{k'})$$

*is an isomorphism, where* $\operatorname{Aut}(B_{k'}, A_{k'})$ *denotes the group of* $k'$*-automorphisms of* $B_{k'}$

*which maps $A_{k'}$ to itself.*

PROOF. We prove this lemma by induction on $n$. Let $\phi$ be an element of $\mathrm{Aut}_{k'}(B_{k'}, A_{k'})$. If $n=2$, then $B_{k'}^1 = \boldsymbol{P}_{k'}^1$ and $A_{k'}^1$ is the set of $k$-rational points. Hence $\phi$ is a $k$-rational linear automorphism. For $n \geq 3$, it suffices to show that $\phi(E)=E$. Indeed, $B_{k'} \setminus E$ is isomorphic to the open subset of $\boldsymbol{P}_{k'}^{n-1}$ whose complement is the union of $k$-rational linear subspaces of codimension two. Hence $\mathrm{Pic}(B_{k'} \setminus E) \cong \boldsymbol{Z}$ and $\phi$ induces an automorphism of the homogeneous coordinate ring of $\boldsymbol{P}_{k'}^{n-1}$. Since $p(E)$ is mapped to itself, it is a $k$-rational linear automorphism.

For $n=3$, the components of $A_{k'}^2$ are nonsingular rational curves with the self-intersection numbers $-q$ or $-1$, where $q$ is the cardinality of $k$. Each component is an exceptional divisor if and only if the number is $-1$. Hence $\phi(E)=E$.

Assume $n>3$. Each point $x$ of $A_{k'}$ is said to be $i$-ple if there exists precisely $i$ irreducible components of $A_{k'}$ which contain $x$. Since $A_{k'}$ is a simple normal crossing divisor, it is at most $(n-1)$-ple. For an $i$-ple point, the $i$ linear subspaces of $\boldsymbol{P}_k^{n-1}$ corresponding to the components form a flag of length $i$. Let $D$ be a component of $A_{k'}$ with $\dim p(D)=s$. Then the number of $(n-1)$-ple points on $D$ is equal to that of full-length $k$-rational flags which contain the linear space $p(D)$ as a member. The number of $(n-1)$-ple points on $D$ is calculated easily to be

$$\prod_{i=1}^{s} \frac{q^{i+1}-1}{q-1} \prod_{i=1}^{n-2-s} \frac{q^{i+1}-1}{q-1}.$$

Since this number is invariant under $\phi$, $\phi(D)$ is a component of $A_{k'}$ with $\dim p(\phi(D))=s$ or $n-2-s$.

Since $D$ is in $E$ if and only if $\dim p(D)<n-2$, it suffices to show that no component $D$ of $A_{k'}$ satisfies $\dim p(D)=0$ and $\dim p(\phi(D))=n-2$. Suppose that a component $D$ satisfies the equalities. Let $p_1 : D \to E_1 \simeq \boldsymbol{P}_{k'}^{n-2}$ be the morphism to the exceptional divisor $E_1$ of the blow-up of $\boldsymbol{P}_{k'}^{n-1}$ at the point $p(D)$, and $p_2 : \phi(D) \to E_2 = p(\phi(D)) \subset \boldsymbol{P}_{k'}^{n-1}$ the restriction of $p$. By the construction of $B_{k'}$, we see that these morphisms are both isomorphic to the morphism $B_{k'}^{n-2} \to \boldsymbol{P}_{k'}^{n-2}$. By the induction hypothesis, the isomorphism $D \simeq \phi(D)$ is induced by a $k$-rational isomorphism $E_1 \simeq E_2$. Let $l_1$ and $l_2$ be general lines of $E_1$ and $E_2$, $l_1'$ and $l_2'$ their proper transforms in $D$ and $\phi(D)$, respectively. Here, we may replace $k'$ by its algebraic closure in order to take sufficiently general lines. Then $l_2'$ intersects the exceptional divisor at $(q^{n-1}-1)/(q-1)$ points while $l_1'$ does not intersect the exceptional divisor other than $D$. Hence the intersection numbers of $D \cdot l_1'$ and $\phi(D) \cdot l_2'$ are $-1$ and $1-(q^{n-1}-1)/(q-1) = -(q+\cdots+q^{n-2})$, respectively. This is a contradiction, since we may assume $l_2' = \phi(l_1')$. q.e.d.

