

COMPLEX EXTENSORS AND LAGRANGIAN SUBMANIFOLDS IN COMPLEX EUCLIDEAN SPACES

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Abstract. Lagrangian H -umbilical submanifolds are the “simplest” Lagrangian submanifolds next to totally geodesic ones in complex-space-forms. The class of Lagrangian H -umbilical submanifolds in complex Euclidean spaces includes Whitney’s spheres and Lagrangian pseudo-spheres. For each submanifold M of Euclidean n -space and each unit speed curve F in the complex plane, we introduce the notion of the complex extensor of M in the complex Euclidean n -space via F . The main purpose of this paper is to classify Lagrangian H -umbilical submanifolds of the complex Euclidean n -space by utilizing complex extensors. We prove that, except the flat ones, Lagrangian H -umbilical submanifolds of complex Euclidean n -space with n greater than 2 are Lagrangian pseudo-spheres and complex extensors of the unit hypersphere of the Euclidean n -space. For completeness we also include in the last section the classification of flat Lagrangian H -umbilical submanifolds of complex Euclidean spaces.

1. Introduction. Let $f: M \rightarrow \tilde{M}^m$ be an immersion from a Riemannian n -manifold M into a complex m -dimensional Kaehler manifold \tilde{M}^m . M is called a *totally real* submanifold if the almost complex structure J of \tilde{M}^m carries each tangent space of M into its corresponding normal space. The totally real submanifold M of \tilde{M}^m is called *Lagrangian* if $n=m$.

An n -dimensional submanifold M of a Riemannian manifold N is said to be totally umbilical (respectively, totally geodesic) if its second fundamental form h in N satisfies $h(X, Y) = \langle X, Y \rangle H$ (respectively, $h=0$ identically), where $H = (1/n)\text{trace } h$ is the mean curvature vector of M in N and $\langle \cdot, \cdot \rangle$ denotes the inner product associated with the Riemannian metrics on M as well as on N . For a totally umbilical submanifold the shape operator A_H at H has exactly one eigenvalue; moreover, $A_\xi = 0$ for each normal vector ξ perpendicular to H .

Totally umbilical submanifolds, if they exist, are the simplest submanifolds next to totally geodesic submanifolds in a Riemannian manifold. However, it was proved in [7] that a complex-space-form of complex dimension ≥ 2 admits no totally umbilical Lagrangian submanifolds except the totally geodesic ones.

In view of above facts it is natural to look for the “simplest” Lagrangian submanifolds next to the totally geodesic ones in complex-space-forms. In order to do so the author introduced in [4] the notion of *Lagrangian H -umbilical submanifolds*. In [4]

he classified Lagrangian H -umbilical submanifolds of complex projective spaces and also of complex hyperbolic spaces. In particular, he proved that, except some exceptional classes, Lagrangian H -umbilical submanifolds of complex projective spaces or of complex hyperbolic spaces are obtained from Legendre curves via Hopf's fibration in some natural ways (see [4] for details).

According to [4], a Lagrangian H -umbilical submanifolds of a Kaehler manifold \tilde{M}^n is a non-totally geodesic Lagrangian submanifold whose second fundamental form takes the following simple form:

$$(1.1) \quad \begin{aligned} h(e_1, e_1) &= \lambda J e_1, & h(e_2, e_2) &= \dots = h(e_n, e_n) = \mu J e_1, \\ h(e_1, e_j) &= \mu J e_j, & h(e_j, e_k) &= 0, \quad j \neq k, \quad j, k = 2, \dots, n \end{aligned}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field. It is obvious that condition (1.1) is equivalent to

$$(1.1)' \quad \begin{aligned} h(X, Y) &= \alpha \langle JX, H \rangle \langle JY, H \rangle H \\ &\quad + \beta \langle H, H \rangle \{ \langle X, Y \rangle H + \langle JX, H \rangle JY + \langle JY, H \rangle JX \} \end{aligned}$$

for any vectors X, Y tangent to M , where

$$(1.2) \quad \alpha = \frac{\lambda - 3\mu}{\gamma^3}, \quad \beta = \frac{\mu}{\gamma^3}, \quad \gamma = \frac{\lambda + (n-1)\mu}{n}$$

where $H \neq 0$. Clearly, a non-minimal Lagrangian H -umbilical submanifold satisfies the following two conditions:

- (a) JH is an eigenvector of the shape operator A_H and
- (b) the restriction of A_H to $(JH)^\perp$ is proportional to the identity map.

