SOME TRIGONOMETRICAL SERIES, V

SHIN-ICHI IZUMI

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1. Let $f(x)$ be any integrable function defined in $(0,1)$ and $f(x)=f(x+1)$ for all real x. Let us put

$$F_n(x)=F_n(x,f)=\frac{1}{n}\sum_{k=1}^{n} f\left(x+\frac{k}{n}\right)$$

which is known as Riemann sum of $f(x)$.

Jessen has proved that if (n_k) is a sequence of integers such that $n_k | n_{k+1}$ $(k=1,2,\ldots)$, then $F_{n_k}(x,f)$ converges to the integral of $f(x)$ for almost all x as $k\to\infty$. But, in general, $F_n(x,f)$ does not converge almost everywhere for integrable function $f(x)$. Further, Marcinkiewicz and Zygmund [1] proved that there is a function $f(x)$ belonging to the class (L^p) $(1\leqq p<2)$ such that its Riemann sum does not converge. His example is essentially

$$f(x)\sim\sum_{k=1}^{\infty}\frac{\log k}{\sqrt{k}}\cos 2\pi kx.$$

Recently, T. Tsuchikura [2] proved that the Riemann sum of the function

$$f(x)\sim\sum_{k=2}^{\infty}\frac{\cos 2\pi kx}{\sqrt{k}(\log k)^{1+\varepsilon}}\quad(\varepsilon>0)$$

converges almost everywhere, and proposed the problem "Does the Riemann sum of the function

$$f(x)\sim\sum_{k=2}^{\infty}\frac{\cos 2\pi kx}{\sqrt{k}\log k}$$

converge almost everywhere?"

This is positive. We can prove, more generally, that the Riemann sum of the function

$$f(x)\sim\sum_{k=2}^{\infty}\frac{\cos 2\pi kx}{\sqrt{k}(\log k)^{\alpha}}$$

diverges almost everywhere for $\alpha\leqq 1/2$, and converges almost everywhere for $\alpha>1/2$.

2. We prove

THEOREM 1. *Let $f(x)$ be an integrable function with period 1 and its Fourier series be*

$$f(x)\sim\sum_{n=2}^{\infty}\frac{\cos 2\pi nx}{\sqrt{n}(\log n)^{\alpha}}\cdot\tag{1}$$

Then the Riemann sum of $f(x)$

$$F_n(x) = F_n(x,f) = \frac{1}{n}\sum_{k=1}^{n} f\left(x + \frac{k}{n}\right) \tag{2}$$

diverges almost everywhere for $\alpha \leqq 1/2$ *and converges almost everywhere for* $\alpha > 1/2$, *as* $n \to \infty$.

For the proof of this theorem, we need a lemma, due to Khintchine:

LEMMA 1. *If* $f(x)$ *is a positive increasing function defined in* $(0, \infty)$ *such that* $\int^{\infty} f(x)dx = \infty$, *then there is an infinite number of solutions* p/q *such that*

$$|x - p/q| < f(q)/q \tag{3}$$

for almost all x, *but if* $\int^{\infty} f(x)dx$ *converges, then the number of solutions of* (3) *is finite for almost all* x.

Let us now prove the theorem. We can easily see that

$$F_n(x) = \sum_{k=1}^{\infty} \frac{\cos 2\pi knx}{\sqrt{kn}\,(\log (kn))^{\alpha}}$$

for $x \not\equiv 0 \pmod 1$. The last series, by Abel's lemma, is

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\cos 2\pi knx}{\sqrt{kn}\,(\log (kn))^{\alpha}} \\
&= -\frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}\,k^{3/2}(\log (kn))^{\alpha}} \sum_{\lambda=1}^{n} \cos 2\pi\lambda nx\,(1 + o(1)) \\
&= \frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}\,k^{5/2}(\log (kn))^{\alpha}} \frac{\sin^2 (k+1)2\,nx}{2\sin^2 \pi\,nx}(1 + o(1)) \\
&= \frac{3}{4} S\,(1 + o(1)),
\end{aligned} \tag{4}$$

say. In order to estimating S, we distinguish three cases.

a) $\alpha < 1/2$. By the lemma, for almost all x, there is an infinite number of integers n such that

$$|(nx)| < 1/n \log n, \tag{5}$$

(y) denoting the difference of y and the nearest integer. For such n

$$\frac{\sin^2 (k+1)2\pi nx}{2\sin^2 \pi\,nx} > k^2/2 \qquad \text{for } k \leqq An\log n, \tag{6}$$

A being a constant, and then

$$S \geqq \sum_{1 \leqq k < n\log n} \frac{1}{\sqrt{kn}\,(\log (kn))^{\alpha}} \geqq A\frac{\sqrt{n\log n}}{\sqrt{n}\,(\log n)^{\alpha}} = A\,(\log n)^{1/2-\alpha}$$

Hence, for almost all x,

$$\limsup_{n\to\infty} F_n(x) = \infty. \tag{7}$$

b) $\alpha = 1/2$. By the lemma, we can replace $\log n$ in (5) by $\log n$ loglog n, and then we obtain, for almost all x,

$$S \geqq A\sqrt{\log\log n}$$

for infinitely many n. Thus we get (7) in this case.

