GENERALIZED FOURIER INTEGRALS

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Introduction. Concerning the representation of functions by the generalized Fourier integrals, H. Hahn [4] proved the following theorem (cf. Titchmarsh [7, p. 14]):

Let $f(x)/(1+|x|)$ belong to $L(-\infty,\infty)$ and write

$$\Phi_1(v)=\int_{-\infty}^{\infty} f(y)\frac{\sin vy}{y}\,dy,$$

$$\Psi_1(v)=\int_{-1}^{+1} f(y)\frac{1-\cos vy}{y}\,dv-\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right) f(y)\frac{\cos vy}{y}\,dy.$$

Let $f(t)$ satisfy a condition of convergence of Fourier series in the neighbourhood of $t=x$. Then

$$f(x)=\frac{1}{\pi}\int_{\to 0}^{\to\infty}\{\cos vx\,d\,\Phi_1(v)+\sin vx\,d\,\Psi_1(v)\},$$

where the integral is defined appropriately.

Another Hahn's generalization [3] is of the following form:

Let $f(x)/(1+|x|^2)$ belong to $L(-\infty,\infty)$ and write

$$\Phi_2(v)=\int_{-\infty}^{\infty} f(y)\frac{1-\cos vy}{y^2}\,dy,$$

$$\Psi_2(v)=\int_{-1}^{1} f(y)\frac{vy-\sin vy}{y^2}\,dy-\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)f(y)\frac{\sin vy}{y^2}\,dy,$$

for $v\geqq 0$, and

$$\Phi_2(v)=\Psi_2(v)=0,$$

for $v<0$. Then

$$f(x)=\frac{1}{\pi}(C,1)\int_{-0}^{\infty}\left\{\cos vx\frac{d^2\Phi_2(v)}{dv}+\sin vx\frac{d^2\Psi_2(v)}{dv}\right\}$$

for almost all x, where

$$(C,1)\int_{-0}^{\infty}(\ \ )\ \text{means}\ \lim_{\lambda\to\infty}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\lambda}\left(1-\frac{v}{\lambda}\right)(\ \ ).$$

Professor S. Izumi [6] further generalized these theorems for the case $f(x)/(1+|x|^{\alpha})\in L(-\infty,\infty)$, where $1<\alpha<2$, or $\alpha=2,3,4,\ldots\ldots$ Although Professor Izumi used the mean convergence for $\alpha=3,4,\ldots,$ this is seemingly unnatural. The object of this paper is to generalize these theorems for the arbitrary α. Since the equivalence of Bessel summation and Cesàro summation is well known (cf. Hardy-Littlewood [5]), we shall use Bessel summation. This is suitable for the behaviour of $f(x)$ at the infinity. In § 1, we summalize some results of Bessel summation. The generalized Stieltjes integral of Hahn-Izumi type is defined in § 2. The main theorem is proved in § 3. In § 4 we extend Burkill's theorem. Our theorems are refinement and generalization of Izumi's. In the last paragraph we shall treat the case where

$$|f(x)|^{p}/(1+|x|^{\alpha}) \text{ belongs to } L(-\infty,\infty) \text{ for } p>1 \text{ and } \alpha>1.$$

1. **Bessel summation.** K. Chandrasekharan and O. Szàsz [2] derived many valuable results concerning Bessel summability. For the use of the following articles, we present some results of this summability.

Let $J_{\mu}(t)$ denote the Bessel function of order μ:

$$J_{\mu}(t)=\frac{t^{\mu}}{2^{\mu}}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{t^{2\nu}}{2^{2\nu}\nu!\,\Gamma(\mu+\nu+1)}, \qquad \mu>-\frac{1}{2},$$

and let

$$\alpha_{\mu}(t)=2^{\mu}\Gamma(\mu+1)J_{\mu}(t)/t^{\mu},$$

then we have

$$\alpha_{\mu}(0)=1$$

and

$$\alpha_{\mu}(t)=O(t^{-\mu-\frac{1}{2}}), \text{ as } t\to+\infty.$$

Since

$$\int_0^{\infty}2^{\mu-\frac{1}{2}}\Gamma\left(\mu+\frac{1}{2}\right)x^{-\mu}J_{\mu}(x)\cos xt\ dt=\begin{cases}(1-t^2)^{\mu-\frac{1}{2}}, & 0\leqq t\leqq 1,\\ 0, & t>1,\end{cases}$$

we have

$$C(\mu)\int_0^{\infty}\alpha_{\mu}(x)\cos x\,t\ dx=\begin{cases}(1-t^2)^{\mu-\frac{1}{2}}, & 0\leqq t\leqq 1\\ 0\quad, & t>1\end{cases}$$

where

$$C(\mu)=\Gamma\left(\mu+\frac{1}{2}\right)\Big/(\sqrt{\pi}\,\Gamma(\mu+1)).$$

Putting $t=0$, we get

$$C(\mu)\int_0^{\infty}\alpha_{\mu}(x)\,dx=1.$$

From the inversion formula we get

$$\frac{2}{\pi}\int_0^1 (1-t^2)^{\mu-\frac{1}{2}} \cos xt\, dt = C(\mu)\alpha_\mu(x).$$

Under these preparations we can prove

LEMMA 1. *If* $f(x)/(1+|x|^\alpha)$ *belongs to* $L(-\infty,\infty)$ *for* $\alpha>1$, *then we have*

$$f(x) = \frac{1}{\pi}\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(y)dy \int_0^\lambda \left\{1-\left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1} \cos v(y-x)\, dv, \quad a.e.$$

PROOF. From the above formula,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f(y)dy \int_0^\lambda \left\{1-\left(\frac{v}{\lambda}\right)^2\right\}^{\mu-\frac{1}{2}} \cos v(y-x)dv$$

$$= \frac{C(\mu)\lambda}{2}\int_{-\infty}^{\infty} f(y)\,\alpha_\mu\{\lambda(y-x)\}dy$$

$$= \int_{-\infty}^{\infty} f(y)\,K_\mu\{\lambda(y-x)\}dy,$$

say. Since $\alpha_\mu(t)$ is bounded in the neighbourhood of $t=0$,

$$K_\mu\{\lambda(y-x)\} = O(\lambda), \text{ for } |y-x| \leqq \frac{1}{\lambda}$$

and

$$K_\mu\{\lambda(y-x)\} = O(\lambda^{-\mu+\frac{1}{2}}|y-x|^{-\mu-\frac{1}{2}}), \text{ for } |y-x| > \frac{1}{\lambda}.$$

