SOME TRIGONOMETRICAL SERIES VIII.

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1. G. Sunouchi has proved the following theorem [1]:

THEOREM 1. Let $\Delta = \gamma/\beta \geqq 1$. If

(1) $$\varphi_\beta(t) = o(t^\gamma) \quad (t \to 0),$$

where $\varphi_\beta(t)$ is the β th integral of $\varphi(t)$, and further if

(2) $$\int_0^t |d(u^\Delta \varphi(u))| = O(t) \quad (0 < t < \eta),$$

then the Fourier series of $\varphi(t)$ converges at $t = 0$.

He proved this by a Tauberian theorem due to F. T. Wang. We give a direct proof of this theorem and some generalizations.

2. PROOF OF THEOREM 1. The method of proof is similar as G. Sunouchi's [1], except that the Tauberian theorem is not used.

Let $\alpha = 1/\varepsilon n^{1/\Delta}$, and let

$$\int_0^\pi \varphi(t) \frac{\sin nt}{t} dt = \int_0^\alpha + \int_\alpha^\pi \varphi(t) \frac{\sin nt}{t} dt = I + J.$$

If we put

$$\theta(t) = t^\Delta \varphi(t),\ \Theta(t) = \int_0^t |d\theta(u)|,$$

then we have, by (1) and (2),

(3) $$\Theta(t) = O(t),\ \ \theta(t) = O(t).$$

Hence we have

$$J = \int_\alpha^\pi \varphi(t) \frac{\sin nt}{t} dt = \int_\alpha^\pi \theta(t) \frac{\sin nt}{t^{1+\Delta}} dt$$

$$= -\int_\alpha^\pi \theta(t) d\Lambda(t),$$

where

$$\Lambda(t) = \int_t^\pi \frac{\sin nt}{t^{1+\Delta}} dt = \frac{1}{t^{1+\Delta}} \int_t^\xi \sin nt\, dt = O(1/nt^{1+\Delta}).$$

By integration by parts, we have

$$-J = \Big[\theta(t)\Lambda(t)\Big]_{\alpha}^{\pi} + \int_{\alpha}^{\pi} \Lambda(t)d\theta(t) = J_1 + J_2,$$

say. Then

$$J_1 = O(\alpha/n\alpha^{\Delta+1}) = O(1/n\alpha^{\Delta}) = O(\varepsilon^{\Delta}),$$

$$J_2 = O\Big(\frac{1}{n}\int_{\alpha}^{\pi} \frac{|d\theta(t)|}{t^{1+\Delta}}\Big)$$

$$= O\Big(\frac{1}{n}\Big[\frac{\Theta(t)}{t^{1+\Delta}}\Big]_{\alpha}^{\pi} + \frac{1}{n}\int_{\alpha}^{\pi} \Theta(t)\frac{dt}{t^{2+\Delta}}\Big)$$

$$= O(1/n\alpha^{\Delta}) + O(1/n) + O\Big(\frac{1}{n}\int_{\alpha}^{\pi} \frac{dt}{t^{\Delta+1}}\Big)$$

$$= O(1/n\alpha^{\Delta}) + o(1) = O(\varepsilon^{\Delta}) + o(1).$$

Thus $J = J_1 + J_2$ tends to zero as $n \to \infty$ and then $\varepsilon \to 0$.

Let us now estimate I. For this purpose we distinguish the cases concerning β. Firstly, let $0 < \beta < 1$. By integration by parts, we have

$$I = \int_0^{\alpha} \varphi(t)\frac{\sin nt}{t}\,dt$$

$$= \Big[\varphi_1(t)\,\frac{\sin nt}{t}\Big]_0^{\alpha} + \int_0^{\alpha} \varphi_1(t)\frac{nt\cos nt - \sin nt}{t^2}\,dt$$

