Tôhoku Math. J.
50 (1998), 469–489

BEST BOUNDS OF AUTOMORPHISM GROUPS OF HYPERELLIPTIC FIBRATIONS

ZHI-JIE CHEN

(Received February 20, 1997)

Abstract. For a relatively minimal hyperelliptic fibration, the best bounds of the orders of its automorphism group are obtained.

Let S be a smooth projective surface over the complex number field. A hyperelliptic fibration is a morphism $f: S \to C$ where C is a projective curve such that a general fiber of f is a smooth hyperelliptic curve.

DEFINITION 0.1. An automorphism of the fibration $f: S \to C$ is a pair of automorphisms $(\tilde{\alpha}, \alpha)$ with $\tilde{\alpha} \in \mathrm{Aut}(S)$, $\alpha \in \mathrm{Aut}(C)$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\tilde{\alpha}"] \arrow[d, "f"'] & S \arrow[d, "f"] \\ C \arrow[r, "\alpha"] & C \end{tikzcd}$$

commutes.

The automorphism group of a fibration f will be denoted by $\mathrm{Aut}(f)$. Let G be a subgroup of $\mathrm{Aut}(f)$, G. Xiao has obtained upper bounds for the order of G:

PROPOSITION 0.1 ([6, Proposition 1]). *Suppose S is a complete surface of general type over the complex number field with a relatively minimal fibration $f: S \to C$ whose general fiber is of genus $g \geq 2$. Then*

$$|G| \leq \begin{cases} 882K_S^2 & \text{if } g(C) \geq 2 \\ 168(2g+1)(K_S^2+8g-8) & \text{otherwise}. \end{cases}$$

When $g=2$, we have shown the following result.

THEOREM 0.1 ([3, Theorem 0.1]). *Suppose S is a complete surface of general type over the complex number field with a relatively minimal genus 2 fibration $f: S \to C$. Then*

$$|G| \leq 504K_S^2 .$$

This work was carried out under the support of NSFC.
1991 *Mathematics Subject Classification.* Primary 14J25; Secondary 14E09.

If S is not locally trivial, then

$$|G|\leq\begin{cases}126K_S^2 & \text{if } g(C)\geq 2\\ 144K_S^2 & \text{if } g(C)=1\\ 120K_{S/C}^2 & \text{if } g(C)=0\,.\end{cases}$$

THEOREM 0.2 ([1, Theorems 1, 2]). *Suppose S is a complete surface of general type over the complex number field with a relatively minimal fibration $f\colon S\to C$ whose general fiber is a hyperelliptic curve of genus $g\geq 2$. If $g(C)\geq 2$, then*

$$|G|\leq\begin{cases}\dfrac{84(g+1)}{g-1}K_S^2 & \text{if } g\neq 2,3,5,9\\ 504K_S^2 & \text{if } g=2\\ 252K_S^2 & \text{if } g=3\\ 315K_S^2 & \text{if } g=5\\ 157.5K_S^2 & \text{if } g=9\,.\end{cases}$$

If $g(C)=1$, then

$$|G|\leq\begin{cases}\dfrac{24(g+1)(2g+1)}{7g-13}K_S^2 & \text{if } g\geq 6\\ 144K_S^2 & \text{if } g=2\\ 84K_S^2 & \text{if } g=3\\ 72K_S^2 & \text{if } g=4\\ 90K_S^2 & \text{if } g=5\,.\end{cases}$$

If $g(C)=0$, then

$$|G|\leq\begin{cases}\dfrac{20(g+1)(2g+1)}{7g-13}K_{S/C}^2 & \text{if } g\geq 6\\ 120K_{S/C}^2 & \text{if } g=2\\ 70K_{S/C}^2 & \text{if } g=3\\ 60K_{S/C}^2 & \text{if } g=4\\ 75K_{S/C}^2 & \text{if } g=5\,.\end{cases}$$

In this paper, by using more detailed analysis of the singular fibers, we will obtain the best upper bounds for the orders of the automorphism groups of hyperelliptic fibrations. The main theorem of this paper is the following:

THEOREM 0.3. *Suppose S is a complete surface of general type over the complex number field with a relatively minimal fibration $f\colon S\to C$ whose general fiber is a hyperelliptic curve of genus $g\geq 2$. If $g(C)\geq 2$ and f is not locally trivial, then*

$$|G| \leq \begin{cases} \dfrac{21(g+1)}{g-1} K_S^2 & \textit{if} \quad g \neq 2, 3, 5, 9 \\ \dfrac{126}{g-1} K_S^2 & \textit{if} \quad g = 2, 3 \\ \dfrac{315}{g-1} K_S^2 & \textit{if} \quad g = 5, 9\,. \end{cases}$$

If $g(C) = 1$, *then*

$$|G| \leq \begin{cases} \dfrac{24(g+1)}{g-1} K_S^2 & \textit{if} \quad g \neq 2, 3, 5, 9 \\ \dfrac{144}{g-1} K_S^2 & \textit{if} \quad g = 2, 3 \\ \dfrac{360}{g-1} K_S^2 & \textit{if} \quad g = 5, 9\,. \end{cases}$$

If $g(C) = 0$, *then*

$$|G| \leq \begin{cases} \dfrac{20(g+1)}{g-1} K_{S/C}^2 & \textit{if} \quad g \neq 2, 3, 5, 9 \\ \dfrac{120}{g-1} K_{S/C}^2 & \textit{if} \quad g = 2, 3 \\ \dfrac{300}{g-1} K_{S/C}^2 & \textit{if} \quad g = 5, 9\,. \end{cases}$$

All these bounds are the best possible. Furthermore, if the equality holds, then the fibration is necessarily equimodular.

1. Preliminaries. Let $f: S \to C$ be a hyperelliptic fibration of genus $g \geq 2$. Then the relative canonical map of f is generically of degree 2. This map determines an involution σ on S whose restriction on a general fiber F of f is a hyperelliptic involution of F. σ is called the *hyperelliptic involution* associated to the hyperelliptic fibration f. We always assume $\sigma \in G$.

Let $\rho: \tilde{S} \to S$ be the composite of all the blow-ups of isolated fixed points of the involution σ, and let $\tilde{\sigma}$ be the induced involution on $\tilde{S}$. The factor space $\tilde{P} = \tilde{S}/\langle\tilde{\sigma}\rangle$ is a smooth surface, and f induces a ruling on $\tilde{P}$:

$$\tilde{\pi}: \tilde{P} \to C\,.$$

The projection from $\tilde{S}$ to $\tilde{P}$ is a smooth double cover $\tilde{\theta}: \tilde{S} \to \tilde{P}$ which is determined by the pair $(\tilde{R}, \tilde{\delta})$ where $\tilde{R}$ is the branch locus of $\tilde{\theta}$ and $\tilde{\delta}$ is the divisor such that

$$\tilde{\theta}_* \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{\tilde{P}} \oplus \mathcal{O}_{\tilde{P}}(-\tilde{\delta})\,.$$

LEMMA 1.1. *There exist contractions of ruled surfaces $\tilde{\psi}: \tilde{P} \to \hat{P}$ and $\hat{\psi}: \hat{P} \to P$, such that $\pi: P \to C$ is a minimal ruled surface and $\tilde{\pi} = \pi\hat{\psi}\tilde{\psi}$. Let $(\hat{R}, \hat{\delta})$ and (R, δ) be the images of $(\tilde{R}, \tilde{\delta})$ in $\hat{P}$ and P, respectively. Then $\hat{\psi}\tilde{\psi}: \tilde{P} \to P$ is a minimal even resolution of singularities of R, and $\hat{\psi}: \hat{P} \to P$ is a minimal even resolution of non-negligible singularities of R.*

The proof is obvious. Note that the minimal ruled surface P need not be unique, but $\hat{P} \to C$ is uniquely determined by $\tilde{\pi}: \tilde{P} \to C$ and $(\tilde{R}, \tilde{\delta})$.