Further,

$$\lim_{\lambda\to\infty}\int_x^\infty K_\mu\{\lambda(y-x)\}dy = \lim_{\lambda\to\infty}\int_x^\infty \frac{C(\mu)\lambda}{2}\alpha_\mu\{\lambda(y-x)\}dy$$

$$= \frac{C(\mu)}{2}\int_0^\infty \alpha_\mu(t)dt = \frac{1}{2}$$

and

$$\lim_{\nu\to\infty}\int_{-\infty}^x K_\mu\{\lambda(y-x)\}dy = \frac{1}{2}.$$

If $f(x)/(1+|x|^{\mu+\frac{1}{2}}) \in L(-\infty,\infty)$, and $\mu+\frac{1}{2}>1$, then we have

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(y)K_\mu\{\lambda(y-x)\}dy = f(x), \text{ a.e.},$$

by the general convergence theorem (cf. Titchmarsh [7, p.28]). Let $\mu+\frac{1}{2}=\alpha$, we get the desired results.

2. **Generalized Stieltjes integral.** H. Hahn [3] defined the following Stieltjes integral. Let $f(x)$ and $g(x)$ be defined in (a, b), and suppose that $g(x)$ is absolutely continuous and $f(x)$ is bounded and continuous in (a, b). Then $g'(x)f(x)$ is integrable in (a, b), and we define the integral of $g(x)$ with respect to $f(x)$ by the equation:

$$\int_a^b g(x)df(x) = g(b)f(b) - g(a)f(a) - \int_a^b g'(x)f(x)dx.$$

It is easily seen that this integral coincides with the ordinary Lebesgue-Stieltjes integral when $f(x)$ is of bounded variation. Prof. Izumi [6] generalized this definition. Further we generalize it in the following manner. Let α be any positive real number, then there is an interger k such that

$$k-1<\alpha\leqq k,$$

and let

$$h=k-\alpha.$$

Put the integral

$$I^h f(x) = \begin{cases} \dfrac{1}{\Gamma(\lambda)}\displaystyle\int_0^x (x-y)^{h-1}f(y)dy, & \text{if } h>0 \\ f(x), & \text{if } h=0 \end{cases}$$

and denote

$$\Delta_\epsilon^k I^h f(x) = \sum_{\nu=0}^{k} A_\nu^{(-k-1)} I^h(x+\nu\varepsilon),$$

$$\Delta_{\epsilon *}^k I^h f(x) = \sum_{\nu=0}^{k} A_\nu^{(-k-1)} I^h(x-\nu\varepsilon),$$

where

$$A_n^{(m)} = \binom{m+n}{n}.$$

Suppose that $g(x)$ is everywhere differentiable $(k-1)$ times and $g^{(k-1)}(x)$ is absolutely continuous in (a, b), and that $f(x)$ is continuous and bounded in (a, b). If the limits

$$\lim_{\epsilon\to 0} g^{(r)}\{a+(k-r-1)\varepsilon\}\,\Delta_\epsilon^{k-r-1} I^h f(a)/\varepsilon^{k-r-1},$$

$$\lim_{\epsilon\to 0} g^{(r)}\{b-(k-r-1)\mathcal{E}\}\underset{\epsilon}{\Delta_*^{k-r-1}} I^h f(b)/\mathcal{E}^{k-r-1}$$
$$(r=0,1,2,\ldots,k-1)$$

exist and are finite, we define

$$\int_a^b g(x)\frac{d^\alpha f(x)}{dx^{\alpha-1}} = g(b)I^{1-\alpha}f(b) - g(a)I^{1-\alpha}f(a)$$
$$-\int_a^b g'(x)I^{1-\alpha}f(x)\,dx,$$

if $0<\alpha\leqq 1$, and for any positive α, we define by induction

$$\int_a^b g(x)\frac{d^\alpha f(x)}{dx^{\alpha-1}} = \lim_{\epsilon\to 0} g\{b-(k-1)\mathcal{E}\}\underset{\epsilon}{\Delta_*^{k-1}} I^h f(b)/\mathcal{E}^{k-1}$$
$$-\lim_{\epsilon\to 0} g\{a+(k-1)\mathcal{E}\}\underset{\epsilon}{\Delta^{k-1}} I^h f(a)/\mathcal{E}^{k-1} - \int_a^b g'(x)\frac{d^{\alpha-1}f(x)}{dx^{\alpha-2}}.$$

Then we have

LEMMA 2. *If $f(x)$ is differentiable $(\alpha-1)$ times at $x=a$ and $x=b$, then*

$$\int_a^b g(x)\frac{d^\alpha f(x)}{dx^{\alpha-1}} = g(b)f_-^{(\alpha-1)}(b) - g(a)f_+^{(\alpha-1)}(a) - \int_a^b g'(x)\frac{d^{\alpha-1}f(x)}{dx^{\alpha-2}}.$$

Proof. Immediate.

LEMMA 3. *If $f(x)$ is everywhere differentiable $(\alpha-1)$ times, and $f^{(\alpha-1)}(x)$ is absolutely continuous in (a,b), then*

$$\int_a^b g(x)\frac{d^\alpha f(x)}{dx^{\alpha-1}} = \int_a^b g(x)f^{(\alpha)}(x)dx.$$

PROOF. This is proved by the repeated use of integration by parts.

LEMMA 4. *Let $\{f_n(x)\}$ be a uniformly bounded sequence of continuous functions, such that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

uniformly in (a,b) and further

$$\lim_{n\to\infty}\lim_{\epsilon\to 0} g^{(r)}\{a+(k-r-1)\mathcal{E}\}\underset{\epsilon}{\Delta^{k-r-1}} I^h f_n(a)/\mathcal{E}^{k-r-1}$$
$$=\lim_{\epsilon\to 0} g^{(r)}\{a+(k-r-1)\mathcal{E}\}\underset{\epsilon}{\Delta^{k-r-1}} I^h f(a)/\mathcal{E}^{k-r-1},$$

$$\lim_{n\to\infty}\lim_{\epsilon\to 0} g^{(r)}\{b-(k-r-1)\mathcal{E}\}\underset{\epsilon}{\Delta_*^{k-r-1}} I^h f_n(b)/\mathcal{E}^{k-r-1}$$

$$= \lim_{\epsilon\to 0} g^{(r)}\{b-(k-r-1)\mathcal{E}\}\Delta_{\epsilon}^{*\,k-r-1} I^h f(b)/\mathcal{E}^{k-r-1}$$

$$(r = 0, 1, 2, \ldots, k-2),$$

then we have $$\lim_{n\to\infty}\int_a^b g(x)\frac{d^\alpha f_n(x)}{dx^{\alpha-1}} = \int_a^b g(x)\frac{d^\alpha f(x)}{dx^{\alpha-1}}.$$

PROOF. This is proved by the definition of the integral and Lebesgue's convergence theorem. (See Izumi [6]).