LEMMA 3.4. *Let $\Theta_B(-\log A)$ be the sheaf of algebraic vector fields with logarithmic zeros along $A$. Then* $\mathrm{H}^0(\Theta_B(-\log A)) = \{0\}$.

PROOF. The restriction $\phi : B \setminus E \to \boldsymbol{P}_k^{n-1}$ of $p$ is clearly an open immersion.

Let $\Lambda_0$ be the union of $k$-rational hyperplanes of $\boldsymbol{P}_k^{n-1}$. Then $\Theta_B(-\log A)|_{B\setminus E}$ is equal to $\phi^*\Theta_{\boldsymbol{P}_k^{n-1}}(-\log A_0)$. We have

$$\mathrm{H}^0(\Theta_B(-\log A)|_{B\setminus E})\simeq \mathrm{H}^0(\Theta_{\boldsymbol{P}_k^{n-1}}(-\log A_0))\,,$$

since the sheaf $\Theta_{\boldsymbol{P}_k^{n-1}}(-\log A_0)$ is reflexive and the complement of the image of $\phi$ is of codimension two. Hence, it suffices to show that $\mathrm{H}^0(\Theta_{\boldsymbol{P}_k^{n-1}}(-\log A_0))=\{0\}$.

For each $0\leq i\leq n-1$, let $D_i$ be the hyperplane $X_i=0$. For $D=D_0\cup\cdots\cup D_{n-1}$, $\Theta_{\boldsymbol{P}_k^{n-1}}(-\log D)$ is a free sheaf of rank $n-1$ (cf. [18, Prop. 3.1]). Let $(u_1,\dots,u_{n-1})$ be the coordinate of the affine space $(X_0\neq 0)$ defined by $u_i=X_i/X_0$ for $i=1,\dots,n-1$. Then $\mathrm{H}^0(\Theta_{\boldsymbol{P}_k^{n-1}}(-\log D))$ is the $k$-vector space with basis

$$\left\{u_1\frac{\partial}{\partial u_1},\dots,u_{n-1}\frac{\partial}{\partial u_{n-1}}\right\}.$$

The restriction of the hyperplane $H=(X_0+\cdots+X_{n-1}=0)$ to this affine space is defined by the equation $1+u_1+\cdots+u_{n-1}=0$. Let

$$\delta=a_1u_1\frac{\partial}{\partial u_1}+\cdots+a_{n-1}u_{n-1}\frac{\partial}{\partial u_{n-1}}$$

be an element of $\mathrm{H}^0(\Theta_{\boldsymbol{P}_k^{n-1}}(-\log D))$. Since $\delta(1+u_1+\cdots+u_{n-1})=a_1u_1+\cdots+a_{n-1}u_{n-1}$, $\delta$ has logarithmic zero along $H$ if and only if $\delta=0$. Hence $\mathrm{H}^0(\Theta_{\boldsymbol{P}_k^{n-1}}(-\log(D+H)))=\{0\}$. Since $D+H\subset A_0$, we have $\mathrm{H}^0(\Theta_{\boldsymbol{P}_k^{n-1}}(-\log A_0))=\{0\}$. q.e.d.

LEMMA 3.5. *If an $R'$-automorphism $\phi$ of the formal scheme $\hat{\Omega}_{R'}$ fixes all the irreducible components, then $\phi$ is the identity.*

PROOF. For each $\Lambda\in\Delta_0$, we denote by $B(\Lambda)$ the corresponding component of $\hat{\Omega}_{R'}$ with the reduced scheme structure. Set

$$A(\Lambda)=B(\Lambda)\cap\left(\bigcup_{\Lambda'\neq\Lambda}B(\Lambda')\right).$$

Then the pair $(B(\Lambda), A(\Lambda))$ is isomorphic to $(B_{k'}, A_{k'})$ (cf. 1.6 and 1.8). Hence the automorphism of the pair $(B(\Lambda), A(\Lambda))$ induced by $\phi$ is the pull-back of a linear automorphism of $\boldsymbol{P}(\Lambda)_0$ by Lemma 3.3. Since $\phi$ fixes the components neighboring $B(\Lambda)$, this $k'$-linear automorphism fixes all $k$-rational points of the projective space over $k'$. Hence the induced automorphism of $B(\Lambda)$ is the identity for an arbitrary $\Lambda\in\Delta_0$.