Now we obtain a diagram as follows.

$$(*) \qquad \begin{tikzcd}
\tilde{S} \arrow[rr, "\tilde{\theta}"] \arrow[dd, "\rho"'] \arrow[dddr, "\tilde{f}"] & & \tilde{P} \arrow[d, "\tilde{\psi}"] \\
& & \hat{P} \arrow[d, "\hat{\psi}"] \\
S \arrow[dr, "f"'] & & P \arrow[dl, "\pi"] \\
& C &
\end{tikzcd}$$

In this paper the linear and numerical equivalence will be denoted by "$\equiv$" and "$\sim$", respectively. Let

$$R \sim -(g+1)K_{P/C} + nF.$$

Then

$$\delta \sim -\frac{g+1}{2}K_{P/C} + \frac{n}{2}F.$$

Since $K_{P/C}^2 = 0$ and $K_{P/C}F = -2$, we have

$$\delta^2 = (g+1)n, \qquad \delta K_{P/C} = -n.$$

$\hat{\psi}$ can be decomposed into a series of blow-ups. Suppose that the center of the i-th blow-up is a singular point of multiplicity m_i in the corresponding even resolution of R. Let $k_i = [m_i/2]$. Denote the total transform in $\hat{P}$ of the exceptional curve of the i-th blow-up by $\mathfrak{E}_i$. Then we have

$$\hat{\delta} \equiv \hat{\psi}^*\delta - \sum k_i \mathfrak{E}_i, \qquad K_{\hat{P}/C} \equiv \hat{\psi}^* K_{P/C} + \sum \mathfrak{E}_i.$$

By the formulas for double covering, we have

$$(1) \qquad \begin{aligned} \chi_{\tilde{f}} &= \chi(\mathcal{O}_{\tilde{S}}) - (g-1)(g(C)-1) \\ &= \frac{1}{2}(\hat{\delta}^2 + \hat{\delta}K_{\hat{P}/C}) = \frac{1}{2}gn - \frac{1}{2}\sum k_i(k_i - 1), \end{aligned}$$

(2)
$$K_{\tilde{S}/C}^2 = K_{\tilde{S}}^2 - 8(g-1)(g(C)-1)$$
$$= 2(\hat{\delta} + K_{\hat{P}/C})^2 = 2(g-1)n - 2\sum(k_i - 1)^2 .$$

$\hat{R}$ may contain isolated (-2)-curves which are produced during desingularization of $(2k-1 \to 2k-1)$ singular points. We have the following lemma.

LEMMA 1.2. *The images under $\tilde{\theta}$ of vertical (-1)-curves of $\tilde{S}$ are isolated (-2)-curves in $\tilde{R}$, which are the isolated (-2)-curves in $\hat{R}$ as well. If $\hat{R}$ contains l vertical (-2)-curves, then*

$$K_{S/C}^2 = K_{\tilde{S}/C}^2 + l .$$

PROOF. The vertical (-1)-curves in $\tilde{S}$ are produced by blowing up the isolated fixed points in S with respect to the involution σ. The restriction of $\tilde{\sigma}$ to these (-1)-curves is the identity map. Therefore the images of these (-1)-curves under the double covering $\tilde{\theta}$ must be contained in the branch locus $\tilde{R}$. Since $\tilde{R}$ is a smooth divisor, these images are isolated (-2)-curves in $\tilde{R}$. These (-2)-curves cannot be produced during the desingularization of $\hat{R}$ because the singularities in $\hat{R}$ are rational singularities after a double covering. Hence they are isolated (-2)-curves in $\hat{R}$ as well. □

If we take away from the branch locus $\hat{R}$ all the isolated vertical (-2)-curves, we obtain a divisor $\hat{R}_p$ which is called the *principal part* of $\hat{R}$. The *second singularity index* (or more precisely, the index of negligible singularities) $s_2(f)$ of the hyperelliptic fibration $f\colon S \to C$ is defined as

$$s_2(f) = \hat{R}_p^2 + \hat{R}_p K_{\hat{P}/C} .$$

Since $\hat{R} - \hat{R}_p$ is the sum of l isolated vertical (-2)-curves, we have

$$(\hat{R} - \hat{R}_p)^2 = \hat{R}(\hat{R} - \hat{R}_p) = -2l ,$$
$$(\hat{R} - \hat{R}_p) K_{\hat{P}/C} = 0 .$$

Hence

(3)
$$s_2(f) = \hat{R}^2 + \hat{R} K_{\hat{P}/C} + 2l = 4\tilde{\delta}^2 + 2\tilde{\delta} K_{\hat{P}/C} + 2l$$
$$= 2n(2g+1) - 4\sum k_i^2 + 2\sum k_i + 2l .$$

Substituting (1) and (2) by (3), we have the following proposition.

PROPOSITION 1.1 (cf. [7, Theorem 5.1.7]). *If $f\colon S \to C$ be a relatively minimal hyperelliptic fibration of genus $g \geq 2$, then*

$$(2g+1)\chi_f = (2g+1)\chi_{\tilde{f}} = \frac{g}{4} s_2(f) - \frac{g}{2} l - \frac{1}{2} \sum k_i(k_i - 1) ,$$

$$(2g+1) K_{S/C}^2 = (g-1) s_2(f) + 3l + \frac{1}{2} \sum (3g^2 - 2g - 1 - 3(g+1-2k_i)^2) .$$

Since afterwards we need only the formula for $K_{S/C}^2$, for simplicity by abuse of language, we define the *higher order singularity index* (or more exactly, the index of non-negligible singularities) $s_h(f)$ as

$$s_h(f)=3l+\frac{1}{2}\sum(3g^2-2g-1-3(g+1-2k_i)^2)\,.$$

The contributions of each fiber F of π to the singularity indices $s_h(f)$ or $s_2(f)$ are referred to as $s_h(F)$ or $s_2(F)$ respectively.

Now we will show how to calculate $s_2(f)$ and $s_2(F)$. Let $f\colon S\to C$ be a fibration, and D an effective divisor on S. If D is a non-vertical (i.e., $f(D)=C$) smooth irreducible curve, then $f|_D\colon D\to C$ is a finite cover of C with degree DF where F is a fiber of f. The ramification index of this cover can be calculated by the following formula:

$$D^2+DK_{S/C}=2p_g(D)-2-(2g(C)-2)DF\,.$$

If D is vertical (i.e., $f(D)$ is a point), then

$$D^2+DK_{S/C}=2p_g(D)-2=-\chi_{\mathrm{top}}(D)\,.$$

For any reduced effective divisor D, we will define the *relative ramification index* of the divisor D with respect to C as $D^2+DK_{S/C}$. Let $\tilde{f}\colon \tilde{S}\to S$ be an embedded resolution of singularities of D in S and $\tilde{D}$ the strict transform of D. Then $\tilde{D}$ is a disjoint union of smooth irreducible curves. Assume that the center of the i-th blow-up in $\tilde{f}$ is a singular point of multiplicity m_i in D or in the successive strict transform of D. Then

$$D^2+DK_{S/C}=\tilde{D}^2+\tilde{D}K_{\tilde{S}/C}+\sum m_i(m_i-1)\,.$$

Therefore the relative ramification index of a reduced divisor D is just the sum of the ramification index of $\tilde{D}$ with respect to C and of the double of the difference between the arithmetic genera of D and $\tilde{D}$. If $\tilde{D}$ contains vertical components, then their contribution to the ramification index is equal to the negative of their Euler characteristic. In this way we can calculate explicitly the singularity index $s_2(f)$ and $s_2(F)$.