3. Main theorem and its proof.

THEOREM 1. *If* $f(x)/(1+|x|^\alpha)$ *is absolutely integrable in* $(-\infty, +\infty)$ *for* $\alpha > 1$, *then*

$$f(x) = \frac{1}{\pi}(B, \alpha-1)\int_{-0}^{\infty}\left\{\cos vx\,\frac{d^\alpha \Phi_\alpha(v)}{dv^{\alpha-1}} + \sin vx\,\frac{d^\alpha \Psi^\alpha(v)}{dv^{\alpha-1}}\right\}$$

for almost all x, where $(B, \alpha-1)\int_{-0}^{\infty}(\quad)$ *means*

$$\lim_{\lambda\to\infty}\lim_{\delta\to 0}\int_{-\delta}^{\lambda}\left\{1-\left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1}(\quad);$$

and if k is an integer such that

$$k-1 < \alpha \leqq k, \qquad h = k-\alpha,$$

and if k is even, that is $k = 2m$,

$$\Phi_\alpha(v) = \int_{-1}^{1} f(y)\frac{C_\alpha(vy)}{y^\alpha}dy + \left(\int_{-\infty}^{-1} + \int_1^\infty\right)f(y)\frac{(-1)^m C_{2-h}(vy)}{y^\alpha}dy,$$

$$\Psi_\alpha(v) = \int_{-1}^{1} f(y)\frac{C_{\alpha+1}(vy)}{y^\alpha}dy + \left(\int_{-\infty}^{-1} + \int_1^\infty\right)f(y)\frac{(-1)^m C_{1-h}(vy)}{y}dy,$$

$$(v \geqq 0)$$

and

$$\Phi_\alpha(v) = \Psi_\alpha(v) = 0, \qquad (v < 0)$$

if k is odd, that is $k = 2m+1$,

$$\Phi_\alpha(v) = \int_{-1}^{1} f(y)\frac{C_{\alpha+1}(vy)}{y^\alpha}dy + \left(\int_{-\infty}^{-1} + \int_1^\infty\right)f(y)\frac{(-1)^m C_{1-h}(vy)}{y^\alpha}dy$$

$$\Psi_\alpha(v) = \int_{-1}^{1} f(y)\frac{C_\alpha(vy)}{y^\alpha}dy + \left(\int_{-\infty}^{-1} + \int_1^\infty\right)f(y)\frac{(-1)^m C_{2-h}(vy)}{y^\alpha}dy$$

$$(v \geqq 0)$$

and

$$\Phi_\alpha(v) = \Psi_\alpha(v) = 0 \qquad (v < 0),$$

$C_\alpha(u)$ *being Young's function.*

PROOF. We shall prove for $k = 2m$. The case $k = 2m + 1$ is treated similarly. The following properties of Young's function are well known. They are

$$\left(\int_0^v\right)^\alpha \cos u y\, du\backslash = \frac{C_\alpha(vy)}{\{y^\alpha}$$

$$\frac{d^\alpha}{dv^\alpha}\left\{(-1)^m \frac{C_{2-h}(vy)}{y^\alpha}\right\} = \cos vy$$

and

$$\frac{d^\alpha}{dv^\alpha}\left\{(-1)^m \frac{C_{1-h}(vy)}{y^\alpha}\right\} = -\sin' vy.$$

From Lemma 1,

$$f(x) = \lim_{\lambda\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)dy \int_0^\lambda \left\{1 - \left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1} \cos v(y-x)\, dv = \lim_{\lambda\to\infty} I_\lambda(x),$$

say. Then

$$I_\lambda(x) = \lim_{n\to\infty} \frac{1}{\pi}\int_{-n}^{n} f(y)dy \int_0^\lambda \left\{1 - \left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1} \cos v(y-x)dv$$

$$= \lim_{n\to\infty} \frac{1}{\pi}\left[\int_0^\lambda \left\{1 - \left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1} \cos vx\, dv \int_{-n}^{n} f(y)\cos vy\, dy\right.$$

$$\left. + \int_0^\lambda \left\{1 - \left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1} \sin\ vx\, dv \int_{-n}^{n} f(y)\sin vy dy\right].$$

If we put

$$\Phi_{\alpha,n}(v) = \int_{-1}^{1} f(y)\frac{C_\alpha(vy)}{y^\alpha}dy + \left(\int_{-n}^{-1} + \int_1^n\right) f(y)(-1)^m \frac{C_{2-h}(vy)}{y^\alpha}dy, \quad (v \geqq 0)$$
$$= 0 \qquad (v < 0);$$

and

$$\Psi_{\alpha,n}(v) = \int_{-1}^{1} f(y)\frac{C_{\alpha+1}(vv)}{y^\alpha}dy - \left(\int_{-n}^{-1} + \int_1^n\right) f(y)\frac{(-1)^m C_{1-h}(vy)}{y^\alpha}dy, \quad (v \geqq 0)$$
$$= 0 \qquad (v < 0),$$

then, from Lemma 3 and by the properties of Young's function, we have

$$I_\lambda(x) = \frac{1}{\pi}\lim_{n\to\infty}\left[\left\{\int_{-\delta}^{\lambda} \left\{1 - \left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1} \cos vx \frac{d^\alpha \Phi_{\alpha,n}(v)}{dv^{\alpha-1}}\right.\right.$$

$$+\int_{-\delta}^{\lambda}\left\{1-\left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1}\sin vx\,\frac{d^\alpha\Phi_{\alpha,n}(v)}{dv^{\alpha-1}}\Bigg],$$

where $\delta > 0$.

