ON THE CESÀRO SUMMABILITY OF DOUBLE FOURIER SERIES

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1. F.T. Wang [3], [4] has proved that, if $\varphi_x(t) = \frac{1}{2}[f(x+t)+f(x-t)-2s]$ and

$$\int_0^t \varphi_x(u)\,du = o\left(t\Big/\log\frac{1}{t}\right) \qquad \text{as } t\to 0,$$

then the Fourier series of $f(t)$ is summable $(C,1)$ at $t=x$; and if $0<\alpha<1$ and

$$\int_0^t \varphi_x(u)du = o(t^{1/\alpha}) \qquad \text{as } t\to 0,$$

then the Fourier series of $f(t)$ is summable (C,α) at $t=x$. In this note, we generalize these results to the double Fourier series.

THEOREM 1. *Suppose that the function $f(u,v)$ is integrable in the Lebesgue sense, over the square $(-\pi,\pi;-\pi,\pi)$ and is periodic with period 2π in each variable. Let*

$$\varphi(u,v)=\varphi_{x,y}(u,v)=\frac{1}{4}\Big[f(x+u,y+v)+f(x+u,y-v)+f(x-u,y+v)+f(x-u,y-v)-4s\Big].$$

If, as $u\to+0, v\to+0$,

$$\Phi(u,v)=\int_0^u ds\int_0^v \varphi(s,t)\,dt = o\left(uv\Big/\log\frac{1}{u}\log\frac{1}{v}\right),$$

$$\int_0^\pi dt\left|\int_0^u \varphi(s,t)\,ds\right| = O\left(u\Big/\log\frac{1}{u}\right),$$

$$\int_0^\pi ds\left|\int_0^v \varphi(s,t)\,dt\right| = O\left(v\Big/\log\frac{1}{v}\right),$$

then the double Fourier series of $f(u,v)$ is summable $(C,1,1)$ to sum s at $u=x$, $v=y$.

THEOREM 2. *If $0<\alpha<1, 0<\beta<1$ and as $u\to+0, v\to+0$,*

$$\Phi(u,v)=o(u^{\frac{1}{\alpha}}v^{\frac{1}{\beta}}),$$

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$$\int_0^\pi dt \left| \int_0^u \varphi(s,t)\, ds \right| = o(u^{\frac{1}{\alpha}}),$$

$$\int_0^\pi ds \left| \int_0^v \varphi(s,t)\, dt \right| = o(v^{\frac{1}{\beta}}),$$

then the double Fourier series of $f(u, v)$ is summable (C, α, β) to sum s at $u = x$, $v = y$.

2. To prove these theorems we need the following lemmas.

LEMMA 1. *Let $K_m(t)$ be the Fejér kernel of order 1 and $0 < r < \frac{1}{2}$. Then*[1]

$$0 \leqq K_m(t) = O(m^{-1}t^{-2}), \qquad (m^{-1} \leqq t \leqq \pi),$$
$$0 \leqq K_m(t) = O(m^{-1+2v}), \qquad (m^{-r} \leqq t \leqq \pi),$$
$$\frac{d}{dt}K_m(t) \equiv K'_m(t) = O(mt^{-1}), \qquad (0 < t \leqq m^{-1}),$$
$$K'_m(t) = O(t^{-2}), \qquad (m^{-1} \leqq t \leqq \pi).$$

LEMMA 2. *If $0 < \alpha < 1$ and $K_m^\alpha(t)$ is the Fejér kernel of order α, then*[1]

$$K_m(t) = O(m^{-\alpha}t^{-1-\alpha}), \qquad (m^{-1} \leqq t \leqq \pi),$$
$$\frac{d}{dt}K_m^\alpha(t) \equiv [K_m^\alpha(t)]' = O(n^2), \qquad (0 \leqq t \leqq \pi),$$
$$[K_m^\alpha(t)]' = O(m^{1-\alpha}t^{-1-\alpha}), \qquad (m^{-1} \leqq t \leqq \pi).$$

Lemma 1 is obvious, for $K_m(t) = \dfrac{1}{2(n+1)}\left(\dfrac{\sin\frac{m+1}{2}t}{\sin\frac{1}{2}t}\right)^2$, and Lemma 2 is known [2].