For a subgroup $G\subseteq\mathrm{Aut}(f)$ we have two exact sequences

$$1\longrightarrow K\longrightarrow G\longrightarrow H\longrightarrow 1\,,$$
$$1\longrightarrow \boldsymbol{Z}_2\longrightarrow K\longrightarrow \tilde{K}\longrightarrow 1\,,$$

where $H\subseteq\mathrm{Aut}(C)$, $K=\{(\tilde{\alpha},\mathrm{id})\in G\}$, and $\tilde{K}\subseteq\mathrm{Aut}(\pi\hat{\psi}\tilde{\psi})$ is induced by K.

LEMMA 1.3. *$\tilde{P}$ can be contracted to a minimal ruled surface $\pi\colon P\to C$ (see the diagram $(*)$) which satisfies the following conditions*:

(1) *Let (R,δ) be the image of $(\tilde{R},\tilde{\delta})$ in P. Then $\hat{\psi}\tilde{\psi}\colon\tilde{P}\to P$ is the minimal even resolution of R.*

(2) *Let R_h be the non-vertical part of R. Then the multiplicity of any singular point in R_h cannot be greater than $g+1$.*

(3) *There exists a finite subset $\Sigma=\{p_1,\dots,p_s\}\subseteq C$ such that after having blown*

up the singular points on each fiber $\pi^{-1}(p_i)$ *which have the highest multiplicity, one gets a ruled surface* $\bar{\pi}: \bar{P} \to C$ *such that* $\bar{\pi}$ *is compatible with* $\tilde{K}$, *i.e.,* $\tilde{K}$ *can induce a subgroup* $\bar{K} \subseteq \mathrm{Aut}(\bar{\pi})$. *If* $\Sigma = \emptyset$, *then the minimal ruled surface is said to be compatible with* $\tilde{K}$.

Proof. Let $\tilde{F}$ be a fiber of $\hat{\psi}\tilde{\psi}: \tilde{P} \to C$ which is not irreducible. Let $\hat{\psi}\tilde{\psi}(\tilde{F}) = p$. The set of (-1)-curves on $\tilde{F}$ can be divided into $\tilde{K}$-orbits. We have the following three cases.

(a) Some $\tilde{K}$-orbit contains more than one (-1)-curves and they meet one another. Then we have $\tilde{F} = E + E'$, $EE' = 1$ and $\{E, E'\}$ forms a $\tilde{K}$-orbit. We can choose one of them, for example, E.

(b) $\tilde{F}$ has more than one (-1)-curves and all (-1)-curves in any $\tilde{K}$-orbit are disjoint. We can choose a $\tilde{K}$-orbit such that the intersection number of its (-1)-curve with $\tilde{R}_h$ (the horizontal part of $\tilde{R}$) is minimal.

(c) $\tilde{F}$ has only one (-1)-curve E which is stable under $\tilde{K}$. The multiplicity of E in $\tilde{F}$ is greater than 1. We will choose E.

Therefore for the so chosen (-1)-curve E we have

$$\tilde{R}_h E \leq \frac{1}{2} \tilde{R}\tilde{F} = g + 1 .$$

Contracting the above chosen (-1)-curve (or all the (-1)-curves in a $\tilde{K}$-orbit as in the Case (b)), we get a morphism $\psi_1: \tilde{P} \to P_1$. P_1 is still a ruled surface and $\tilde{R}$, $\tilde{R}_h$, will be contracted to divisors R_1, $R_{h,1}$ in P_1. The multiplicities of singular points in $R_{h,1}$ will not be greater than $g+1$. In the Cases (b) and (c) $\tilde{K}$ induces a subgroup of the automorphism group of $P_1 \to C$. In the Case (a) $\tilde{F}$ will be contracted to a projective line in P_1 and $\tilde{K}$ induces a subgroup of the automorphism group of $P_1 - \psi_1(\tilde{F}) \to C - \{p\}$. Replacing $\tilde{P}$, $\tilde{R}$ and $\tilde{R}_h$ by P_1, R_1 and $R_{h,1}$ respectively, this process can be continued inductively. Finally we will obtain a needed minimal ruled surface.

The statement of (1) is obvious by the uniqueness of minimal even resolution. □

2. Local analysis.

PROPOSITION 2.1. *Suppose that* f *is not locally trivial and that there exists a minimal ruled surface* $\pi: P \to C$ *which is compatible with* $\tilde{K}$. *If* R_h *is étale, then*

$$|G| \leq \begin{cases} \dfrac{4(g+1)}{g-1} rK_{S/C}^2 & \text{if } g \neq 2, 3, 5, 9 \\[2ex] \dfrac{24}{g-1} rK_{S/C}^2 & \text{if } g = 2, 3 \\[2ex] \dfrac{60}{g-1} rK_{S/C}^2 & \text{if } g = 5, 9 \end{cases}$$

where

$$r=\min_{s_2(F)\neq 0}|\mathrm{Stab}_H\pi(F)|\,.$$

PROOF. Let F_0 be a fiber of π with $s_2(F_0)\neq 0$ such that $r=|\mathrm{Stab}_H\pi(F_0)|$. Then F_0 must be a component of R. Hence $s_2(F_0)=2(2g+1)$, we have

$$K_{S/C}^2\geq 2(g-1)\frac{|H|}{r}\,,$$

$$|G|=2|\bar{K}||H|\leq\frac{|\bar{K}|}{g-1}rK_{S/C}^2\,.$$

By [4] and [5] we have

$$|\bar{K}|\leq\begin{cases}4g+4 & \text{if}\quad g\neq 2,3,5,9\\ 24 & \text{if}\quad g=2,3\\ 60 & \text{if}\quad g=5,9\,.\end{cases}$$

□

PROPOSITION 2.2. *Suppose that the minimal ruled surface $\pi\colon P\to C$ satisfies the conditions of Lemma* 1.3. *If there exists a fiber F of π such that $s_h(F)\neq 0$ and $\pi(F)\notin\Sigma$, then*

$$|G|<\frac{4(g+1)}{g-1}rK_{S/C}^2\,,$$

where

$$r=\min_{\substack{s_h(F)\neq 0\\ \pi(F)\notin\Sigma}}|\mathrm{Stab}_H\pi(F)|\,.$$

PROOF. Let F_0 be a fiber of π with $s_h(F_0)\neq 0$ such that $r=|\mathrm{Stab}_H\pi(F_0)|$ and $\pi(F_0)\notin\Sigma$. Then $\bar{K}$ must be a dihedral group or a cyclic group. If there is a non-negligible singular point of R outside the poles of F_0, then we must have $k_i\geq 2$ or $k_1=1$ but $k_2=2$ (i.e., a $(3\to 3)$ singular point). Thus

$$s_h(F_0)\geq\frac{1}{2}(3g^2-2g-1-3(g+1-4)^2)\cdot\frac{|\bar{K}|}{2}\cdot\frac{|H|}{r}=(4g-7)\frac{|\bar{K}||H|}{r}\,.$$