On the other hand, since

$$\frac{C^\alpha(vy)}{y^\alpha} = O(1), \quad \text{for } |y| \leqq 1$$

and

$$\frac{C_{2-h}(vy)}{y^\alpha} = O(y^{-\alpha}),\ \frac{C_{1-h}(vy)}{y^\alpha} = O(y^{-\alpha}) \text{ for } |y| > 1,$$

$\Phi_{\alpha,n}(v)$ and $\Psi_{\alpha,n}(v)$ are uniformly bounded in any finite interval of v, and tend uniformly to $\Phi(v)$ and $\Psi(v)$ in that interval.

In order to verify the condition of remainder terms, for $r=0$ we put

$$\Gamma^{(0)}_{\lambda,\Phi}(\varepsilon, n) = \left[1-\left\{\frac{\lambda-(k-1)\varepsilon}{\lambda}\right\}^2\right]^{\alpha-1}\cos\{\lambda-(k-1)\varepsilon\}x\,\frac{\Delta_*^{k-1}{}_{\epsilon}\, I^h\Phi_{\alpha,n}(\lambda)}{\varepsilon^{k-1}}$$

$$= \left[1-\left\{\frac{\lambda-(k-1)\varepsilon}{\lambda}\right\}^2\right]^{\alpha-1}\frac{\cos\{\lambda-(k-1)\varepsilon\}x}{\varepsilon^{k-1}}\left\{\int_{-1}^{1} f(x)\frac{C_\alpha(\lambda y)}{y^\alpha}dy\right.$$

$$\left.+\left(\int_{-n}^{-1}+\int_{1}^{n}\right)f(y)\frac{(-1)^m C_{2-h}(\lambda y)}{y^\alpha}dy\right\} = J_1 + J_2.$$

$$J_1 \equiv \left[1-\left\{\frac{\lambda-(k-1)\varepsilon}{\lambda}\right\}^2\right]^{\alpha-1}\frac{\cos\{\lambda-(k-1)\varepsilon\}x}{\varepsilon^{k-1}}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}$$

$$\times\frac{1}{\Gamma(h)}\int_0^{\lambda-j\epsilon}\{(\lambda-j\varepsilon)-u\}^{h-1}du\int_{-1}^{1} f(y)\frac{C_\alpha(uy)}{y^\alpha}dy$$

$$= O(\varepsilon^{-h})\left[\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\int_{-1}^{1} f(y)dy\frac{1}{\Gamma(h)}\int_0^1(1-u)^{h-1}(\lambda-j\varepsilon)^h\frac{C_\alpha\{(\lambda-j\varepsilon)uy\}}{y^\alpha}du\right]$$

$$= O(\varepsilon^{1-h})\left\{\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\int_{-1}^{1} f(y)dy\frac{1}{\Gamma(h)}\int_0^1(1-u)^{h-1}\left[\frac{d}{d\lambda}\frac{\lambda^h C_\alpha(\lambda u y)}{y^\alpha}\right]_{\lambda=\lambda'-j\epsilon}du\right\}$$

$$(\lambda-\varepsilon<\lambda'<\lambda).$$

$$= O(\varepsilon^{1-h}).$$

Similarly

$$J_2 = O(\varepsilon^{-h})\left(\int_{-n}^{-1}+\int_1^n\right)\frac{f(y)}{y^\alpha}dy\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\frac{1}{\Gamma(h)}\int_0^1(1-u)^{h-1}(\lambda-j\varepsilon)^h$$

$$\times C_{2-h}\{(\lambda-j\varepsilon)uy\}du$$

$$= O(\mathcal{E}^{-h})\Big[\int_{-n}^{-1} + \int_{1}^{n}\Big]\frac{f(y)}{y^{\alpha}}dy\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\frac{1}{\Gamma(h)}$$

$$\times\int_0^1 (1-u)^{h-1}\Big[\{\lambda-(j+1)\mathcal{E}\}^h C_{2-h}(\{\lambda-(j+1)\mathcal{E}\}uy) - (\lambda - j\mathcal{E})^h$$

$$\times C_{2-h}\{(\lambda - j\mathcal{E})uy\}\Big]du$$

$$= \Big[\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}O(\mathcal{E}^{-h})\Big\{\int_{-n}^{-1} + \int_{1}^{n}\Big\}\frac{f(y)}{y^{\alpha}}$$

$$\times\frac{\cos[\{\lambda-(j+1)\mathcal{E}\}y] - \cos\{(\lambda - j\mathcal{E})y\}}{y^h}dy,$$

$$= \sum_{j=0}^{k-1}(-1)^{j+1}\binom{k-2}{j}K_j,$$

say.

$$K_0 = O(\mathcal{E}^{-h})\Big\{\int_{-n}^{-1} + \int_{1}^{n}\Big\}\frac{f(y)}{y^{\alpha}}\frac{\cos\lambda y - \cos\{(\lambda-\mathcal{E})y\}}{y^h}dy = L_n + M_n$$

$$\lim_{n\to\infty} M_n = O(\mathcal{E}^{-h})\int_1^{\infty}\frac{f(y)}{y^{\alpha}}\frac{\cos\lambda y - \cos\{(\lambda-\mathcal{E})y\}}{y^h}dy$$

$$= O(\mathcal{E}^{-h})\Big(\int_1^{n_0} + \int_{n_0}^{1/\epsilon} + \int_{1/\epsilon}^{\infty}\Big)\frac{f(y)}{y^{\alpha}}\frac{\cos\lambda y - \cos\{(\lambda-\mathcal{E})y\}}{y^h}dy$$

$$= N_1 + N_2 + N_3,$$

say, where n_0 is determined for given $\eta > 0$ such that

$$\int_{n_0}^{\infty}\frac{|f(y)|}{y^{\alpha}}dy < \eta, \quad \text{for all } n \geqq n_0.$$

Then

$$|I_1| = O(\mathcal{E}^{1-h})\left|\int_1^{n_0}\frac{f(y)}{y^{\alpha}}\frac{y[\sin\lambda' y]}{y^h}dy\right|$$

$$= O(\mathcal{E}^{1-h})\int_1^{n_0}\frac{|f(y)|}{y^{\gamma-1}}dy = O(\mathcal{E}^{1-h}),$$

where $\lambda - \mathcal{E} < \lambda' < \lambda$.