On the other hand, because the second fundamental form of a Lagrangian submanifold satisfies (cf. [8])

$$(1.3) \quad \langle h(X, Y), JZ \rangle = \langle h(Y, Z), JX \rangle = \langle h(Z, X), JY \rangle$$

for vectors X, Y, Z tangent to M , Lagrangian H -umbilical submanifolds are indeed the simplest Lagrangian submanifolds satisfying both Conditions (a) and (b). In this way we can regard Lagrangian H -umbilical submanifolds as the simplest Lagrangian submanifolds in a complex-space-form next to the totally geodesic ones.

Given an immersion $G: M \rightarrow E^m$ of a manifold into Euclidean m -space E^m and a unit speed curve $F: I \rightarrow C$ in the complex plane, we may extend immersion $G: M \rightarrow E^m$ to an immersion of $I \times M$ into complex Euclidean m -space C^m by utilizing the tensor product of F and G . We call this extension the *complex extensor* of G via F . Whitney's spheres and Lagrangian pseudo-spheres are nice examples of complex extensors of the ordinary unit hypersphere (see Section 2 for details).

In Section 2 we provide some basic properties of complex extensors of an immersion. In particular, we prove that every complex extensor of the unit hypersphere of E^n is a

Lagrangian H -umbilical submanifold of C^n . Furthermore, we provide examples of Lagrangian H -umbilical submanifolds of C^n satisfying (1.1) with $\lambda = 3\mu$, $\lambda = 2\mu$, $\lambda = \mu$ and $\lambda = 0$, respectively. Section 3 gives a simple geometric characterization of Lagrangian pseudo-spheres; namely, a Lagrangian submanifold of C^n is a Lagrangian pseudo-sphere if and only if it satisfies (1.1) with $\lambda = 2\mu$. In Section 4, we obtain classification theorems for Lagrangian H -umbilical submanifolds of C^n . In particular, we prove that, except the flat ones, Lagrangian H -umbilical submanifolds of C^n with $n \geq 3$ are Lagrangian pseudo-spheres and complex extendors of the unit hypersphere of E^n . For completeness we include in the last section the classification of flat Lagrangian H -umbilical submanifolds of complex Euclidean spaces.

2. Geometry of complex extendors. In this section we introduce the notion of complex extendors of an immersion and provide some of their basic properties.

Let $G: M^{n-1} \rightarrow E^m$ be an isometric immersion of a Riemannian $(n-1)$ -manifold into Euclidean m -space E^m and $F: I \rightarrow C$ a unit speed curve in the complex plane. We extend the immersion $G: M^{n-1} \rightarrow E^m$ to an immersion of $I \times M^{n-1}$ into complex Euclidean m -space C^m given by

$$(2.1) \quad \phi = F \otimes G: I \times M^{n-1} \rightarrow C \otimes E^m = C^m,$$

where $F \otimes G$ is the tensor product immersion of F and G defined by

$$(2.2) \quad (F \otimes G)(s, p) = F(s) \otimes G(p), \quad s \in I, \quad p \in M^{n-1}.$$

We call such an extension $F \otimes G$ of the immersion G a *complex extensor* of G (or of submanifold M^{n-1}) via F .

An immersion $f: N \rightarrow E^m$ is called *spherical* (respectively, *unit spherical*) if N is immersed into a hypersphere (respectively, unit hypersphere) of E^m centered at the origin. The complex extensor $\phi = F \otimes G: I \times M^{n-1} \rightarrow C^m$ is called *F-isometric* if, for each $p \in M^{n-1}$, the immersion $F \otimes G(p): I \rightarrow C^m: s \mapsto F(s) \otimes G(p)$ is isometric. Similarly, the complex extensor is called *G-isometric* if, for each $s \in I$, the immersion $F(s) \otimes G: M^{n-1} \rightarrow C^m: p \mapsto F(s) \otimes G(p)$ is isometric.