It suffices to prove that the automorphism of

$$\hat{\Omega}_i=\hat{\Omega}\otimes_R(R'/\zeta^{i+1})$$

induced by $\phi$ is the identity for all nonnegative integers $i$, since $\phi$ is their projective limit. We proceed by induction on $i$. The assertion is clear for $i=0$, since $\hat{\Omega}_0$ is a reduced scheme with support $\bigcup_{\Lambda\in\Delta_0}B(\Lambda)$.

Suppose that the automorphism is the identity for $i=d-1\geq 0$. Then $\phi$ induces an automorphism $\phi_d^*$ of the sheaf $\mathcal{O}_{\hat{\Omega}_d}$ of $R'/(\zeta^{d+1})$-algebras. Then $\phi_d^*-1$ defines a $k'$-derivation of $\mathcal{O}_{\hat{\Omega}_0}$:

$$\mathcal{O}_{\hat{\Omega}_0} \to \mathcal{O}_{\hat{\Omega}_{R'}} \otimes_{R'} (\zeta^d)/(\zeta^{d+1}) \simeq \mathcal{O}_{\hat{\Omega}_0}.$$

Since $\hat{\Omega}_0$ is a normal crossing union of the nonsingular $k'$-varieties $B(\Lambda)$ for $\Lambda \in \Delta_0$, the sheaf $\mathcal{D}er_{k'}(\mathcal{O}_{\hat{\Omega}_0})$ has a natural injective homomorphism to

$$\bigoplus_{\Lambda \in \Delta_0} \Theta_{B(\Lambda)}(-\log A(\Lambda))$$

(cf. [11, Thm. 2.1]). Since $\mathrm{H}^0(\Theta_{B(\Lambda)}(-\log A(\Lambda)))=\{0\}$ by Lemma 3.4, for every $\Lambda \in \Delta_0$, there is no nontrivial $k'$-derivation of $\mathcal{O}_{\hat{\Omega}_0}$. Hence $\phi_d^*=1$, and the assertion is true for $i=d$. q.e.d.

Proof of Proposition 3.1. For any finite extension $K''/K'$, there exists a natural injective homomorphism

$$\mathrm{Aut}_{R'}(\hat{\Omega}_{R'}) \to \mathrm{Aut}_{R''}(\hat{\Omega}_{R''}),$$

where $R''$ is the integer ring of $K''$. Hence, we may assume that $K'/K$ is a Galois extension by the condition $\operatorname{char} K=0$. Let $G$ be the Galois group. Let $\phi$ be an $R'$-automorphism of $\hat{\Omega}_{R'}$. For any element $\sigma \in G$, $\sigma^{-1}\phi^{-1}\sigma\phi$ is an $R'$-automorphism of $\hat{\Omega}_{R'}$. Since $\sigma$ fixes all components of the formal scheme, this $R'$-automorphism also fixes them. Hence $\sigma^{-1}\phi^{-1}\sigma\phi$ is the identity by Lemma 3.5. This implies that $\phi$ descends to an $R$-automorphism of the formal scheme $\hat{\Omega}$. We are done since the homomorphism

$$PGL(n, K) \to \mathrm{Aut}_R(\hat{\Omega})$$

is an isomorphism by Mustafin [17, Prop. 4.2]. (A precise proof of this proposition of Mustafin is given in [10].) q.e.d.

## 4. Proof of Theorem 0.2.

In this section, we prove the following theorem which is equal to Theorem 0.2.