$$|I_2| = O(\varepsilon^{-h})\left|\int_{n_0}^{\frac{1}{\epsilon}} \frac{f(y)}{y^\alpha}\frac{\varepsilon y[\sin\lambda'' y]}{y^h}dy\right|$$

$$= O(\varepsilon^{1-h})\int_{n_0}^{1/\epsilon} \frac{|f(y)|}{y^\alpha}y^{1-h}dy$$

$$= O(\varepsilon^{1-h}\varepsilon^{h-1})\int_{n_0}^{1/\epsilon}\frac{|f(y))|}{y^\alpha}dy$$

$$= O(\eta)$$

$$|I_3| = O(\varepsilon^{-h})\int_{1/\epsilon}^{\infty}\frac{|f(y)|}{y^\alpha}\frac{2}{y^h}dy \leqq O(\varepsilon^{-h}\varepsilon^h)\int_{1/\epsilon}^{\infty}\frac{|f(y)|}{y^\alpha}dy$$

$$= O(1)\int_{1/\epsilon}^{\infty}\frac{|f(y)|}{y^\alpha}dy.$$

Thus we get

$$\left|\lim_{\epsilon\to 0}\lim_{n\to\infty} M_n\right| = \lim_{\epsilon\to 0}\left\{O(\varepsilon^{1-h}) + O(\eta) + O(1)\int_{1/\epsilon}^{\infty}\frac{|f(y)|}{y^\alpha}dy\right\} = O(\eta).$$

Since η is any small number, we have

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} M_n = 0.$$

Consequently the required formula

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\Gamma^{(0)}_{\lambda,\Phi}(\varepsilon, n) = \lim_{\epsilon\to 0}\lim_{n\to\infty}\Gamma^{(0)}_{\lambda,\Phi}(\varepsilon, n) = 0$$

is derived easily. Similarly we obtain

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\Gamma^{(0)}_{\lambda,\Psi}(\varepsilon, n) = \lim_{\epsilon\to 0}\lim_{n\to\infty}\Gamma^{(0)}_{\lambda,\Psi}(\varepsilon, n) = 0.$$

We can treat the case $r = 1, 2, 3, \ldots, k-2$ analogously and then we get the theorem from Lemma 1 and Lemma 3.

4. Burkill's generalization of Fourier integral.

LEMMA 5. *Suppose that,* (1°) $F(0) = 0$, (2°) $F(x)$ *is continuous in any finite interval,* (3°) $F(x) = o(|x|^\alpha)$ *as* $|x| \to \infty$ *and* (4°) $F(x)/(1 + |x|^\alpha) \in L(-\infty, \infty)$, *for* $\alpha > 1$, *then we have*

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} dF(y)\int_0^x K_\mu\{\lambda(y-u)\}du = F(x),$$

where

$$K_\mu(\lambda x) = \frac{2C(\mu)}{\lambda}\alpha_\mu(\lambda x), \quad \mu + \frac{1}{2} = \alpha.$$

PROOF. We have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} dF(y)\int_0^x K_\mu\{\lambda(y-u)\}\,du = \lim_{n\to\infty}\frac{1}{\pi}\int_{-n}^{n} dF(y)\int_0^x K_\mu\{\lambda(y-u)\}du$$

$$= \lim_{n\to\infty} I_n(x), \text{ say.}$$

$$I_n(x) = \left[\frac{1}{\pi}F(y)\int_0^x K_\mu\{\lambda(y-u)\}du\right]_{-n}^{n} - \frac{1}{\pi}\int_{-n}^{n} F(y)\left\{\frac{d}{dy}\int_0^x K_\mu\{\lambda(y-u)\}du\right\}dy$$

$$= \frac{F(n)}{\pi}\int_0^x K_\mu\{\lambda(n-u)\}du - \frac{F(-n)}{\pi}\int_0^x K_\mu\{\lambda(n-u)\}du$$

$$+\frac{1}{\pi}\int_{-n}^{n} F(y)\left\{\frac{d}{dy}\int_{\lambda y}^{\lambda(y-x)}\frac{K^\mu(v)dv}{\lambda}\right\}dy$$

$$= \frac{F(n)}{\pi}\int_0^x K_\mu\{\lambda(n-u)\}du - \frac{F(-n)}{\pi}\int_0^x K_\mu\{\lambda(n-u)\}du$$

$$+\frac{1}{\pi}\int_{-n}^{n} F(y)[K_\mu\{\lambda(y-x)\} - K_\mu(\lambda y)]dy$$

$= J_1 - J_2 + J_3$, say.

Then

$$J_1 = \frac{1}{\pi}F(n)\cdot O\left(\int_0^x \lambda^{-\mu+\frac{1}{2}}(n-u)^{-\mu-\frac{1}{2}}\,du\right)$$

$$= O(F(n)/\lambda^{\mu-\frac{1}{2}}n^{\mu+\frac{1}{2}}) = O(F(u)/\lambda^{\alpha-1}n^{\alpha})$$

$$= o(1), \text{ as } n\to\infty,$$

and $J_2 = o(1)$ is proved similarly. Concerning J_3, we get from Lemma 1,

$$\lim_{\lambda\to\infty}\lim_{n\to\infty} J_3 = F(x) - F(0) = F(x),$$

for $F(0) = 0$.

THEOREM 2. *Suppose that* (1°) $F(0) = 0$, (2°) $F(x)$ *be continuous in any finite interval of* $(-\infty,\infty)$, (3°) $F(x) = o(|x|^\alpha)$ *as* $|x|\to\infty$ *for* $\alpha > 1$ *and* (4°) $F(x)/(1+|x|^\alpha)$ *be absolutely integrable in* $(-\infty,+\infty)$. *Let k be an integer such as* $k-1<\alpha\leqq k$, $k-\alpha = h$. *If* $k-2 = 2m$ $(m = 0,1,2\dots)$, *we put*

$$\varphi_\alpha(v) = \int_{-1}^{1}\frac{C_\alpha(vy)}{y^\alpha}dF(y) + \left(\int_{-\infty}^{-1} + \int_1^{\infty}\right)\frac{(-1)^m C_{2-h}(vy)}{y^\alpha}dF(y)$$

$$\psi_\alpha(v) = \int_{-1}^{1}\frac{C_{\alpha+1}(vy)}{y^\alpha}dF(y) + \left(\int_{-\infty}^{-1} + \int_1^{\infty}\right)\frac{(-1)^m C_{1-h}(vy)}{y^\alpha}dF(y),$$

and if $k-2=2m+1$ $(m=0,1,2,\dots)$, *we put*

$$\varphi_\alpha(v)=\int_{-1}^{1}\frac{C_{\alpha+1}(vy)dF(y)}{y^\alpha}-\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)\frac{(-1)^n C_{1-h}(vy)}{y^\alpha}dF(y),$$

$$\psi_\alpha(v)=\int_{-1}^{1}\frac{C_\alpha(vy)}{y^\alpha}dF(y)+\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)\frac{(-1)^m C_{2-h}(vy)}{y^\alpha}dF(y).$$