LEMMA 2.1. *Let $G: M^{n-1} \rightarrow E^m$ be an isometric immersion of a Riemannian $(n-1)$ -manifold into Euclidean m -space E^m and $F: I \rightarrow C$ a unit speed curve in the complex plane. Then*

- (1) *the complex extensor $\phi = F \otimes G$ is F-isometric if and only if G is unit spherical,*
- (2) *the complex extensor $\phi = F \otimes G$ is G-isometric if and only if F is unit spherical,*
and
- (3) *the complex extensor $\phi = F \otimes G$ is totally real if and only if either G is spherical or $F(s) = cf(s)$ for some $c \in C$ and real-valued function f .*

PROOF. Statements (1) and (2) are easy to verify. For Statement (3), recall that an immersion $f: N \rightarrow C^m$ is totally real if the complex structure J on C^m carries each

tangent space of N into its corresponding normal space. By a direct computation it is easy to see that the complex extensor $\phi = F \otimes G$ is totally real if and only if, for any $s \in I, p \in M^{n-1}$ and $Y \in T_p M^{n-1}$, we have

$$(2.3) \quad \operatorname{Re}(iF(s)\bar{F}'(s))\langle G(p), Y \rangle = 0,$$

where \bar{F}' denotes the complex conjugate of F' and $\operatorname{Re}(iF\bar{F}')$ the real part of $iF\bar{F}'$. Hence, we have either $\operatorname{Re}(iF(s)\bar{F}'(s))=0$ for all $s \in I$ or $\langle G(p), Y \rangle=0$ for all $p \in M^{n-1}, Y \in T_p M^{n-1}$. If the first case occurs, $F=cf(s)$ for some $c \in \mathbb{C}$ and if the second case occurs, G is spherical. \square

A submanifold M^{n-1} of E^m is said to be of *essential codimension one* if locally M^{n-1} is contained in an affine n -subspace of E^m .

PROPOSITION 2.2. *Let $G: M^{n-1} \rightarrow E^m$ be an isometric immersion of a Riemannian $(n-1)$ -manifold into Euclidean m -space E^m and $F: I \rightarrow \mathbb{C}$ a unit speed curve. Then the complex extensor $\phi = F \otimes G: I \times M^{n-1} \rightarrow \mathbb{C}^m$ is totally geodesic (with respect to the induced metric) if and only if one of the following two cases occurs:*

- (1) $G: M^{n-1} \rightarrow E^m$ is of essential codimension one and $F(s)=(s+a)c$ for some real number a and some unit complex number c .
- (2) $n=2$ and G is a line in E^m .

PROOF. Under the hypothesis we have

$$(2.4) \quad \phi_s = F'(s) \otimes G, \quad Y\phi = F \otimes Y, \quad \phi_s = \frac{\partial \phi}{\partial s}, \quad \phi_{ss} = \frac{\partial^2 \phi}{\partial s^2},$$

$$(2.5) \quad \phi_{ss} = F''(s) \otimes G, \quad Y\phi_s = F'(s) \otimes Y,$$

$$(2.6) \quad YZ\phi = F \otimes \nabla_Y Z + F \otimes h_G(Y, Z),$$

where ∇ denotes the Levi-Civita connection of M^{n-1} , Y, Z vector fields tangent to the second component of $I \times M^{n-1}$, and h_G the second fundamental form of $G: M^{n-1} \rightarrow E^m$. In this article, we shall regard each tangent vector of M^{n-1} also as a tangent vector of the product manifold $I \times M^{n-1}$ in a natural way.

Since $F: I \rightarrow \mathbb{C}$ is a unit speed curve, $F'(s)$ and $F''(s)$ are orthogonal. Thus, for any unit normal vector ξ of M^{n-1} in E^m , the vector $F''(s) \otimes \xi$ is normal to $I \times M^{n-1}$ in \mathbb{C}^m (via ϕ).

Suppose $\phi = F \otimes G$ is totally geodesic. Then, by definition, $\phi_{ss}, Y\phi_s$ and $YZ\phi$ are tangent to $I \times M^{n-1}$ in \mathbb{C}^m . Thus (2.6) yields

$$(2.7) \quad \langle F''(s) \otimes \xi, F(s) \otimes h_G(Y, Z) \rangle = 0,$$

for any vector fields Y, Z tangent to the M^{n-1} and for any $s \in I$. This implies that either G is totally geodesic or $\langle F'', F \rangle = 0$ identically, where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product of the complex plane.