Therefore

$$|G|\leq\frac{2(2g+1)}{4g-7}rK_{S/C}^2<\frac{4(g+1)}{g-1}rK_{S/C}^2\qquad\text{when}\quad g\geq 2\,.$$

Now suppose that R has a non-negligible singular point at the poles of F_0. The strategy of the proof can be described as follows. Let $p\in F_0$ be a singular point at a pole. Let $s_2(p)$, $s_h(p)$ be the contribution of p to the singularity indices of F_0. Then we have

$$K_{S/C}^2 \geq \frac{1}{2g+1}((g-1)s_2(F_0)+s_h(F_0)) \cdot \frac{|H|}{r},$$

$$|G| \leq 2|\bar{K}||H| \leq \frac{2(2g+1)|\bar{K}|}{(g-1)s_2(F_0)+s_h(F_0)} rK_{S/C}^2.$$

The following inequality we need

$$\frac{2(2g+1)|\bar{K}|}{(g-1)s_2(F_0)+s_h(F_0)} < \frac{4(g+1)}{g-1}$$

is equivalent to the inequality

$$2(g+1)((g-1)s_2(F_0)+s_h(F_0))-(g-1)(2g+1)|\bar{K}|>0.$$

In fact we will show that

$$2(g+1)((g-1)s_2(p)+s_h(p))-(g-1)(2g+1)|\bar{K}|>0.$$

The group $\bar{K}$ may be a cyclic group $\boldsymbol{Z}_m$ or a dihedral group D_{2m}. If $\bar{K} \cong D_{2m}$, then the two poles are isomorphic singular points. Hence $s_2(F_0) \geq 2s_2(p)$ and $s_h(F_0) \geq 2s_h(p)$. It is evident that if we can show the inequality for the group $\boldsymbol{Z}_m$, the same is true for D_{2m}. So from now on in the proof we will assume $\bar{K} \cong \boldsymbol{Z}_m$.

Since p is a pole of $\boldsymbol{P}^1$, the leading part of the local equation of R_h at p is x^a+t^b or $x(x^a+t^b)$, where the local equation of F_0 is $t=0$ and $|\bar{K}|\,|\,a$. Since the singularity index at p of the case x^a+t^b is less than other cases (for example, the cases $x(x^a+t^b)$, $t(x^a+t^b)$ or $xt(x^a+t^b)$). We assume that the leading part of the local equation of R at p is x^a+t^b. Then we have several cases.

(a) $a<b$ and $a \leq g+1$. If $a=3$, there is a $(3 \to 3)$ singular point at p, hence $k_1=1$, $k_2=2$, $l \geq 1$. We have

$$s_h(p) \geq \frac{1}{2}[(3g^2-2g-1-3(g+1-2)^2)+(3g^2-2g-1-3(g+1-4)^2)]+3=10g-13.$$

Hence

$$2(g+1)(10g-13)-(g-1)(2g+1) \cdot 3 \geq 14g^2-3g-23>0 \qquad \text{if} \quad g \geq 2.$$

If $a \geq 4$, then $k_1 \geq (a-1)/2$ and $g \geq 3$. We have

$$\begin{aligned} s_h(p) &\geq \frac{1}{2}(3g^2-2g-1-3(g+1-(a-1))^2) \\ &= -1.5a^2+3(g+2)a-7g-6.5. \end{aligned}$$

Hence

$$\begin{aligned} &2(g+1)s_h(p)-(g-1)(2g+1) \cdot a \\ &\quad \geq -3(g+1)a^2+(4g^2+19g+13)a-14g^2-27g-13>0 \qquad \text{if} \quad g \geq 3. \end{aligned}$$

(b) $4 \leq b \leq g+1$ and $a=b+u$ where $0 \leq u \leq b-1$. Let $c=[b/2]$.
If $u \geq 4$, then we have $k_1=c$, $k_2=2$ and $|\bar{K}| \leq a \leq 2b-1 \leq 4c+1$.

$$s_h(p) \geq \frac{1}{2}[(3g^2-2g-1-3(g+1-2c)^2)+(3g^2-2g-1-3(g-3)^2)]$$
$$= -6c^2+6(g+1)c+4g-16 .$$

Hence

$$2(g+1)s_h(p)-(g-1)(2g+1)(4c+1)$$
$$\geq -12(g+1)c^2+4(g^2+7g+4)c+6g^2-23g-31>0 \qquad \text{if} \quad g \geq 3 .$$

If $u=3$ and $c \geq 3$, then we have $k_1=c$, $k_2=1$, $k_3=2$, $l \geq 1$ and $|\bar{K}| \leq a \leq 2c+4$.

$$s_h(p) \geq -6c^2+6(g+1)c+6g-15 .$$

Hence

$$2(g+1)s_h(p)-(g-1)(2g+1)(2c+4)$$
$$\geq -12(g+1)c^2+2(4g^2+13g+7)c+4g^2-14g-26>0 \qquad \text{if} \quad g \geq 3 .$$

If $u=3$ and $c=2$, then we have $k_1=2$, $s_2(p) \geq 6$ and $|\bar{K}| \leq a \leq 8$.

$$s_h(p) \geq 2(g-1) .$$

Hence

$$2(g+1)s_h(p)+2(g+1)(g-1)s_2(p)-(g-1)(2g+1)\cdot 8 \geq 8(g-1)>0 \qquad \text{if} \quad g \geq 2 .$$

If $u=2$, then we have $k_1=c$, $s_2(p) \geq 2$ and $|\bar{K}| \leq a \leq 2c+3$.

$$s_h(p) \geq -6c^2+6(g+1)c-4g-2 .$$

Hence

$$2(g+1)s_h(p)-(g-1)(2g+1)(2c+3)$$
$$\geq -12(g+1)c^2+2(4g^2+13g+7)c-10g^2-9g-5>0 \qquad \text{if} \quad g \geq 3 .$$

If $0 \leq u \leq 1$, then we have $k_1=c$ and $|\bar{K}| \leq a \leq 2c+2$.

$$s_h(p) \geq -6c^2+6(g+1)c-4g-2 .$$

Hence

$$2(g+1)s_h(p)-(g-1)(2g+1)(2c+2)$$
$$\geq -12(g+1)c^2+2(4g^2+13g+7)c-12g^2-10g-2>0 \qquad \text{if} \quad g \geq 4 .$$

If $g=3$ in this case, then $b=4$ and $|\bar{K}| \leq a \leq 5$. Hence

$$2(g+1)s_h(p)-(g-1)(2g+1)\cdot 5 \geq 10 .$$

(c) $4 \le b = 2c \le g+1$ and $a = qb + u$ where $0 \le u \le b-1$, $q \ge 2$. In this case $|\bar{K}| \le a \le 2c(q+1)-1$. We have $k_1 = \cdots = k_q = c$.

$$s_h(p) \ge \frac{1}{2}(3g^2 - 2g - 1 - 3(g+1-2c)^2)q$$
$$= (-6c^2 + 6(g+1)c - 4g - 2)q .$$