Then

$$F(x)=\frac{1}{\pi}(B,\alpha)\int_0^x\left\{\frac{\sin vx}{v}\frac{d^\alpha\varphi_\alpha(v)}{dv^{\alpha-1}}+\frac{1-\cos vx}{v}\frac{d\psi_\alpha(v)}{dv^{\alpha-1}}\right\}.$$

PROOF. From Lemma 5,

$$\begin{aligned}I_n&=\frac{1}{\pi}\int_{-n}^{n}dF(y)\int_0^x K_{\alpha-\frac{1}{2}}\{\lambda(y-u)\}du\\&=\frac{1}{\pi}\int_{-n}^{n}dF(y)\int_0^x du\int_0^\lambda\left\{1-\left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1}\cos v(y-u)\,dv\\&=\frac{1}{\pi}\left[\int_0^\lambda\left\{1-\left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1}\frac{\sin vx}{v}dv\int_{-n}^{n}\cos vy\ dF(y)\right.\\&\quad\left.+\int_0^\lambda\left\{1-\left(\frac{v}{\lambda}\right)^2\right\}^{\alpha-1}\frac{1-\cos vx}{v}dv\int_{-n}^{n}\sin vy\,dF(y)\right].\end{aligned}$$

The proof is almost identical with that of Theorem 1, But the lower limit of integral is not -0, but it is exactly 0. This is an essentially different point. To clarify this circumstance, we prove the case $k=2$ for the sake of simplicity.

Let us put

$$\varphi_{2,n}(v)=\int_{-n}^{n}\frac{1-\cos vy}{y^2}dF(y),$$

$$\psi_{2,n}(v)=\int_{-1}^{+1}\frac{vy-\sin vy}{y^2}dF(y)-\left(\int_{-n}^{-1}+\int_{1}^{n}\right)\frac{\sin vy}{y^2}dF(y).$$

Then integrating by parts, we get

$$\begin{aligned}I_n=&\int_0^\lambda\left(1-\frac{v}{\lambda}\right)\frac{\sin vx}{v}\frac{d^2\varphi_{2,n}(v)}{dv}\\&+\int_0^\lambda\left(1-\frac{v}{\lambda}\right)\frac{1-\cos vx}{v}\frac{d^2\psi_{2,n}(v)}{dv}\end{aligned}$$

We shall show that

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\Gamma_\varphi(\mathcal{E},n)=\lim_{\epsilon\to 0}\lim_{n\to\infty}\Gamma_\varphi(\mathcal{E},n),$$

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\Gamma_\psi(\mathcal{E},n)=\lim_{\epsilon\to 0}\lim_{n\to\infty}\Gamma_\psi(\mathcal{E},n),$$

where

$$\Gamma_\varphi(\mathcal{E},n)=\left(1-\frac{\mathcal{E}}{\lambda}\right)\frac{\sin\mathcal{E}x}{\mathcal{E}^2}\int_{-n}^{n}\frac{1-\cos\mathcal{E}y}{y^2}\,dF(y),$$

$$\Gamma_\psi(\mathcal{E},n)=\left(1-\frac{\mathcal{E}}{\lambda}\right)\frac{1-\cos\mathcal{E}x}{\mathcal{E}^2}\left\{\int_{-1}^{1}\frac{\mathcal{E}y-\sin\mathcal{E}y}{y^2}\,dF(y)\right.$$

$$\left.-\left(\int_{-n}^{-1}+\int_{1}^{n}\right)\frac{\sin\mathcal{E}v}{y^2}\,dF(y)\right\}.$$

Now

$$\Gamma_\varphi(\mathcal{E},n)=\left(1-\frac{\mathcal{E}}{\lambda}\right)x\frac{\sin\mathcal{E}x}{\mathcal{E}x}\left\{\left[\frac{1-\cos\mathcal{E}y}{\mathcal{E}y^2}F(y)\right]_{-n}^{n}\right.$$

$$\left.-\int_{-n}^{n}\frac{1}{\mathcal{E}}\left(\frac{1-\cos\mathcal{E}y}{y^2}\right)'F(y)dy\right\}$$

$$=J_1-J_2,\ \text{say}.$$

$$\lim_{\epsilon\to 0}J_1=\lim_{\epsilon\to 0}\left(1-\frac{\mathcal{E}}{\lambda}\right)x\cdot\frac{\sin\mathcal{E}x}{\mathcal{E}x}\cdot\mathcal{E}\left[\frac{1-\cos\mathcal{E}y}{\mathcal{E}^2y^2}F(y)\right]_{-n}^{n}=0,$$

$$\lim_{\epsilon\to 0}J_2=\left(1-\frac{\mathcal{E}}{\lambda}\right)x\frac{\sin\mathcal{E}x}{\mathcal{E}x}\frac{1}{\mathcal{E}}\int_{-n}^{n}\left\{\frac{y^2\mathcal{E}\sin\mathcal{E}y-2y(1-\cos\mathcal{E}y)}{y^4}\right\}F(y)dy$$

$$=0.$$

On the other hand,

$$|J_1|=O\left(\frac{|F(n)|+|F(-n)|}{n^2}\right)=o(1),\ \text{as } n\to\infty.$$

For a given $\eta>0$, we take a large N for a fixed $\mathcal{E}$ such that

$$\left(\int_{-\infty}^{-N}+\int_{N}^{\infty}\right)\left|\frac{1}{\mathcal{E}}\left(\frac{1-\cos\mathcal{E}y}{y^2}\right)'F(y)\right|dy<\eta,$$

and

$$\lim_{\epsilon\to 0}\int_{-N}^{N}\left|\frac{1}{\mathcal{E}}\frac{y^2-\sin\mathcal{E}y-2y(1-\cos\mathcal{E}y)}{y^4}F(y)\right|dy=0.$$

Therefore

$$\overline{\lim_{\epsilon\to 0}}\lim_{n\to\infty}|J_2| < \eta,$$

that is,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} J_2 = 0.$$

Thus we get

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\Gamma_\varphi(n,\mathcal{E}) = \lim_{\epsilon\to 0}\lim_{n\to\infty}\Gamma_\varphi(n,\mathcal{E}).$$

The remaining part is analogous. Burkill [1] proved the case $\alpha = 1$, and the summability is replaced by ordinary convergence. S. Izumi [6] proved the case $\alpha = 2$, but he has put -0 at the lower limit of integral.