3. PROOF OF THEOREM 1.

$$\pi^2\sigma_{m,n}$$
$$= \int_0^\pi \int_0^\pi \varphi(u, v)K_m(u)K_n(v)dudv$$
$$= \left(\int_0^{m^{-r}}\int_0^{n^{-r}} + \int_0^{m^{-r}}\int_{n^{-r}}^{\pi} + \int_{m^{-r}}^{\pi}\int_0^{n^{-r}} + \int_{m^{-r}}^{\pi}\int_{n^{-r}}^{\pi}\right)\varphi(u, v)K_m(u)K_n(v)dudv$$
$$= I_1 + I_2 + I_3 + I_4,$$

say. By Lemma 1,

$$I_4 = O\left(m^{-1+2r}n^{-1+2r}\int_0^\pi \int_0^\pi |\varphi(u, v)|\, dudv\right) = o(1)$$

[1] O is independent of t, m and n.

for $0 < r < \frac{1}{2}$. By partial integration, we have

$$I_3 = \int_{m^{-r}}^{\pi} K_m(u)du \int_0^{n^{-r}} \varphi(u, v)K_n(v)dv$$

$$= \int_{m^{-r}}^{\pi} K_m(u)du\left[\Phi_1(u, n^{-r})K_n(n^{-r}) - \int_0^{n^{-r}} \Phi_1(u, v)K_n'(v)dv \right].$$

where $\Phi_1(u, v) = \int_0^v \varphi(u, t)dt$ provided this integral exists, and is ∞ otherwise.

Since, by Lemma 1,

$$\int_{m^{-r}}^{\pi} K_m(u)\Phi_1(u, n^{-r})K_n(n^{-r})du = O\left(m^{-1+2r}|K_n(n^{-r})| \int_{m^{-r}}^{\pi} |\Phi_1(u, n^{-r})|du\right)$$

$$= O(m^{-1+2r}n^{-1+2r}n^{-r}(\log n^r)^{-1}) = o(1),$$

and

$$\int_{m^{-r}}^{\pi} K_m(u)du \int_0^{n^{-r}} \Phi_1(u, v)K_n'(v)dv$$

$$= O\left(m^{-1+2r}\left[\int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-r}}\right]K_n'(v)dv \int_{m^{-r}}^{\pi} |\Phi_1(u, v)|du\right)$$

$$= o\left\{\left[\int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-r}}\right]|K_n'(v)|v\left(\log\frac{1}{v}\right)^{-1}dv\right\}$$

$$= o\left(n\int_0^{n^{-1}} \left(\log\frac{1}{v}\right)^{-1} dv\right) + o\left(\int_{n^{-1}}^{n^{-r}} \frac{dv}{v\log\frac{1}{v}}\right)$$

$$= o(1) + o(1) = o(1).$$

we have $I_3 = o(1)$. Similarly $I_2 = o(1)$. By partial integration for the double integral [1], we have

$$I_1 = \int_0^{m^{-r}}\int_0^{n^{-r}} \varphi(u, v)K_m(u)K_n(v)\,du\,dv$$

$$= \Phi(m^{-r}, n^{-r})K_m(m^{-r})K_n(n^{-r}) - K_n(n^{-r})\int_0^{m^{-r}} \Phi(u, n^{-r})K_m'(u)du$$

$$- K_m(m^{-r})\int_0^{n^{-r}} \Phi(m^{-r}, v)K_n'(v)\,dv + \int_0^{m^{-r}}\int_0^{n^{-r}} \Phi(u, v)K_m'(u)K_n'(v)du\,dv$$

$$= I_{11} + I_{12} + I_{13} + I_{14},$$

say. Now,

$$I_{11} = O\left(m^{-1+2r} n^{-1+2r} \frac{m^{-r} n^{-r}}{r^2 \log m \log n}\right) = o(1),$$

$$I_{12} = O\left(n^{-1+2r}\left[\int_0^{m^{-1}} + \int_{m^{-1}}^{m^{-r}}\right] |\Phi(u, n^{-r}) K'_m(u)| \, du\right)$$

$$= o\left(m \int_0^{m^{-1}} \frac{du}{\log \frac{1}{u}}\right) + o\left(\int_{m^{-1}}^{m^{-r}} \frac{du}{u \log \frac{1}{u}}\right) = o(1) + o(1) = o(1),$$