Theorem 4.1. *Assume* $\operatorname{char} K=0$. *Let* $\Gamma_1$ *and* $\Gamma_2$ *be torsionfree co-compact subgroups of* $PGL(n, K)$. *Let* $X_{\Gamma_1}$ *and* $X_{\Gamma_2}$ *be the R-schemes obtained by the algebraizations of the formal schemes* $\mathcal{X}_{\Gamma_1}=\hat{\Omega}/\Gamma_1$ *and* $\mathcal{X}_{\Gamma_2}=\hat{\Omega}/\Gamma_2$, *respectively. Then,* $X_{\Gamma_1,\eta}\otimes_K \bar{K}$ *and* $X_{\Gamma_2,\eta}\otimes_K \bar{K}$ *are isomorphic over* $\operatorname{Spec} \bar{K}$ *if and only if* $\Gamma_1$ *and* $\Gamma_2$ *are conjugate in* $PGL(n, K)$, *where* $\bar{K}$ *is the algebraic closure of* $K$.

Proof. The "if" part of the above theorem is clear (cf. [17, §4]). We are going to prove the other part. Suppose we have an isomorphism $X_{\Gamma_1,\eta}\otimes_K \bar{K} \xrightarrow{\sim} X_{\Gamma_2,\eta}\otimes_K \bar{K}$ over $\operatorname{Spec} \bar{K}$. There exists a finite Galois extension $K'/K$ such that this isomorphism descends to an isomorphism $X_{\Gamma_1,\eta}\otimes_K K' \xrightarrow{\sim} X_{\Gamma_2,\eta}\otimes_K K'$ over $K'$ (cf. [EGA4, Cor. 8.8.2.5]). From now on, we use the notation fixed in §2. Consider the normal schemes $X_{\Gamma_i,R'}=X_{\Gamma_i}\otimes_R R'$ over $\operatorname{Spec} R'$ for $i=1, 2$. Since the generic fibers of these schemes are isomorphic, there exists a birational map

$$\varphi: X_{\Gamma_1,R'} \cdots \to X_{\Gamma_2,R'} \tag{5}$$

over $\operatorname{Spec} R'$. First, we wish to show that $\varphi$ is an isomorphism.

We look at the canonical rings

$$\bigoplus_{m\geq 0} \mathrm{H}^0(X_{\Gamma_i,R'}, \omega_{X_{\Gamma_i,R'}/R'}^{\otimes m}) \tag{6}$$

of $X_{\Gamma_i,R'}$ for $i=1$, 2. Since the sheaves $\omega_{X_{\Gamma_i,R'}/R'}$ are ample on $X_{\Gamma_i}$ for $i=1$, 2, it suffices to prove that $\varphi$ induces an isomorphism

$$\bigoplus_{m\geq 0} \mathrm{H}^0(X_{\Gamma_2,R'}, \omega_{X_{\Gamma_2,R'}/R'}^{\otimes m}) \xrightarrow{\sim} \bigoplus_{m\geq 0} \mathrm{H}^0(X_{\Gamma_1,R'}, \omega_{X_{\Gamma_1,R'}/R'}^{\otimes m}) \tag{7}$$

of graded $R'$-algebras. Take the resolutions $X'_{\Gamma_i}$ of $X_{\Gamma_i,R'}$ as in Proposition 2.3 for $i=1$, 2. Since these resolutions produce no discrepancy with respect to the relative dualizing sheaves, we have isomorphisms of graded $R'$-algebras

$$\bigoplus_{m\geq 0} \mathrm{H}^0(X_{\Gamma_i,R'}, \omega_{X_{\Gamma_i,R'}/R'}^{\otimes m}) \xrightarrow{\sim} \bigoplus_{m\geq 0} \mathrm{H}^0(X'_{\Gamma_i}, \omega_{X'_{\Gamma_i}/R'}^{\otimes m})$$

for $i=1$, 2. The map (5) induces a birational map $\varphi': X'_{\Gamma_1} \cdots \to X'_{\Gamma_2}$. Since $X'_{\Gamma_1}$ is regular and $X'_{\Gamma_2}$ is proper over $R$, $\varphi'$ is regular outside a closed subset of codimension not less than two.