§ 5. The L_p ($p > 1$) case.

LEMMA 6. *Let*

$$\int_{-\infty}^{\infty}\frac{|f(x)|^p}{1+|x|^{\alpha+1}}\,dx < \infty \text{ for } p > 1 \text{ and } \alpha > 0.$$

If (1°) $\displaystyle\int_{-\infty}^{\infty} K(x)\,dx = 1,$

(2°) $|K(x)| \leqq M$ *for* $|x| \leqq 1,$

and

(3°) $|x^{1+\beta}K(x)| \leqq M$ *as* $|x| \to \infty$, *for* $\beta > \dfrac{\alpha}{p}$,

then

$$\lim_{\lambda\to\infty}\lambda\int_{-\infty}^{\infty} f(y)K\{\lambda(y-x)\}dy = f(x), \quad a.e.$$

PROOF. We may assume without loss of generality that $x = 0$ is the Lebesgue point of $f(x)$ and $f(0) = 0$. Therefore it is sufficient to prove

$$\lim_{\lambda\to\infty}\lambda\int_0^{\infty} f(y)\,K(\lambda y)dy = 0.$$

Let us put

$$I = \lambda\int_0^\infty f(x)K(\lambda x)dx = \int_0^{1/\lambda} + \int_{1/\lambda}^{\eta} + \int_\eta^\infty = I_1 + I_2 + I_3,$$

say. Since $x = 0$ is the Lebesgue point,

$$I_1 = O\left(\lambda\int_0^{1/\lambda}|f(x)|\,dx\right) = o(1), \text{ as } \lambda\to\infty.$$

If we write

$$\int_0^t |f(x)|dx = F(t),$$

then $|F(t)/t| < \varepsilon$, for all t such as $0 < t \leqq \eta$. Therefore

$$|I_2| \leqq \lambda \int_{1/\lambda}^{\eta} |f(x)||K(\lambda x)|\,dx \leqq M\lambda^{-\beta} \int_{1/\lambda}^{\eta} |f(x)| \frac{dx}{x^{\beta+1}}$$

$$= M\lambda^{-\beta}\Big[\frac{F(x)}{x^{\beta+1}}\Big]_{1/\lambda}^{\eta} + M\lambda^{-\beta}(\beta+1)\int_{1/\lambda}^{\eta} \frac{F(t)}{t^{\beta+2}}\,dt$$

$$= I_4 + I_5.$$

Then

$$|I_4| = \left|\lambda^{-\beta}\frac{F(\eta)}{\eta^{\beta+1}}\right| + \left|\lambda^{-\beta}\frac{F(1/\lambda)}{(1/\lambda)^{\beta+1}}\right|$$

$$\leqq \left|\frac{F(\eta)}{\eta}\right| + \left|\frac{F(1/\lambda)}{1/\lambda}\right| \leqq 2\varepsilon,$$

$$|I_5| \leqq (\beta+1)\lambda^{-\beta}\int_{1/\lambda}^{\eta} \frac{\varepsilon}{t^{\beta+1}} \leqq 4\varepsilon.$$

On the other hand, putting $1/p + 1/q = 1$, we have

$$\int_{\eta}^{\infty} |x^{(q-1)(\alpha+1)}K^{(p)}(x)|\,dx \leqq M\int_{\eta}^{\infty} |x|^{(q-1)(\alpha-1)-(\beta+1)q}dx$$

$$\leqq M\int_{\eta}^{\infty} x^{-(1+\delta)}dx < \infty,$$

where $\delta = \beta - \alpha/p > 0$. Therefore

$$|I_3| \leqq \Big(\int_{\eta}^{\infty} \frac{|f(x)|^p}{|x|^{\alpha+1}}dx\Big)^{1/p}\Big(\int_{\lambda\eta}^{\infty} \lambda^{q-\frac{q(\alpha+1)}{p}-1}u^{\frac{q(\alpha+1)}{p}}|K(u)|^q du\Big)^{1/q}$$

$$\leqq \Big(\int_{\eta}^{\infty} \frac{|f(x)|^p}{|x|^{\alpha+1}}dx\Big)^{1/p}\Big(\int_{\lambda\eta}^{\infty} u^{(q-1)(\alpha+1)}|K(u)|^q du\Big)^{1/q}$$

$$\to 0, \text{ as } \lambda \to \infty.$$

Thus we get the lemma.

LEMMA 7. *Let*

$$\int_{-\infty}^{\infty} \frac{|f(x)|^p}{1+|x|^\alpha} dx < \infty \text{ for } p > 1 \text{ and } \alpha > 1,$$

then

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(y)dy \int_0^{\lambda} \left\{1 - \left(\frac{v}{\lambda}\right)^2\right\}^{\frac{\alpha-1}{p}+\delta} \cos v(y-x)dv = f(x), a.e.$$

PROOF. Substituting

$$k_\beta \alpha_{\alpha+\frac{1}{2}}(x), \text{ where, } k_\beta = \frac{\Gamma(\beta+1)}{2\sqrt{\pi}\,\Gamma(\beta+\frac{3}{2})}, \quad \beta = \frac{\alpha-1}{p} + \delta,$$

in the place of $K(x)$ in Lemma 6, we can prove the lemma similarly as Lemma 1.

LEMMA 8. *The inverse difference*

$$S_k^{-\mu}(\cos\lambda x) = \sum_{\nu=0}^{k} A_{k-\nu}^{-\mu-1} \cos\{\lambda - (n-\nu)\mathcal{E}\}x$$

satisfies

(1) $$S_k^{-\mu}\left\{S_k^{-\nu}(\cos\lambda x)\right\} = \Delta_{*\,\epsilon}^{k} \cos\lambda x, \quad k = \mu + \nu,$$

(2) $$S^{-\mu}(\cos\lambda x) = O(\mathcal{E}^{\mu} x^{\mu}) \quad as\ \mathcal{E} \to 0.$$

Proof. Immediate.