Hence

$$\begin{aligned} &2(g+1)s_h(p) - (g-1)(2g+1)(2c(q+1)-1) \\ &\quad \ge -12q(g+1)c^2 + 2[(4q-2)g^2 + (13q+1)g + 7q + 1]c \\ &\qquad -2(4q-1)g^2 - (12q+1)g - 4q - 1 > 0 \qquad \text{if} \quad g \ge 3 . \end{aligned}$$

(d) $5 \le b = 2c+1 \le g+1$ and $a = 2qb + u$ where $0 \le u \le b-1$, $q \ge 1$. In this case $|\bar{K}| \le a \le (2c+1)(2q+1)-1$. We have $k_1 = k_3 = \cdots = k_{2q-1} = c$, $k_2 = k_4 = \cdots = k_{2q} = c+1$, $l \ge q$.

$$s_h(p) \ge (-12c^2 + 12gc - 2g - 1)q .$$

Hence

$$\begin{aligned} &2(g+1)s_h(p) - (g-1)(2g+1)((2c+1)(2q+1)-1) \\ &\quad \ge -24q(g+1)c^2 + 2(8qg^2 - 2g^2 + 14qg + g + 2q + 1)c \\ &\qquad -4qg(2g+1) > 0 \qquad \text{if} \quad g \ge 4 . \end{aligned}$$

(e) $5 \le b = 2c+1 \le g+1$ and $a = (2q+1)b + u$ where $0 \le u \le b-1$, $q \ge 1$. In this case $|\bar{K}| \le a \le (2c+1)(2q+2)-1$. We have $k_1 = k_3 = \cdots = k_{2q+1} = c$, $k_2 = k_4 = \cdots = k_{2q} = c+1$, $l \ge q$.

$$s_h(p) \ge -6(2q+1)c^2 + 6(2qg + g + 1)c - 2qg - 4g - q - 2 .$$

Hence

$$\begin{aligned} &2(g+1)s_h(p) - (g-1)(2g+1)((2c+1)(2q+2)-1) \\ &\quad \ge -12(2q+1)(g+1)c^2 + 4(4qg^2 + g^2 + 7qg + 7g + q + 4)c \\ &\qquad -8qg^2 - 10g^2 - 4qg - 11g - 3 > 0 \qquad \text{if} \quad g \ge 4 . \end{aligned}$$

(f) $b = 3 \le g+1$ and $a = (2q+1)b + u$ where $0 \le u \le 2$, $q \ge 0$. In this case $|\bar{K}| \le a \le 6q+5$. We have $k_1 = k_3 = \cdots = k_{2q-1} = c$, $k_2 = k_4 = \cdots = k_{2q} = c+1$, $l \ge q$.

$$s_h(p) \ge (10g - 13)q .$$

Hence

$$\begin{aligned} &2(g+1)s_h(p) - (g-1)(2g+1)(6q+5) \\ &\quad \ge (8q+2)g^2 + 5g - 20q - 7 > 0 \qquad \text{if} \quad g \ge 2 . \end{aligned}$$

□

PROPOSITION 2.3. *Suppose that there exists a minimal ruled surface* $\pi: P\to C$ *which is compatible with* $\tilde{K}$. *If* R_h *is not étale and* $s_h(f)=0$, *then*

(1) $|G|<(4(g+1)/(g-1))rK_{S/C}^2$, *where*

$$r=\max_{s_2(F)\neq 0}|\mathrm{Stab}_H(F)|.$$

(2) $|G|<(20(g+1)/(g-1))K_{S/C}^2$ *if* $g(C)=0$.

PROOF. (1) First we fix some notation. Let C_0 be a section of π with the least self-intersection number $C_0^2=-e$, and F a general fiber of π. Let $R_h\sim 2(g+1)C_0+nF$. Since R_h is not étale, we have $n>0$.

$\bar{K}$ may be a dihedral group or a cyclic group. Assume that $\bar{K}=\boldsymbol{Z}_m$. If R_h has a singular point $p\in F_0$ which is not a pole, then $s_2(p)\geq 2$, $s_2(F_0)\geq 2m$. If p is a pole, then $s_2(p)\geq 2+(m-1)=m+1$. Hence

$$\begin{aligned}&(g-1)[2(g+1)s_2(F_0)-(2g+1)m]\\&\quad\geq(g-1)[2(m+1)(g+1)-m(2g+1)]>0\,.\end{aligned}$$

Now assume that R_h is smooth. If F_0 is tangent to R_h at a point outside the poles (or at any point when $m=1$), then $s_2(F_0)\geq m$. We have

$$(g-1)[2(g+1)s_2(F_0)-(2g+1)m]\geq m(g-1)>0\,.$$

Let $C_\infty\sim C_0+eF$ be a section which is stable under the action of $\bar{K}$. Since $C_\infty R_h=n>0$, there exists a fiber F_0 tangent to R_h at a pole p_0, hence $s_2(p_0)=m-1$. If C_0 meets R_h, then there exists a fiber F_1 tangent to R_h at a pole p_1. Hence

$$\begin{aligned}K_{S/C}^2&\geq\frac{g-1}{2g+1}\left(\frac{m-1}{r_0}+\frac{m-1}{r_1}\right)|H|\\&\geq\frac{2(g-1)(m-1)}{2g+1}\cdot\frac{|H|}{r},\\|G|&\leq 2\cdot m\cdot\frac{2g+1}{2(g-1)(m-1)}rK_{S/C}^2\\&<\frac{4(g+1)}{g-1}rK_{S/C}^2\,.\end{aligned}$$

Next assume that C_0 does not meet R_h. If C_0 is not a component of R_h, then $n=2(g+1)e$, i.e., $R_h\sim 2(g+1)(C_0+eF)$. Then relative ramification index of R_h will be

$$R_h^2+R_hK_{P/C}=4(g+1)[(g+1)e+2g(C)-2]\,.$$

Since the sum of ramification indices on C_∞ is equal to $n(m-1)=2(g+1)(m-1)e<R_h^2+R_hK_{P/C}$, there exists another fiber of π which is tangent to R_h at a point outside poles. If C_0 is a component of R_h, then $n=(2g+1)e$, i.e., $R_h-C_0\sim(2g+1)(C_0+eF)$.

The relative ramification index of $R_h - C_0$ will be

$$(2g+1)[(2g+1)e+4g(C)-4]\,.$$

Since the sum of ramification indices on C_∞ is equal to $(2g+1)(m-1)e$ which is less than the relative ramification index of R_h (note that $m \leq 2g+1$), there exists another fiber of π which is tangent to R_h at a point outside poles.

Therefore we have shown

$$|G| < \frac{4(g+1)}{g-1}\, r K_{S/C}^2$$

when $\bar{K} \cong \boldsymbol{Z}_m$.