$$= I_1 + I_2,$$

say. Since

$$\varphi_1(t) = o(t^{1+\gamma-\beta}) = o(t) \qquad (t \to 0),$$

we have $I_1 = o(1)$. On the other hand

$$I_2 = \int_0^{\alpha} \frac{nt\cos nt - \sin nt}{t^2}\,dt \int_0^{t} \varphi_\beta(u)(t-u)^{-\beta}\,du$$

$$= \int_0^{\alpha} \varphi_\beta(u)\,du \int_u^{\alpha} \frac{nt\cos nt - \sin nt}{t^2}(t-u)^{-\beta}\,dt,$$

where the inner integral becomes

$$n^{1+\beta}\int_{nu}^{n\alpha} \frac{\tau\cos\tau - \sin\tau}{\tau^2}(\tau - nu)^{-\beta}d\tau = O(n^{\beta}/u).$$

Thus we have

$$I_2 = O\left(n^\beta \int_0^\alpha |\varphi_\beta(u)| u^{-1} du \right) = o(n^\beta \alpha^\gamma) = o(1)$$

and hence $I = I_1 + I_2 = o(1)$.

Secondly consider the case $1 < \beta < 2$. We have

$$I = \left[\varphi_1(t) \frac{\sin nt}{t} \right]_0^\alpha + \left[\varphi_2(t) \frac{d}{dt} \frac{\sin nt}{t} \right]_0^\alpha$$

$$- \int_0^\alpha \varphi_2(t) \frac{d^2}{dt^2} \left(\frac{\sin nt}{t} \right) dt = I_1' + I_2' + I_3',$$

say. Since $\varphi(t) = O(t^{1-\Delta})$ by (3), we have, by the convexity theorem due to G. Sunouchi,

$$\varphi_1(t) = o(t^{1+(\gamma-\beta)/\beta^2}) = o(t)$$

and

$$\varphi_2(t) = o(t^{2+\gamma-\beta}).$$

Hence we get

$$I_1' = o(1),$$

$$I_2' = o(\alpha^{2+\gamma-\beta}(n/\alpha)) = o(\alpha^{1+\gamma-\beta} n) = o(1),$$

$$I'_3 = \int_0^\alpha \frac{d^2}{dt^2} \left(\frac{\sin nt}{t} \right) dt \int_0^t \varphi_\beta(u)(t-u)^{(2-\beta)-1} du$$

$$= \int_0^\alpha \varphi_\beta(u)\, du \int_u^\alpha \frac{d^2}{dt^2} \left(\frac{\sin nt}{t} \right) (t-u)^{1-\beta} dt$$

$$= n^{1+\beta} \int_0^\alpha \varphi_\beta(u)\, du \int_u^\alpha \frac{d^2}{dt^2} \left(\frac{\sin \tau}{\tau} \right) (\tau - nu)^{1-\beta} d\tau$$

$$= o\left(n^\beta \int_0^\alpha u^{\gamma-1} du \right) = o(n^\beta \alpha^\gamma) = o(1).$$

Thus $I = I_1' + I_2' + I_3' = o(1)$.

The proof of the general case $k < \beta < k+1$ $(k \geqq 0)$ is now in hand. It is sufficient to use that, if $\varphi_\beta(t) = o(t^\gamma)$, then

$$\varphi_\nu(t) = o(t^{1+(\nu-1)\gamma/(\beta+\nu(\gamma-\beta)/\beta^2)})$$
$$= o(t^{1+(\nu-1)\Delta})$$

for $0 < \nu < \beta$, and that

$$\frac{d^k}{dt^k} \frac{\sin nt}{t} = O(n^k/t) \qquad (k = 1, 2, \dots.)$$

The case $\beta = k$ (integer) is easy, so That the theorem 1 is proved.

3. As a generalization of Theorem 1, we get

THEOREM 2. Let $\Delta = \gamma/\beta \geqq 1$. If

(1) $$\varphi_\beta(t) = o(t^\gamma) \qquad (t \to 0)$$

and

(5) $$\lim_{k\to\infty} \limsup_{n\to\infty} \int_{1/(kn)^{1/\Delta}}^{\eta} \left| \frac{\varphi(t)}{t} - \frac{\varphi(t+\pi/n)}{t+\pi/n} \right| dt = 0,$$

then the Fourier series of $\varphi(t)$ converges at $t = 0$.