Case (i): G is non-totally geodesic. In this case, $F''(s)$ is perpendicular to both $F(s)$ and $F'(s)$ for any $s \in I$. Since F is a unit speed plane curve, this implies $F''(s) = 0$. Hence, $F(s) = sc + e$ for some $c, e \in \mathbb{C}$ with $|c| = 1$. Therefore, we obtain

$$(2.8) \quad \phi_s = c \otimes G, \quad Y\phi = (sc + e) \otimes Y,$$

$$(2.9) \quad \phi_{ss} = 0, \quad Y\phi_s = c \otimes Y,$$

$$(2.10) \quad YZ\phi = (sc + e) \otimes \nabla_Y Z + (sc + e) \otimes h_G(Y, Z),$$

for Y, Z tangent to the second component M^{n-1} .

Since the complex extensor ϕ is totally geodesic, (2.8) and (2.9) imply that, for each vector Y tangent to M^{n-1} , there is a vector Z tangent to M^{n-1} satisfying

$$(2.11) \quad c \otimes (Y - \alpha(s)G) = \beta(s) \otimes (sc + e)Z$$

for some real-valued functions α and β , which is impossible unless $e = ac$ for some $a \in \mathbb{R}$. Thus $F(s) = (s + a)c$, where a is a real number and c a unit complex number.

Because ϕ is assumed to be totally geodesic, (2.8), (2.10) and $e = ac$ imply that, for vectors Y, Z tangent to M^{n-1} , there exist a tangent vector W tangent to M^{n-1} and functions $\gamma(s)$ and $\delta(s)$ such that

$$(2.12) \quad (s + a)h_G(Y, Z) = \gamma(s)G + (s + a)\delta(s)W.$$

(2.12) implies that $h_G(Y, Z)$ is in the direction of G^\perp , where G^\perp denotes the normal component of G in E^m . Thus for each point $p \in M^{n-1}$ the first normal space of G at p is at most one-dimensional. Furthermore, by taking the covariant derivative of (2.12) and applying (2.12) again, we also know that the first normal spaces are parallel in the normal bundle with respect to the normal connection. Consequently, M^{n-1} is of essential codimension one in E^m .

Case (ii): G is totally geodesic. In this case, since ϕ is totally geodesic, (2.4) and (2.5) imply that, for any Y tangent to M^{n-1} , we have

$$(2.13) \quad (F''(s) - \alpha(s)F'(s)) \otimes G = \beta(s)F(s) \otimes Y,$$

where α, β are real-valued functions. Thus, either $F''(s) = \alpha(s)F'(s)$ for all $s \in I$ or G is parallel to every tangent vector of M^{n-1} in E^m which is impossible unless $n = 2$.

If the first case occurs, we have $F''(s) = 0$, since $F''(s)$ is also perpendicular to $F'(s)$. Thus, $F(s)$ is linear in s . Hence, by applying the same argument given in Case (i), we conclude that $F(s) = (s + a)c$ for some real number a and unit complex number c . If $n = 2$, G is a line in E^m . In this case, the complex extensor ϕ of G is an open portion of a complex plane which is clearly totally geodesic in \mathbb{C}^m .

The converse can be verified easily. □

THEOREM 2.3. *Let $\iota: S^{n-1} \rightarrow E^n$ be the inclusion of the unit hypersphere of E^m (centered at the origin). Then every complex extensor of ι via a unit speed curve F in C is a Lagrangian H -umbilical submanifold of \mathbb{C}^n unless $F(s) = (s + a)c$ for some real number*

a and some unit complex number *c*.

PROOF. Statement (3) of Lemma 2.1 implies that every complex extensor of the unit hypersphere centered at the origin in E^n is a Lagrangian submanifold in C^n .

Now we prove that every complex extensor of the unit hypersphere of E^n is a Lagrangian *H*-umbilical submanifold of C^n .

Since $F: I \rightarrow C$ is a unit speed curve, we may put

$$(2.14) \quad F'(s) = e^{if(s)},$$

for some real-valued function *f* on *I*. Therefore, *F* takes the following form:

$$(2.15) \quad F(s) = \int_a^s e^{if(t)} dt$$

for some real number *a*.

Let $\{x_2, \dots, x_n\}$ be a local coordinate chart on S^{n-1} . Then $\{s, x_2, \dots, x_n\}$ is a local coordinate chart on $I \times S^{n-1}$. Since ι is the unit hypersphere, (2.2), (2.15) and a direct computation imply

$$(2.16) \quad \phi_s = e^{if(s)} \otimes \iota, \quad Y\phi = F \otimes Y,$$

$$(2.17) \quad \begin{aligned} \phi_{ss} &= if'(s)e^{if(s)} \otimes \iota, \quad Y\phi_s = e^{if(s)} \otimes Y, \\ YZ\phi &= F \otimes \nabla_Y Z - \langle Y, Z \rangle (F \otimes \iota), \end{aligned}$$

where *Y*, *Z* are vector fields tangent to the second component of $I \times S^{n-1}$.