Let $(X'_{\Gamma_1})^\circ$ be the open subscheme of $X'_{\Gamma_1}$ defined as the intersection of the smooth locus over $R'$ and the regular locus of the rational map $\varphi'$. Clearly $X'_{\Gamma_1,\eta} \subset (X'_{\Gamma_1})^\circ$, and $(X'_{\Gamma_1})^\circ_0$ is open dense in $X'_{\Gamma_1,0}$. Let $(X'_{\Gamma_2})^\circ$ be the smooth locus of $X'_{\Gamma_2}$ over $R'$.

CLAIM. $\varphi'((X'_{\Gamma_1})^\circ) \subset (X'_{\Gamma_2})^\circ$.

PROOF. Let $x$ be a closed point of $(X'_{\Gamma_1})^\circ$ and let $y=\varphi'(x) \in X'_{\Gamma_2}$. Let $(\mathcal{O}_x, \mathfrak{m}_x)$ and $(\mathcal{O}_y, \mathfrak{m}_y)$ be the local rings of $x$ and $y$, respectively. Then we have dominations of regular local rings

$$R' \subset \mathcal{O}_y \subset \mathcal{O}_x .$$

Since $X'_{\Gamma_1}$ is smooth over $R'$ at $x$, $\mathcal{O}_x/\zeta\mathcal{O}_x$ is a regular local ring. Hence $\zeta \notin \mathfrak{m}_x^2$. On the other hand, $\mathfrak{m}_y^2 \subset \mathfrak{m}_x^2$ since $\mathfrak{m}_y \subset \mathfrak{m}_x$. Hence $\zeta \notin \mathfrak{m}_y^2$ and $\mathcal{O}_y/\zeta\mathcal{O}_y$ is a regular local ring. Since $X'_{\Gamma_2}$ is smooth at all points of $(X'_{\Gamma_2})_0$ which is regular in $(X'_{\Gamma_2})_0$, $X'_{\Gamma_2}$ is smooth at $y$. q.e.d.

Since $\varphi': (X'_{\Gamma_1})^\circ \to (X'_{\Gamma_2})^\circ$ is a morphism of smooth $R'$-schemes, the pull-back homomorphism

$$(\varphi')^*\omega_{(X'_{\Gamma_2})^\circ/R'} \to \omega_{(X'_{\Gamma_1})^\circ/R'}$$

of sheaves on $(X'_{\Gamma_1})^\circ$ is defined naturally. For any $\alpha \in \bigoplus_{m\geq 0} \mathrm{H}^0(X'_{\Gamma_2}, \omega_{X'_{\Gamma_2}/R'}^{\otimes m})$, the pull-back $(\varphi')^*\alpha$ can be expressed as a regular differential form defined on a Zariski open set of $X'_{\Gamma_1}$ whose complement is of codimension at least two. Hence we can prolong $(\varphi')^*\alpha$ to an element in $\bigoplus_{m\geq 0}\mathrm{H}^0(X'_{\Gamma_1}, \omega^{\otimes m}_{X'_{\Gamma_1}/R'})$, since $\omega^{\otimes m}_{X'_{\Gamma_1}/R'}$ is invertible and $X'_{\Gamma_1}$ is regular. Thus, we have an injective homomorphism

$$\bigoplus_{m\geq 0}\mathrm{H}^0(X'_{\Gamma_2}, \omega^{\otimes m}_{X'_{\Gamma_2}/R'}) \to \bigoplus_{m\geq 0}\mathrm{H}^0(X'_{\Gamma_1}, \omega^{\otimes m}_{X'_{\Gamma_1}/R'})$$

of graded $R$-algebras. This is an isomorphism, since the inverse is obtained similarly by $\varphi^{-1}$. Hence, we have the isomorphism (7). This implies that we can prolong (5) to an isomorphism $\varphi\colon X_{\Gamma_1,R'} \xrightarrow{\sim} X_{\Gamma_2,R'}$ over $\operatorname{Spec} R'$.