THEOREM 3. Let $\Delta = \gamma/\beta \geqq 1$. If

(1) $$\varphi_\beta(t) = o(t^\gamma) \qquad (t \to 0)$$

and

(6) $$\lim_{k\to\infty} \limsup_{u\to 0} \int_{(ku)^{1/\Delta}}^{\eta} \frac{|\varphi(t) - \varphi(t+u)|}{t} dt = 0,$$

then the Fourier series of $\varphi(t)$ converges at $t = 0$.

There are generalization of not only Theorem 1 but theorems due to Gergen [2] and Sunouchi [3], which includes Pollard's theorem and many others.

By proving that (5) implies (6), we deduce Theorem 2 from Theorem 3. Proof of Theorem 3 is analogous to that of Gergen's theorem, which is the case $\Delta = 1$.

4. PROOF OF THEOREM 3. We need a lemma, which is a simple modification of a lemma due to J. J. Gergen [2].

LEMMA 1. Under the conditions of Theorem 3

(4) $$\varphi_\nu(t) = o(t^{1+(\nu-1)\Delta}) \qquad (t \to 0)$$

for integral $\nu, 0 < \nu < \beta$.

PROOF. Let β be non-integral and $\mu = [\beta] + 1$. Then, by (1), we have

$$\varphi_\mu(t) = o(t^{\gamma+(\mu-\beta)}) = o(t^{1+(\mu-1)\Delta}).$$

In order to prove the lemma, it is sufficient to prove that $\varphi_{r+1}(t) = o(t^{1+r\Delta})$ implies $\varphi_r(t) = o(t^{1+(r-1)\Delta})$ for $0 < r < \mu$. For this purpose we consider the integral

$$\int_0^{(hu)^\Delta} dt \int_{(kt)^{1/\Delta}}^{u-rt} \left\{ \sum_{\nu=0}^{r} (-1)^{r+\nu} \binom{r}{\nu} \varphi_{r-1}(w+\nu t) \right\} dw,$$

$(hu)^\Delta$ th multiple of which is the sum of φ_r, linear combination of φ_{r+1} and the term majorated by the integral in (6). Thus we get the required. The detail will be seen from the Gergen's paper [2].

Let us now prove Theorem 3. Let $u = \pi/n, \alpha = (ku)^{1/\Delta}$ If we put

$$s_n = \int_0^\eta \varphi(t) \frac{\sin nt}{t} dt,$$

$$\beta(n,k) = \left(\int_0^{\alpha} + 2\int_0^{\alpha+u} + \int_0^{\alpha+2u} - 2\int_{\eta}^{\eta+u} - \int_{\eta}^{\eta+2u}\right) \varphi(t)\frac{\sin nt}{t}\, dt,$$

then, by Lemma 1, we can prove as in the proof of Theorem 1 that

$$\lim_{k\to\infty} \limsup_{u\to 0} \beta(n,k) = 0.$$

We have also

$$4s_n - \beta(n,k) = \left(\int_{\alpha}^{\eta} + 2\int_{\alpha+u}^{\eta+u} + \int_{\alpha+2u}^{\eta+2u}\right)\varphi(t)\frac{\sin nt}{t} dt$$

$$= 2u^{\eta}\int_{\alpha}^{\eta} \frac{\varphi(t+u)t}{t(t+u)(t+2u)} \sin nt\, dt$$

$$+ \int_{\alpha}^{\eta} \left\{ \frac{\varphi(t+2u) - \varphi(t+u)}{t+2u} - \frac{\varphi(t+u) - \varphi(t)}{t} \right\} \sin nt\, dt$$

$$= 2\gamma(n,k) + \delta(n,k),$$

say. By the condition (6)

$$\lim_{k\to\infty} \limsup_{n\to\infty} \delta(n,k) = 0.$$

We have also

$$\lim_{k\to\infty} \limsup_{n\to\infty} \gamma(n,k) = 0$$

by (4) and the integration by parts. Thus the Theorem is completely proved.