Now assume that $\bar{K} = D_{2m}$. In this case we have $e=0$. If R_h has a singular point, then by the same argument as above we can show

$$|G| < \frac{4(g+1)}{g-1}\, r K_{S/C}^2\,.$$

Now we assume R_h is smooth. Since $C_0 R_h = n > 0$, there exists a fiber F_0 of π which meets R_h at 2 poles. The second singularity index of these points is greater than or equal to $m-1$. There exist also fibers F_1 and F_2 which meet R_h at m points whose stabilizer in $\bar{K}$ is isomorphic to $\boldsymbol{Z}_2$. Let $r_i = |\mathrm{Stab}_H \pi(F_i)|$, $i=0, 1, 2$. Then

$$\begin{aligned} K_{S/C}^2 &\geq \frac{g-1}{2g+1}\left(\frac{2(m-1)}{r_0} + \frac{m}{r_1} + \frac{m}{r_2}\right)|H| \\ &\geq \frac{2(g-1)(2m-1)}{2g+1} \cdot \frac{|H|}{r}\,, \end{aligned}$$

$$\begin{aligned} |G| &\leq 2\cdot(2m)\cdot\frac{2g+1}{2(g-1)(2m-1)}\, r K_{S/C}^2 \\ &< \frac{4(g+1)}{g-1}\, r K_{S/C}^2\,. \end{aligned}$$

(2) If H is not dihedral or cyclic, then we have $r=5$, and by the inequality of (1), the conclusion of (2) is true. Now assume that H is dihedral or cyclic. Since a rational fibration has at least three singular fibers, we may assume that F_0 is a fiber with $s_2(F_0) > 0$ with $|\mathrm{Stab}_H \pi(F_0)| \leq 2$. Let $|\bar{K}| = m$. Then $s_2(F_0) \geq m/2$. Hence

$$K_{S/C}^2 \geq \frac{g-1}{2g+1} \cdot \frac{m}{2} \cdot \frac{|H|}{2} = \frac{g-1}{8(2g+1)}\,|G|\,,$$

$$|G| \leq \frac{8(2g+1)}{g-1}\, K_{S/C}^2 < \frac{20(g+1)}{g-1}\, K_{S/C}^2\,.$$

□

PROPOSITION 2.4. *Suppose that the minimal ruled surface* $\pi: P \to C$ *satisfies the conditions of Lemma* 1.3 *with* $\Sigma \neq \varnothing$. *Then*

(1) $|G| < (4(g+1)/(g-1))rK_{S/C}^2$, *where*

$$r = \max_{p \in C} |\mathrm{Stab}_H(p)| .$$

(2) $|G| < (20(g+1)/(g-1))K_{S/C}^2$ *if* $g(C)=0$.

PROOF. By Lemma 1.3(3), there exists a finite subset $\Sigma = \{p_1, \dots, p_s\} \subseteq C$ such that after having blown up the singular points on each fiber $\pi^{-1}(p_i)$ which have the highest multiplicity, one gets a ruled surface $\bar{\pi}: \bar{P} \to C$ such that $\bar{\pi}$ is compatible with $\tilde{K}$, i.e., $\tilde{K}$ can induce a subgroup $\bar{K} \subseteq \mathrm{Aut}(\bar{\pi})$. Let $\bar{\pi}^{-1}(p_1) = \Gamma_1 + \Gamma_2$ with $\Gamma_i R_h = g+1$, $i=1,2$. Blowing down Γ_1, we get $F_1 = \pi^{-1}(p_1)$ which has a singular point of R_h with multiplicity $g+1$. If $g+1$ is an odd number, then one and only one of the components Γ_1 and Γ_2 will belong to the ramification divisor $\bar{R}$. This is impossible because these 2 components are symmetric. Thus $g+1$ must be even. We have

$$\begin{aligned} s_h(F_1) &\geq \frac{1}{2}(3g^2 - 2g - 1 - 3(g+1-(g+1))^2) \\ &= 1.5g^2 - g - 0.5 . \end{aligned}$$

If Σ contains more than one H-orbits, then

$$K_{S/C}^2 \geq \frac{1.5g^2 - g - 0.5}{2g+1} \cdot 2 \cdot \frac{|H|}{r} ,$$

$$|G| \leq \frac{4(2g+1)(g+1)}{3g^2 - 2g - 1} rK_{S/C}^2 < \frac{4(g+1)}{g-1} rK_{S/C}^2 .$$

From now on we suppose that Σ itself is an H-orbit. If there is a fiber F_2 of π with $s_h(F_2) > 0$ and $\pi(F_2) \notin \Sigma$, then by Proposition 2.2 we are done. By assumption there is an $\alpha \in \tilde{K}$ such that $\alpha(\Gamma_1) = \Gamma_2$. Let

$$N = \{\gamma \in \tilde{K} \mid \gamma(\Gamma_i) = \Gamma_i, i=1,2\} .$$

We have an exact sequence

$$1 \to N \to \tilde{K} \to \mathbf{Z}_2 \to 1 .$$

Since the intersection point of Γ_1 with Γ_2 is fixed by the action of N, N must be cyclic. Similarly, $\tilde{K}$ must be cyclic or dihedral. There exist two sections $\bar{C}_0$ and $\bar{C}_\infty$ of $\bar{\pi}$ which are stable by N. Let $\bar{p}_0 = \Gamma_1 \cap \bar{C}_0$, $\bar{p}_\infty = \Gamma_2 \cap \bar{C}_\infty$. If $\tilde{K}$ is cyclic, then $\alpha^2 \in N$. But α stabilizes the sections $\bar{C}_0$ and $\bar{C}_\infty$, so $\alpha(\bar{p}_0) = \bar{p}_0$, $\alpha(\bar{p}_\infty) = \bar{p}_\infty$. This contradicts the fact that $\alpha(\Gamma_1) = \Gamma_2$. Therefore $\tilde{K}$ must be dihedral and α is an involution.

Since Σ is an H-orbit, the factor space $P' = \bar{P}/\langle\alpha\rangle$ is a minimal ruled surface. Let $\pi': P' \to C$ be the ruling, and let $\bar{P} \to P'$ be the double covering with branch locus

$B' \sim 2C_0' + mF'$ where C_0' is a section of π' having the least self-intersection number $(C_0')^2 = -e'$. As $\bar{P}$ is smooth, the branch locus B' is smooth as well. Moreover, B' is tangent to the fibers of π' over Σ. Thus we have $e' > 0$ and $m \geq 2e'$. Since the two sections $\bar{C}_0$ and $\bar{C}_\infty$ in $\bar{P}$ do not meet the ramification divisor, there is at least one section in P' which does not meet B'. Hence $m = 2e'$. Let $r_1 = |\mathrm{Stab}_H(p_1)|$. Then we have

$$B'^2 + B'K_{P'/C} = 2e' = \frac{|H|}{r_1}.$$

Let R_h' be the image of $\bar{R}_h$ in P'. Then

$$R' \sim (g+1)C_0' + sF'.$$

Now we distinguish two cases.

(a) There is a fiber F_2 of π, $\pi(F_2) \notin \Sigma$ such that B' meets R_h' on the image of F_2. Then we have $s_2(F_2) \geq |\bar{K}|/2$. Let $r_2 = |\mathrm{Stab}_H \pi(F_2)|$. Then we have

$$\begin{aligned} K_{S/C}^2 &\geq \frac{1.5g^2 - g - 0.5}{2g+1} \cdot \frac{|H|}{r_1} + \frac{(g-1)|\bar{K}|}{2(2g+1)} \cdot \frac{|H|}{r_2} \\ &\geq \frac{(g-1)(3g+1+|\bar{K}|)}{4(2g+1)|\bar{K}|} \cdot \frac{|G|}{r}. \end{aligned}$$

Since $|\bar{K}| \leq 2g+2$, we have

$$(g+1)(3g+1+|\bar{K}|) - (2g+1)|\bar{K}| \geq (g+1)^2 > 0.$$

Namely,

$$|G| \leq \frac{4(2g+1)|\bar{K}|}{(g-1)(3g+1+|\bar{K}|)} rK_{S/C}^2 < \frac{4(g+1)}{g-1} rK_{S/C}^2.$$