Then we lift the isomorphism $\varphi$ to an automorphism $\tilde{\varphi}$ of the universal covering formal scheme $\hat{\Omega}_{R'}$. Then we have $\tilde{\varphi}\Gamma_1\tilde{\varphi}^{-1}=\Gamma_2$. By Theorem 3.1, $\tilde{\varphi}$ comes from an element of $PGL(n, K)$. Hence the groups $\Gamma_1$ and $\Gamma_2$ are conjugate in $PGL(n, K)$ as desired. q.e.d.

**5. Fake projective planes.** In this section, we consider fake projective planes as an application of our main theorem. A *fake projective plane* is, by definition, a nonsingular projective complex surface of general type with

$$p_g=q=0\,, \quad c_1^2=3c_2=9\,. \tag{8}$$

The underlying topological space of a fake projective plane has the same Betti numbers as the projective plane $\boldsymbol{CP}^2$ (cf. [1, V, Rem. 1.2]). In [16], Mumford constructed a fake projective planes as follows. He showed that, in case $n=3$, the invariants of the surface $X_{\Gamma,\eta}$ satisfies (8) if $K=\boldsymbol{Q}_2$ and $\Gamma$ acts on $\Delta_0$ transitively. Then, for a fixed isomorphism $\bar{\boldsymbol{Q}}_2 \xrightarrow{\sim} \boldsymbol{C}$ of fields, $X_{\Gamma,\eta}\otimes_{\boldsymbol{Q}_2}\boldsymbol{C}$ is a fake projective plane. Mumford [16] constructed one example of torsionfree co-compact subgroup $\Gamma$ of $PGL(3, \boldsymbol{Q}_2)$ which acts transitively on $\Delta_0$. Theorem 4.1 implies that, if we find another such $\Gamma$ which is not conjugate to that of Mumford, we get another example of fake projective planes.

5.1. Here, we refer to the recent work of Cartwright, Mantero, Steger, and Zappa on discrete subgroups of $PGL(3, K)$ (cf. [4], [5]).

Let $\Delta$ be the Bruhat-Tits building attached to $PGL(3, K)$. Let $\Lambda_0=[M_0]$ be a vertex. The labeling $\tau:\Delta_0\to\boldsymbol{Z}/3\boldsymbol{Z}$ is defined as follows. For $\Lambda\in\Delta$, we choose a representative $M$ of $\Lambda$ contained in $M_0$. Then $\tau(\Lambda)$ is defined to be the length modulo 3 of the $R$-module $M_0/M$. It is easy to see that this definition does not depend on the choice of $M$. We denote by $\mathcal{N}_0$ the set of all adjacent vertices of $\Lambda_0$. We set $P:=\{\Lambda\in\mathcal{N}_0 \mid \tau(\Lambda)=1\}$ and $L:=\{\Lambda\in\mathcal{N}_0 \mid \tau(\Lambda)=2\}$. $P$ and $L$ correspond to the sets of $k$-rational points and lines of $\boldsymbol{P}(M_0/\pi M_0)\simeq\boldsymbol{P}_k^2$, respectively.

Let $\Gamma$ be a discrete subgroup of $PGL(3, K)$ which may not be torsionfree. We assume that $\Gamma$ acts simply transitively on $\Delta_0$. Then, for each $\Lambda\in\Delta_0$, there exists a unique element $g_\Lambda\in\Gamma$ such that $g_\Lambda(\Lambda_0)=\Lambda$. The correspondence $\Lambda\mapsto\lambda(\Lambda):=g_\Lambda^{-1}(\Lambda_0)$ defines a bijection $\lambda\colon P\to L$, which is called a *point-line correspondence* (cf. [4, §1]). Set

$$\mathscr{F}=\{(\Lambda, \Lambda', \Lambda'')\in P^3 \mid g_\Lambda g_{\Lambda'} g_{\Lambda''}=1\}\,.$$

Then it was shown that the group $\Gamma$ is generated by $\{g_\Lambda \mid \Lambda\in P\}$ with the fundamental relation $\{g_\Lambda g_{\Lambda'} g_{\Lambda''}=1 \mid (\Lambda, \Lambda', \Lambda'')\in\mathscr{F}\}$ (cf. [4, Thm. 3.1]). The combinatorial data $(\lambda, \mathscr{F})$ is called the *triangle presentation* of $\Gamma$. In the rest of this section, we assume $q=2$. Then there exist exactly eight combinatorial possibilities of triangle presentations for $\boldsymbol{P}_k^2$ up to projective transformations and correlations (cf. [5, Thm. 1]). These were named $A1$, $A2$, $A3$, $A4$, $B1$, $B2$, $B3$ and $C1$ in [5].