c) $\alpha > 1/2$. For almost all but fixed x, there is an integer n_0 such that

$$|(nx)| > 1/n(\log n)^\beta \qquad (n \geqq n_0)$$

where $2\alpha > \beta > 1$, by the second part of the lemma. Now

$$S = \sum_{k<n(\log n)^\beta} - \sum_{k\geqq n(\log n)^\beta} \equiv S_1 + S_2,$$

say. Then, for $n \geqq n_0$, we have

$$S_1 \leqq A \sum_{k<n(\log n)^\beta} \frac{1}{\sqrt{nk}(\log(kn))^\alpha} = O\left(\frac{\sqrt{n}(\log n)^{3/2}}{\sqrt{n}(\log n)^\alpha}\right)$$

$$= O\left(\frac{1}{(\log n)^{\alpha-\beta/2}}\right) = o(1).$$

and

$$S_2 < \frac{1}{2\sin^2 \pi nx} \sum_{k\geqq n(\log n)^\beta} \frac{1}{\sqrt{n}\, k^{5/2}(\log(kn))^\alpha}$$

$$< \frac{1}{2\sin^2 \pi/n(\log n)^3} \frac{1}{\sqrt{n}(\log n)^\alpha} \sum_{k\geqq n(\log n)^\beta} \frac{1}{k^{5/2}}$$

$$< A\frac{n^2(\log n)^{2\beta}}{\sqrt{n}(\log n)^\alpha} \frac{1}{(n(\log n)^3)^{3/2}} = O\left(\frac{1}{(\log n)^{\alpha-\beta/2}}\right) = o(1)$$

Thus we get $S_2 = o(1)$, and then

$$\lim_{n\to\infty} F_n(x) = 0$$

almost everywhere.

3. Concerning the convergence of the Riemann sum of (1) in the stronger sense than the ordinary one, we get

THEOREM 2. *Let $f(x)$ be defined by* (1), *then the series*

$$\sum_{n=1}^{\infty} F_n^2(x, f)$$

converges almost everywhere for $\alpha > 1$ and diverges almost everywhere for $\alpha \leqq 1$.

PROOF. We have

$$\int_0^1 F^2(x,f)\,dx = \sum_{k=1}^{\infty} \frac{1}{kn(\log(kn))^{2\alpha}},$$

$$\sum_{n=2}^{\infty} \int_0^1 F_n^2(x,f)\,dx = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{kn(\log(kn))^{2\alpha}}$$

$$= \sum_{m=1}^{\infty} \frac{d(m)}{m(\log m)^{2\alpha}},$$

where $d(m)$ donotes the number of divisors of m. Putting $\tau(m) = \sum_{i=1}^{m} d(i)$, the series

$$\sum \frac{d(m)}{m(\log m)^{2\alpha}} \quad \text{and} \quad \sum \frac{\tau(m)}{m^2(\log m)^{2\alpha}}$$

converge or diverge simultaneously. It is known that

$$\tau(m) = m \log m + o(m).$$

Hence the last series converges or diverges according as $\alpha > 1$ or $\alpha \leqq 1$. Thus the theorem is proved.

4. We can generalize Theorem 1 in the following form:

THEOREM 3. *Let*

$$f(x) \sim \sum_{k=1}^{\infty} a_k \cos 2\pi kx, \tag{8}$$

where (a_k) *is a convex null sequence. If*

$$\frac{1}{n} \sum_{k=1}^{[n^2 \log n]} a_k \neq O(1) \qquad (n \to \infty), \tag{9}$$

or more generally

$$\frac{1}{n} \sum_{k=1}^{[n^2 \log n \, \log\log n]} a_k \neq O(1) \qquad (n \to \infty),$$

then the Riemann sum of (8) *diverges almost everywhere. On the other hand, if, for some* $\beta > 1$,

$$\frac{1}{n} \sum_{k=1}^{[n^2(\log n)^{\beta}]} a_k \to 0 \qquad (n \to \infty).$$

or more generally

$$\frac{1}{n} \sum_{k=1}^{[n^2 \log n(\log\log n)^{\beta}]} a_k \to 0 \qquad (n \to \infty),$$

then the Riemann sum of (8) *converges to zero almost everywhere.*

Proof runs similarly as Theorem 1. As in the case a) in the proof of

Theorem 1, the Riemann sum diverges as

$$\sum_{k=1}^{[n \log n]} k(k+1)\Delta^2 a_{kn} \neq O(1) \qquad (n \to \infty). \tag{10}$$

The left side is

$$\sum_{k=1}^{[n \log n]} k(k+1)\Delta^2 a_{kn} = 2 \sum_{k=1}^{[n \log n]} a_{kn} - n \log n \, a_{[n^2 \log n]} + (n \log n)^2 a_{[n^2 \log n]}$$

and then (9) implies (10), since (a_k) tends to zero monotonously.

References

[1] J. MARCINKIEWICZ and A. ZYGMUND, Fund. Math., 28 (1930).
[2] T. TSUCHIKURA, Tôhoku Math. Journ., (2) vol. 3 (1951).

MATHEMATICAL INSTITUTE, TOKYO TORITSU UINIVERSITY, TOKYO