THEOREM 3. Let

$$\int_{-\infty}^{\infty} \frac{|f(x)|^p}{1+|x|^\alpha} dx < \infty, \text{ for } p > 1,\ \alpha > 1,$$

then

$$f(x) = \left\{(B, (\alpha-1)/p + 1 + \delta)\right\} \frac{1}{\pi} \int_{-0}^{\infty} \left\{\cos vx \frac{d^\alpha \Phi^\alpha(v)}{dv^{\alpha-1}} + \sin vx \frac{d^\alpha \Psi^\alpha(v)}{dv^{\alpha-1}}\right\},$$

a.e., where $\Phi_\alpha(v)$ and $\Psi_\alpha(v)$ are defined in Theorem 1.

PROOF. The method of proof is almost identical with that of Theorem 1. The existence of $\Phi_\alpha(v)$ and $\Psi_\alpha(v)$ is proved by Hölder's inequality. The essentially different point lies in proving the condition of the end points. For instance we shall prove

$$\Gamma_\lambda^{(0)}(\mathcal{E}, n) = O\left(\mathcal{E}^{\frac{\alpha-1}{p}+\delta}\right) \int_1^{\infty} \frac{f(y)}{y^\alpha} \frac{\Delta_{*\,\epsilon}^{(k-1)} \cos\lambda y}{\mathcal{E}^{k-1} y^h} dy = o(1) \text{ as } \mathcal{E}\to 0.$$

Let us put

$$\Gamma_\lambda^{(0)}(\varepsilon, n) = O(\varepsilon^{\frac{\alpha-1}{p}+\delta})\left(\int_1^N + \int_N^{1/\epsilon} + \int_{1/\epsilon}^\infty\right)\frac{f(y)}{y^\alpha}\frac{\Delta_{\epsilon}^{*(k-1)}\cos\lambda y}{\varepsilon^{k-1}y^h}\,dy$$

$$= J_1 + J_2 + J_3,$$

say, where we select N such that

$$\left(\int_N^\infty \frac{|f(x)|^p}{x^\alpha}dx\right)^{1/p} < \eta$$

for arbitrarily small given $\eta > 0$. Then

$$|J_1| = O(\varepsilon^{\frac{\alpha-1}{p}+\delta})\left(\int_1^N \frac{|f(y)|}{|y|^{\alpha+h}}dy\right) = o(1), \text{ as } \varepsilon\to 0.$$

Using Hölder's inequality

$$|J_3| = O\left(\varepsilon^{\frac{\alpha-1}{p}+\delta-k+1}\right)\left(\int_{1/\epsilon}^\infty \frac{|f(y)|^p}{y^\alpha}dy\right)^{1/p}\left(\int_{1/\epsilon}^\infty \frac{1}{y^\alpha}\left|\frac{\Delta_{\epsilon}^{*(k-1)}\cos\lambda y}{y^h}\right|^q dy\right)^{1/q}$$

$$= O\left(\varepsilon^{\frac{\alpha-1}{p}+\delta-k+1+\frac{\alpha-1}{q}+h-\delta}\right)\left(\int_{1/\epsilon}^\infty \frac{dy}{y^{1+q\delta}}\right)^{1/q}\left(\int_{1/\epsilon}^\infty \frac{|f(y)|^p}{y^\alpha}dy\right)^{1/p}$$

and as

$$(\alpha - 1/p) + \delta - k + 1 + (\alpha - 1/q) + h - \delta = 0, \text{ and } 1 + q\delta > 1,$$

we obtain

$$|J_3| = O\left(\int_{1/\epsilon}^\infty \frac{dy}{y^{1+q\delta}}\right)^{1/q}\left(\int_{1/\epsilon}^\infty \frac{|f(y)|^p}{y^\alpha}dy\right)^{1/p} \to 0, \text{ as } \varepsilon \to 0.$$

Lastly from Lemma 8,

$$|J_2| = O\left(\varepsilon^{\frac{\alpha-1}{p}+\delta-k+1}\right)\cdot\left(\int_N^{1/\epsilon} \frac{|f(y)|^p}{y^\alpha}dy\right)^{1/p}$$

$$\times\left(\int^{1/\epsilon} \frac{1}{y^\alpha}\left|\frac{\bar{S}_{k-1}^{-((\alpha-1)/p+\delta)}\left\{\bar{S}_{k-1}^{-(k-1-(\alpha-1)/p-\delta)}(\cos\lambda y)\right\}}{y^h}\right|^q dy\right)^{1/q}$$

$$= O\left(\varepsilon^{\frac{\alpha-1}{p}+\delta-k+1}\right)\left(\int_N^{1/\epsilon} \frac{|f(y)|^p}{y^\alpha}dy\right)^{1/p}\times$$

$$\times\left(\int_N^{1/\epsilon}\frac{1}{y^\alpha}\cdot\left|\frac{S_{k-1}^{-((\alpha-1)/p+\delta)}\;O((\varepsilon y)^{k-1-(\alpha-1)/p-\delta})}{y^h}\right|^q dy\right)^{1/q}$$

$$=O\left(\int_N^{1/\epsilon}\frac{1}{y^{\alpha+qh+q(\alpha-1/p+\delta-k+1)}}dy\right)\left(\int_N^{1/\epsilon}\frac{|f(y)|^p}{y^\alpha}dy\right)^{1/p},$$

and

$$\alpha+qh+q((\alpha-1)/p+\delta-k+1)=q(\alpha/q+h+(\alpha-1)/p+\delta-k+1)$$
$$=q(-1/p+\delta+1)=1+q\delta>1.$$

Therefore

$$|J_2|=O\left(\int_N^{1/\epsilon}\frac{dy}{y^{1+q\delta}}\right)^{1/q}\left(\int_N^{1/\epsilon}\frac{|f(y)|^p}{y^\alpha}dy\right)^{1/p},$$

$$\overline{\lim_{\epsilon\to 0}}\,|J_2|\leqq\eta.$$

Since η is arbitrarily small, we get

$$\lim_{\epsilon\to 0}\Gamma_\lambda(\varepsilon,n)=0.$$

The other estimations are derived by the similar way.

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