REMARK 5.2. In [5, §2], the classification of triangle presentations was done up to "correlations" of the projective plane. Hence a triangle presentation is identified with its reverse. This identification is not convenient for us, since the group associated to the reverse is not isomorphic but anti-isomorphic to the original group, in general. Among the eight triangle presentations, $A1$, $A2$ and $A3$ are not reverse symmetric, while the others are reverse symmetric. In the table of [5, p. 207], "N" (not reverse symmetric) for $A4$ might be a misprint. In this section, we distinguish $A1$, $A2$ and $A3$ with their reverses which we write $A1'$, $A2'$ and $A3'$, respectively.

THEOREM 5.3 (cf. [5, §3]). (1) *There exist discrete subgroups of* $PGL(3, \boldsymbol{F}_2((X)))$ *with triangle presentations* $A1$, $A2$, $A3$ and $A4$ *which act simply transitively on* $\Delta_0$, *where* $\boldsymbol{F}_2((X))$ *is the quotient field of* $\boldsymbol{F}_2[[X]]$.

(2) *There exist discrete subgroups of* $PGL(3, \boldsymbol{Q}_2)$ *with triangle presentations* $B1$, $B2$, $B3$ *and* $C1$ *which act simply transitively on* $\Delta_0$.

We denote the group associated to $A1$ by $\Gamma_{A1}$, and so on. Then the transpose groups of $\Gamma_{A1}$, $\Gamma_{A2}$ and $\Gamma_{A3}$ in $PGL(3, \boldsymbol{F}_2((X)))$ have the triangle presentations $A1'$, $A2'$ and $A3'$, respectively.

5.4. Since the actions of the groups above preserve the orientation of $\Delta$ defined by the labeling, the restrictions of the actions to $\Delta_1$ are also free. Hence, for each of these groups, the action is free on $\Delta$ if and only if it is free on $\Delta_2$. For each triangle presentation, the number of rows in the table [5, p. 212] is equal to the number of $\Gamma$-orbits in $\Delta_2$. If the action of $\Gamma$ on $\Delta_2$ is free, then the twenty-one $\boldsymbol{F}_2$-rational points of the rational surface $B^2$ are identified to seven triple points of $X_{\Gamma,0}$ (cf. [9, §1]). Otherwise, it has more than seven $\Gamma$-orbits. Hence the action of $\Gamma$ on $\Delta$ is free if and only if the number of the $\Gamma$-orbits in $\Delta_2$ is equal to seven.

(1) By the table on [5, p. 212], we see that $\Gamma_{A1}$ and $\Gamma_{A2}$ act freely on $\Delta$, while $\Gamma_{A3}$ and $\Gamma_{A4}$ are not free on $\Delta_2$.

(2) The group $\Gamma_{C1}$ is equal to the group found by Mumford which is embedded in $PGL(3, \boldsymbol{Q}(\sqrt{-7}))\subset PGL(3, \boldsymbol{Q}_2)$. The groups $\Gamma_{B1}$, $\Gamma_{B2}$ and $\Gamma_{B3}$ are embedded in $PGL(3, \boldsymbol{Q}(\sqrt{-15}))\subset PGL(3, \boldsymbol{Q}_2)$ (cf. [5, §3]). By the same table, we see that $\Gamma_{B3}$ is not free on $\Delta_2$, while other three groups are free on $\Delta$.