5. Proof of Theorem 2. It is sufficient to prove that (5) implies (6). For sufficiently small k

$$\sup_{0<v\leqq u} \int_{(kv)^{1/\Delta}}^{\xi} \left| \frac{\varphi(t)}{t} - \frac{\varphi(t+v)}{t+v} \right| dt \leqq M, \tag{7}$$

by (5). We can suppose $k = 1$ in (7). Let us define a sequence (x_ν) by the relation

$$x_1 = u,\ x_{\nu+1} + x_{\nu+1}^{\Delta} = x_\nu\ (\nu = 2, 3, \ldots.)$$

Then $x_n \downarrow$, and then it converges; the limit must be 0. Let $u_\nu = x_\nu - x_{\nu+1}$ $(\nu = 1, 2, \ldots.)$. For a fixed ξ, we have

$$\int_0^u |\varphi(t)|\, dt = \sum_{\nu=1}^{\infty} \int_{x_{\nu+1}}^{x_\nu} |\varphi(t)|\, dt \leqq \sum_{\nu=1}^{\infty} x_\nu \int_{x_{\nu+1}}^{x_\nu} \frac{|\varphi(t)|}{t}\, dt$$

$$= \sum_{\nu=1}^{\infty} x_\nu \left(\int_{x_{\nu+1}}^{\xi} - \int_{x_\nu}^{\xi+u_\nu} + \int_{\xi}^{\xi+u_\nu} \right) \frac{|\varphi(t)|}{t}\, dt$$

$$= \sum_{\nu=1}^{\infty} x_\nu \left(\int_{x_{\nu+1}}^{\xi} \frac{|\varphi(t)|}{t} dt - \int_{x_{\nu+1}}^{\xi} \frac{|\varphi(t+u_\nu)|}{t+u_\nu} + \int_{\xi}^{\xi+u_\nu} \frac{|\varphi(t)|}{t} dt \right.$$

$$\leqq \sum_{\nu=1}^{\infty} x_\nu \left(\int_{x_{\nu+1}}^{\xi} \left| \frac{\varphi(t)}{t} - \frac{\varphi(t+u_\nu)}{t+u_\nu} \right| dt + \int_{\xi}^{\xi+u} \frac{|\varphi(t)|}{t} d \right.$$

$$\leqq u\left(M + \int_{\xi}^{\xi+u} \frac{|\varphi(t)|}{t} dt \right)$$

Thus we have

$$\int_0^u |\varphi(t)|dt = O(u).$$

Now,

$$\int_{(ku)^{1/\Delta}}^{\xi} \frac{|\varphi(t) - \varphi(t+u)|}{t} dt \leqq \int_{(ku)^{1/\Delta}}^{\xi} \left| \frac{\varphi(t)}{t} - \frac{\varphi(t+u)}{t+u} \right| dt$$

$$+ u \int_{(ku)^{1/\Delta}}^{\xi} \frac{|\varphi t + u)|}{t(t+u)} dt$$

$$\leqq \int_{(ku)^{1/\Delta}}^{\xi} \left| \frac{\varphi(t)}{t} - \frac{\varphi(t+u)}{t+u} \right| dt$$

$$+ \left| \left[\frac{u}{t(t+u)} \int_0^{t+u} |\varphi(w)|dw \right]_{(ku)^{1/\Delta}}^{\xi} \right| + 2u \int_{(ku)^{1/\Delta}}^{\xi} \frac{dt}{t^2(t+u)} \int_0^{t+u} |\varphi(w)|dw$$

$$\leqq \int_{(ku)^{1/\Delta}}^{\xi} \left| \frac{\varphi(t)}{t} - \frac{\varphi(t+u)}{t+u} \right| dt + \frac{Mu}{\xi} + \frac{3Mk^{1-k/\Delta}}{k^{1/\Delta}}$$

Thus (6) implies (5).

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