$$I_{13} = O\left(m^{-1+2r}\left[\int_0^{n-1} + \int_{n^{-1}}^{n^{-r}}\right] |\Phi(m^{-r}, v) K'_n(v) \, dv|\right) = o(1),$$

and

$$|I_{14}| \leqq \left(\int_0^{m^{-1}} \int_0^{n^{-1}} + \int_0^{m^{-1}} \int_{n^{-1}}^{n^{-r}} + \int_{m^{-1}}^{m^{-r}} \int_0^{n^{-1}} + \int_{m^{-1}}^{m^{-r}} \int_{n^{-1}}^{n^{-r}}\right) |\Phi(u, v) K'_m(u) K'_n(v)| \, du \, dv$$

$$= O\left(mn \int_0^{m^{-1}} \int_0^{n^{-1}} \frac{du \, dv}{\log \frac{1}{u} \log \frac{1}{v}}\right) + O\left(m \int_0^{m^{-1}} \frac{du}{\log \frac{1}{u}} \int_{n^{-1}}^{n^{-r}} \frac{dv}{v \log \frac{1}{v}}\right)$$

$$+ O\left(n \int_{m^{-1}}^{m^{-r}} \frac{du}{u \log \frac{1}{u}} \int_0^{n^{-1}} \frac{dv}{\log \frac{1}{v}}\right) + o\left(\int_{m^{-1}}^{m^{-r}} \int_{n^{-1}}^{n^{-r}} \frac{du \, dv}{uv \log \frac{1}{u} \log \frac{1}{v}}\right)$$

$$= o(1) + o(1) + o(1) + o(1) = o(1).$$

Hence $I_1 = o(1)$ and this completes the proof.

4. Proof of the Theorem 2.

$$\pi^2 \sigma_{m,n}^{\alpha,\beta} = \int_0^\pi \int_0^\pi \varphi(u, v) K_m^\alpha(u) K_n^\beta(v) \, du \, dv$$

$$= \left(\int_0^A \int_0^B + \int_0^A \int_B^\pi + \int_A^\pi \int_0^B + \int_A^\pi \int_B^\pi\right) \varphi(u, v) K_m^\alpha(u) K_n^\beta(v) \, du \, dv$$

$$= I_1 + I_2 + I_3 + I_4,$$

say, where $A = \psi_m \cdot m^{-\frac{\alpha}{1+\alpha}}$, $B = \psi_n n^{-\frac{\beta}{1+\beta}}$ and $0 < \psi_n \to \infty$ sufficiently slowly

By Lemma 2,

$$I_4 = O\left(m^{-\alpha}n^{-\beta}\int_A^\pi\int_B^\pi \frac{|\varphi(u,v)|}{u^{1+\alpha}v^{1+\beta}}du\,dv\right)$$

$$= O\left(m^{-\alpha}n^{-\beta}A^{-1-\alpha}B^{-1-\beta}\int_A^\pi\int_B^\pi |\varphi(u,v)|du\,dv\right)$$

$$= O(\psi_m{}^{-1-\alpha}\psi_n{}^{-1-\beta}) = o(1).$$

By partial integration,

$$I_3 = \int_A^\pi\int_0^B \varphi(u,v)K_m^\alpha(u)K_n^\beta(v)\,du\,dv$$

$$= \int_A^\pi K_m^\alpha(u)\,du\left\{\Phi_1(u,B)\,K_n^\beta(B) - \int_0^\beta \Phi_1(u,v)[K_n^\beta(v)]'dv\right\}.$$

Since

$$\int_A^\pi K_m^\alpha(u)\Phi_1(u,B)K_n^\beta(v)dv = O\left(m^{-\alpha}|K_n^\beta(B)|\int_A^\pi \frac{|\Phi_1(u,B)|}{u^{1+\alpha}}du\right)$$

$$= O\left(m^{-\alpha}A^{-1-\alpha}|K_n^\beta(B)|\int_A^\pi |\Phi_1(u,B)|du\right)$$

$$= O\left(\psi^{-1-\alpha}\psi^{-1-\beta}\int_A^\pi |\Phi_1(u,B)|du\right) = o(1),$$

and

$$\int_A^\pi K_m^\alpha(u)du\int_0^B \Phi_1(u,v)[K_n^\beta(v)]'\,dv$$

$$= O\left(m^{-\alpha}A^{-1-\alpha}\int_A^\pi\int_0^B |\Phi_1(u,v)[K_n^\beta(v)]'|du\,dv\right)$$