5.5. Since mutually conjugate groups in $PGL(3, \boldsymbol{Q}_2)$ define equivalent triangle presentations, $\Gamma_{B1}$, $\Gamma_{B2}$ and $\Gamma_{C1}$ are not conjugate to each other. Hence, by Theorem

4.1, the fake projective planes obtained by these groups are mutually distinct. Hence there exist at least three fake projective planes. The Figures 2, 3 and 4 are the configurations of the double curves of $X_{\Gamma,0}$ for these groups. Note that the numbers of double curves with nodes for these three surfaces are mutually distinct.

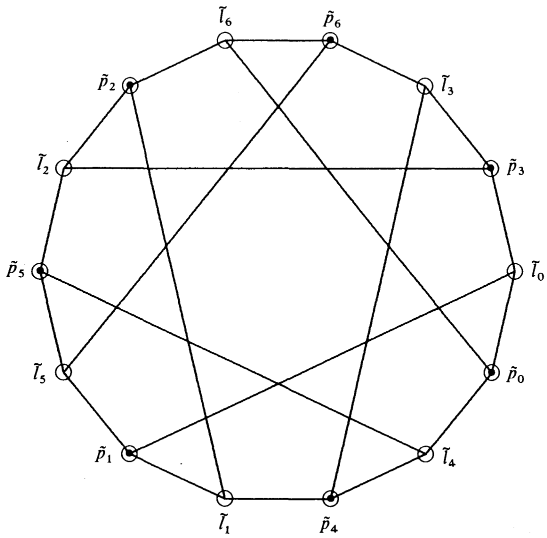


FIGURE 1. The dual graph of the 14 rational curves on $B^2$.

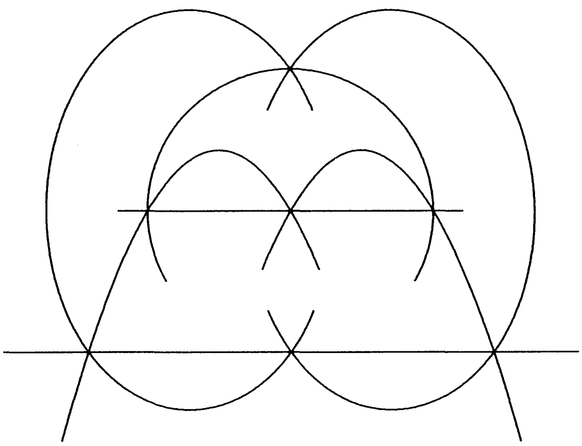

FIGURE 2. Configuration of the double curves in the closed fiber of the uniformized scheme for $\Gamma_{B1}$.

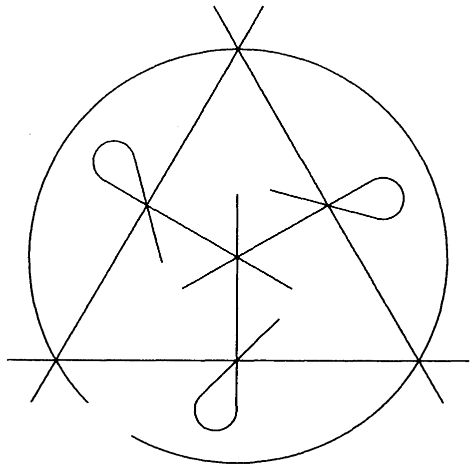

FIGURE 3. Configuration of the double curves in the closed fiber of the uniformized scheme for $\Gamma_{B2}$.

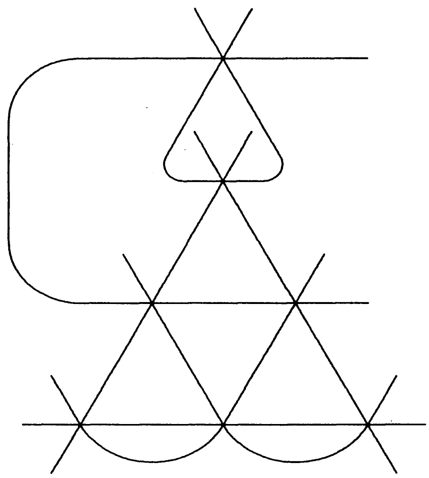

FIGURE 4. Configuration of the double curves in the closed fiber of the uniformized scheme for $\Gamma_{C1}$.

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