(b) All the intersection points of R_h' with B' are on the fibers over Σ. If C_0' is a component of R_h', then $R_h' - C_0' \sim gC_0' + sF'$. Since $s \geq ge'$, we have

$$(R_h' - C_0')B' = 2s \geq 2e' \cdot g.$$

It implies that the multiplicity of $\bar{R}_h$ on the interesection point $\Gamma_1 \cap \Gamma_2$ is at least g. Since $g+1$ is even, we have $k_1 = (g+1)/2$, $k_2 \geq (g-1)/2$, that is,

$$s_h(F_1) \geq 3g^2 - 4g - 7.$$

As $|\bar{K}| \leq 2g$,

$$2(g+1)s_h(F_1) - (g-1)(2g+1)|\bar{K}| \geq g^3 - 10g - 7 > 0 \quad \text{when} \quad g \geq 4.$$

Thus

$$|G| \leq \frac{2(2g+1)|\bar{K}|}{s_h(F_1)} \cdot r_1 K_{S/C}^2 < \frac{4(g+1)}{g-1} rK_{S/C}^2 \quad \text{when} \quad g \geq 4.$$

If $g=3$, then $s_h(F_1)=10$, $s_2(F_1)=6$. We have

$$K_{S/C}^2 \geq \frac{6(g-1)+10}{2g+1} \cdot \frac{|H|}{r_1} \geq \frac{11}{7|\bar{K}|} \cdot \frac{|G|}{r} .$$

$$|G| \leq \frac{42}{11} rK_{S/C}^2 < \frac{4(g+1)}{g-1} rK_{S/C}^2 .$$

If C_0' is not a component of R_h', then $s \geq (g+1)e'$. We have

$$R_h'B' = 2s \geq 2e' \cdot (g+1) .$$

It implies that the multiplicity of $\bar{R}_h$ on the intersection point $\Gamma_1 \cap \Gamma_2$ is at least $g+1$. Since $g+1$ is even, we have $k_1 = k_2 = (g+1)/2$, that is,

$$s_h(F_1) \geq 3g^2 - 2g - 1 .$$

As $|\bar{K}| \leq 2g+2$,

$$2(g+1)s_h(F_1) - (g-1)(2g+1)|\bar{K}| \geq g(g^2-1) > 0 .$$

Thus

$$|G| \leq \frac{2(2g+1)|\bar{K}|}{s_h(F_1)} \cdot r_1 K_{S/C}^2 < \frac{4(g+1)}{g-1} rK_{S/C}^2 .$$

Finally assume that $g(C)=0$. If H is not dihedral or cyclic, then we have $r=5$, hence by the inequality of (1), the conclusion of (2) is true. Now assume that H is dihedral or cyclic. Since a rational fibration has at least three singular fibers, we may assume that F_1 is a fiber such that $\pi(F_1) \in \Sigma$ and $|\mathrm{Stab}_H \pi(F_1)| \leq 2$. We know that $s_h(F_1) \geq 1.5g^2 - g - 0.5$, $|\bar{K}| \leq 2(g+1)$. Hence

$$K_{S/C}^2 \geq \frac{s_h(F_1)}{2g+1} \cdot \frac{|H|}{2} = \frac{s_h(F_1)}{8(2g+1)|\bar{K}|} |G| .$$

Since

$$10(g+1)s_h(F_1) - 2(g-1)(2g+1)|\bar{K}| \geq (g^2-1)(7g+1) > 0 ,$$

we get

$$|G| \leq \frac{8(2g+1)|\bar{K}|}{2s_h(F_1)} K_{S/C}^2 < \frac{20(g+1)}{g-1} K_{S/C}^2 .$$

□

3. Proof of the Main Theorem. By Propositions 2.1 to 2.4, we have

$$|G|\leq\begin{cases}\dfrac{4(g+1)}{g-1}rK^2_{S/C} & \text{if } \ g\neq 2,3,5,9\\[2mm] \dfrac{24}{g-1}rK^2_{S/C} & \text{if } \ g=2,3\\[2mm] \dfrac{60}{g-1}rK^2_{S/C} & \text{if } \ g=5,9\end{cases}$$

where

$$r=\max_{p\in C}|\mathrm{Stab}_H(p)|\,.$$

Or equivalently,

$$|G|\leq\frac{A}{g-1}rK^2_{S/C}\,,$$

where

$$|\bar{K}|\leq A=\begin{cases}4g+4 & \text{if } \ g\neq 2,3,5,9\\ 24 & \text{if } \ g=2,3\\ 60 & \text{if } \ g=5,9\,.\end{cases}$$

We distinguish three cases.

(a) $g(C)\geq 2$. Since H is a subgroup of Aut(C), H determines a finite morphism $\tau\colon C\to X=C/H$. Denote the ramification indices by r_i. Then Hurwitz's theorem implies that

$$2g(C)-2=n(2g(X)-2)+n\sum\left(1-\frac{1}{r_i}\right).$$

Let

$$\varphi(g(X),s,r_1,\dots,r_s)=2g(X)-2+\sum_{i=1}^{s}\left(1-\frac{1}{r_i}\right)>0\,,$$

where $g(X)\geq 0$, $s\geq 0$, $r_i\geq 2$, $i=1,\dots,s$ are integers. By calculation we can see

$$\varphi(0,3,2,3,7)=\frac{1}{42}\,,$$

$$\varphi(0,3,2,3,8)=\frac{1}{24}\,,$$

$$\varphi(g(X),s,r_1,\dots,r_s)\geq\frac{1}{20}\qquad\text{otherwise}\,.$$

Thus

$$\begin{cases} |H|=84(g(C)-1) & \text{when } r_1=2,\quad r_2=3,\quad r_3=7\,, \\ |H|=48(g(C)-1) & \text{when } r_1=2,\quad r_2=3,\quad r_3=8\,, \\ |H|\le 40(g(C)-1) & \text{otherwise}\,. \end{cases}$$

If $|H|=84(g(C)-1)$, then $r=7$. We have

$$\begin{aligned} |G| &\le \frac{A}{g-1}\, rK_{S/C}^2=\frac{7A}{g-1}\,(K_S^2-8(g-1)(g(C)-1)) \\ &=\frac{7A}{g-1}\,K_S^2-\frac{2A|H|}{3}=\frac{7A}{g-1}\,K_S^2-\frac{A}{3|\bar{K}|}\cdot|G| \\ &\le\frac{7A}{g-1}\,K_S^2-\frac{|G|}{3}\,, \end{aligned}$$

hence

$$|G|\le\frac{21}{4}\cdot\frac{A}{g-1}K_S^2\,.$$

If $|H|=48(g(C)-1)$, then $r=8$. Similarly we have

$$|G|\le\frac{8A}{g-1}\,K_S^2-\frac{2|G|}{3}\,,$$

hence

$$|G|\le\frac{25}{5}\cdot\frac{A}{g-1}\,K_S^2<\frac{21}{4}\cdot\frac{A}{g-1}K_S^2\,.$$

If $|H|\le 40(g(C)-1)$ and R_h is not étale, then $|K|\le 4(g+1)$. Hence

$$\begin{aligned} |G| &\le 4(g+1)|H|\le 160(g+1)(g(C)-1)\le\frac{20(g+1)}{g-1}\,K_S^2 \\ &<\frac{21}{4}\cdot\frac{A}{g-1}K_S^2\,. \end{aligned}$$