$$= O\left(\psi_m^{-1-\alpha}\int_A^\pi du\left[\int_0^{n^{-1}} + \int_{n^{-1}}^B\right]|\Phi_1(u,v)[K_n(v)]'dv\right)$$

$$= O\left(\psi_m{}^{-1-\alpha}n^2\int_0^{n^{-1}} v^{\frac{1}{\beta}}dv\right) + O\left(\psi_m^{-1-\alpha}n^{1-\beta}\int_{n^{-1}}^B v^{\frac{1}{\beta}-\beta-1}dv\right)\cdot o(1)$$

$$= O(\psi_m^{-1-\alpha}) + O\left(\psi^{-1-\alpha}\psi_n^{\frac{1-\beta^2}{\beta}}\right)\cdot o(1) = o(1) + o(1) = o(1),$$

since $\psi_n \to \infty$ sufficiently slowly. We have $I_3 = o(1)$.

Similarly, we can show that $I_2 = o(1)$. Integrating by parts [1], we have

$$I_1 = \int_0^A \int_0^B \varphi(u, v) K_m^\alpha(u) K_n^\beta(v) du\, dv$$

$$= \Phi(A, B) K_m^\alpha(A) K_n^\beta(B) - K_n^\beta(B) \int_0^A \Phi(u, B)[K_m^\alpha(u)]' du$$

$$- K_m^\alpha(A) \int_0^B \Phi(A, v)[K_n^\beta(v)]'\, dv + \int_0^A \int_0^B \Phi(u, v)[K_m^\alpha(u)]' [K_n^\beta(v)]'\, du\, dv$$

$$= I_{11} - K_n^\beta(B) I_{12} - K_m^\alpha(A) I_{13} + I_{14}, \text{ say.}$$

Obviously, $I_{11} = o(1)$. Now

$$|I_{12}| \leqq \left(\int_0^{m^{-1}} + \int_{m^{-1}}^A\right) |\Phi(u, B)[K_m^\alpha(u)]'| du$$

$$= O\left(m^2 \int_0^{m^{-1}} u^{\frac{1}{\alpha}} du\right) + o(1)\, O\left(m^{1-\alpha} \int_{m^{-1}}^A u^{\frac{1}{\alpha}-1-\alpha} du\right)$$

$$= O(m^{1-\frac{1}{\alpha}}) + o(1) \cdot O(\psi_m^{\frac{1-\alpha^2}{\alpha}}) = o(1) + o(1) = o(1)$$

for $\psi_m \to \infty$ sufficiently slowly, and similarly, $I_{13} = o(1)$.

Since

$$|I_{14}| \leqq \left(\int_0^{m^{-1}} \int_0^{n^{-1}} + \int_0^{m^{-1}} \int_{n^{-1}}^B + \int_{m^{-1}}^A \int_0^{n^{-1}} + \int_{m^{-1}}^A \int_{n^{-1}}^B\right) |\Phi(u, v)[K_m^\alpha(u)]'[K_n(v)^\beta]' du\, dv$$

$$= o\left(m^2 n^2 \int_0^{m^{-1}} \int_0^{n^{-1}} u^{\frac{1}{\alpha}} v^{\frac{1}{\beta}} du\, dv\right)$$

$$+ o\left(m^2 n^{1-\beta} \int_0^{m^{-1}} u^{\frac{1}{\alpha}} du \int_{n^{-1}}^B v^{\frac{1}{\beta}-\beta-1} dv\right)$$

$$+ o\left(n^2 m^{1-\alpha} \int_0^{n^{-1}} v^{\frac{1}{\beta}} dv \int_{m^{-1}}^A u^{\frac{1}{\alpha}-\alpha-1} du\right)$$

$$+ o\left(m^{1-\alpha} n^{1-\beta} \int_{m^{-1}}^A u^{\frac{1}{\alpha}-\alpha-1} du \int_{n^{-1}}^B v^{\frac{1}{\beta}-\beta-1} dv\right)$$

$$= o(1) + o(1) + o(1) + o(1) = o(1)$$

for $\psi_m \to \infty$ sufficiently slowly, we have $I_1 = o(1)$. The proof is thus completed.

LITERATURE

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