If R_h is étale, then since f is not locally trivial, R must contain some fiber F_0. By Proposition 2.1, $s_2(F_0)=2(2g+1)$. Let $p=f(F_0)$, and $n=|H|$. H determines a finite morphism $\tau\colon C\to X=C/H$. Denote the ramification index of $p\in C$ with respect to τ by r and the other ramification indices by r_i. Then Hurwitz's theorem implies that

$$2g(C)-2=n(2g(X)-2)+n\sum\left(1-\frac{1}{r_i}\right).$$

As the H-orbit of the point p has n/r points, this implies that $s_2(f)\geq 2(2g+1)n/r$. Hence

$$\begin{aligned} K_S^2 &\geq \frac{g-1}{2g+1}\, s_2(f)+8(g-1)(g(C)-1) \\ &\geq \frac{2(g-1)n}{r}+4(g-1)n\left[2g(X)-2+\sum\left(1-\frac{1}{r_i}\right)\right] \\ &=4(g-1)n\left[2g(X)-2+\frac{1}{2r}+\sum\left(1-\frac{1}{r_i}\right)\right]. \end{aligned}$$

It is not difficult to see that the expression $2g(X)-2+1/2r+\sum(1-1/r_i)$ reaches its minimal value 2/21 (under the condition $2g(X)-2+\sum(1-1/r_i)>0$) when $g(X)=0$, $r_1=2$, $r_2=3$, and $r=r_3=7$. Namely,

$$K_S^2\geq \frac{8}{21}(g-1)n=\frac{8}{21}(g-1)|H|=\frac{4}{21|\bar{K}|}(g-1)|G|.$$

Thus

$$|G|\leq \frac{21}{4}\cdot \frac{|\bar{K}|}{g-1}\, K_{S/C}^2\leq \frac{21}{4}\cdot\frac{A}{g-1}\, K_{S/C}^2 .$$

Therefore we have shown

$$|G|\leq \frac{21}{4}\cdot\frac{A}{g-1}\, K_{S/C}^2$$

in all cases when $g(C)\geq 2$.

(b) $g(C)=1$. Then $r=6$, and we get

$$|G|\leq 6\,\frac{A}{g-1}\, K_{S/C}^2=6\,\frac{A}{g-1}\, K_S^2 .$$

(c) $g(C)=0$. If H is not cyclic or dihedral, then $r=5$, and we get

$$|G|\leq 5\,\frac{A}{g-1}\, K_{S/C}^2 .$$

Now assume that H is dihedral or cyclic. Since a rational fibration has at least three singular fibers, we may assume that F_0 is a fiber with $|\mathrm{Stab}_H\pi(F_0)|\leq 2$ such that $s_h(F_0)>0$ or $s_2(F_0)>0$. If R_h is étale or $s_h(F_0)>0$ and $\pi(F_0)\notin\Sigma$, then

$$|G|\leq 2\frac{A}{g-1}\, K_{S/C}^2$$

by Propositions 2.1 and 2.2. Otherwise we have

$$|G|<5\frac{A}{g-1}K_{S/C}^2$$

by Propositions 2.3 and 2.4. □

4. Examples. To construct the fibrations whose automorphism group attains the maximal order is nearly trivial. Let F be a hyperelliptic curve of genus g such that $|\mathrm{Aut}(F)|=2A$. Let $\varphi: F\to X\cong \boldsymbol{P}^1$ be the double cover determined by the canonical linear system $|K_F|$. Let $B\in \mathrm{Div}(X)$ be the corresponding branch locus. Then $\deg B=2g+2$.

Let C_1 be a Hurwitz curve, and $H_1=\mathrm{Aut}(C_1)$. Then C_1 has an H_1-orbit $\{q_1,\dots,q_m\}$ which contains $m=12(g(C)-1)$ points. Let $D_1=\sum q_i\in \mathrm{Div}(C_1)$. Then $\deg D_1=12(g(C)-1)$.

Let C_2 be an elliptic curve with j-invariant $j(C_2)=0$. Fix a $q_1\in C_2$. Then the order of the group of automorphisms $\mathrm{Aut}(C_2, q_1)$ of C_2 leaving q_1 fixed is equal to 6. Let $H_2'\cong \boldsymbol{Z}_m\oplus \boldsymbol{Z}_m$ be a subgroup of translations of $\mathrm{Aut}(C_2)$. Take an extension subgroup $H_2'\subset H_2\subset \mathrm{Aut}(C)$ such that $H_2/H_2'\cong \mathrm{Aut}(C_2, q_1)$. Then $|H_2|=6m^2$. Let $q_1,\dots,q_{m^2}$ be the orbit of q_1 under H_2 and $D_2=\sum q_i\in \mathrm{Div}(C_2)$. Then $\deg D_2=m^2$.

Let $C_3=\boldsymbol{P}^1$, $q_1,\dots,q_{12}$ be the twelve vertices of an icosahedron. Let $H_3\subset \mathrm{Aut}(C_3)$ be the icosahedral group. Let $D_3=\sum q_i\in \mathrm{Div}(C_3)$. Then $\deg D_3=12$, and $|H_3|=60$.

Now let $P=C\times X$ where $C=C_i$, $i=1, 2, 3$. Taking $R=\mathrm{pr}_1^*D+\mathrm{pr}_2^*B$ where $D=D_i$, $i=1, 2, 3$ as the branch locus, we construct double cover of P. After desingularization, we get a smooth surface S with a hyperelliptic fibration of genus g $f: S\to C$.

When $i=1$, we have $g(C)\geq 2$ and

$$|G|=168A(g(C)-1)\,,\qquad K_S^2=32(g-1)(g(C)-1)\,.$$

When $i=2$, we have $g(C)=1$ and

$$|G|=12Am^2\,,\qquad K_S^2=2(g-1)m^2\,.$$

When $i=3$, we have $g(C)=0$ and

$$|G|=120A\,,\qquad K_{S/C}^2=24(g-1)\,.$$

References

[1] T. Arakawa, Bounds for the order of automorphism groups of hyperelliptic fibrations, Tôhoku Math. J. 50 (1998), 317–323.

[2] A. Beauville, Le nombre minimum de fibres singulières d'une courbe stable sur $\boldsymbol{P}^1$, Astérisque 86 (1981), 97–108.

[3] Z. Chen, Bounds of automorphism groups of genus 2 fibrations, Tôhoku Math. J. 46 (1994), 499–521.

[4] M. Heins, On the number of 1-1 directly conformal maps which a multiply-connected plane region of finite connectivity p (>2) admits onto itself, Bull. Amer. Math. Soc. 52 (1946), 454–457.

[5] R. Tsuji, On conformal mapping of a hyperelliptic Riemann surface onto itself, Kodai Math. Sem. Rep. 10 (1958), 127–136.

[6] G. Xiao, On abelian automorphism group of a surface of general type, Invent. Math. 102 (1990), 619–631.
[7] G. Xiao, The Fibrations of Algebraic Surfaces, Shanghai Scientific & Technical Publishers, 1992 (in Chinese).
[8] G. Xiao, Bound of automorphisms of surfaces of general type, I, Ann. of Math. 139 (1994), 51–77.
[9] G. Xiao, Bound of automorphisms of surfaces of general type, II, J. Algebraic Geom. 4 (1995), 701–793.

Department of Mathematics
East China Normal University
Shanghai 200062
People's Republic of China

E-mail address: zjchen@math.ecnu.edu.cn