Since ι is the unit hypersphere in E^n , (2.16) implies that $e_1 = \partial/\partial s$ is a unit vector field tangent to the first component of $I \times S^{n-1}$; moreover, for each *Y* tangent to the second component of $I \times S^{n-1}$, ϕ_s and $Y\phi$ are orthogonal. Therefore, by applying (2.16) and (2.17) we conclude that the second fundamental form of the complex extensor satisfies

$$(2.18) \quad \begin{aligned} h(e_1, e_1) &= \lambda J e_1, \quad h(e_2, e_2) = \dots = h(e_n, e_n) = \mu J e_1, \\ h(e_1, e_j) &= \mu J e_j, \quad h(e_j, e_k) = 0, \quad j \neq k, \quad j, k = 2, \dots, n, \end{aligned}$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal local frame field, and

$$(2.19) \quad \lambda = f'(s), \quad \mu = \frac{\langle\langle e^{if}, iF \rangle\rangle}{\langle\langle F, F \rangle\rangle}.$$

Therefore, the complex extensor $\phi = F \otimes \iota$ is a Lagrangian *H*-umbilical submanifold unless ϕ is totally geodesic, which occurs only when $F(s) = (s+a)c$ for some $a \in R$ and some unit complex number $c \in C$ (cf. Proposition 2.2). □

In the following we provide examples of complex extensors of the unit hypersphere of E^n satisfying (1.1) with $\lambda = 3\mu$, $\lambda = 2\mu$, $\lambda = \mu$ and $\lambda = 0$, respectively.

EXAMPLE 2.1 (Whitney’s sphere). Let $w: S^n \rightarrow C^n$ be the map defined by

$$(2.20) \quad w(y_0, y_1, \dots, y_n) = \frac{1 + iy_0}{1 + y_0^2} (y_1, \dots, y_n), \quad y_0^2 + y_1^2 + \dots + y_n^2 = 1.$$

Then w is a Lagrangian immersion of the n -sphere into C^n which is called the Whitney n -sphere. The Whitney n -sphere is a complex extensor $\phi = F \otimes \iota$ of the unit hypersphere: $\iota: S^{n-1} \rightarrow E^n$ via F , where $F = F(s)$ is an arclength reparametrization of the curve $f: I \rightarrow C$ given by

$$(2.21) \quad f(\varphi) = \frac{\sin \varphi + i \sin \varphi \cos \varphi}{1 + \cos^2 \varphi}.$$

Whitney’s n -sphere is a Lagrangian H -umbilical submanifold satisfies (1.1) with $\lambda = 3\mu$. In fact, up to homothetic transformations, Whitney’s n -sphere is the only Lagrangian H -umbilical submanifold in C^n with $\lambda = 3\mu$ (cf. [1], [2], [10] for geometric characterizations of Whitney’s spheres).

EXAMPLE 2.2 (Lagrangian pseudo-spheres). For a real number $b > 0$, let $F: R \rightarrow C$ be the unit speed curve give by

$$(2.22) \quad F(s) = \frac{e^{2bsi} + 1}{2bi}.$$

With respect to the induced metric, the complex extensor $\phi = F \otimes \iota$ of the unit hypersphere of E^n via F is a Lagrangian isometric immersion of an open portion of an n -sphere $S^n(b^2)$ of sectional curvature b^2 into C^n which is simply called a *Lagrangian pseudo-sphere*.

A Lagrangian pseudo-sphere is a Lagrangian H -umbilical submanifold satisfying (1.1) with $\lambda = 2\mu$. Conversely, we prove in Section 3 that Lagrangian pseudo-spheres are the only Lagrangian H -umbilical submanifolds of C^n which satisfy (1.1) with $\lambda = 2\mu$ (cf. Theorem 3.1).

EXAMPLE 2.3 (Lagrangian-umbilical submanifold). For a nonzero real number a , let

$$(2.23) \quad F(s) = \int^s e^{-ia \ln t} dt,$$

where $\int^s f(t)dt$ denotes an anti-derivative of $f(s)$. Then the complex extensor of the unit hypersphere of E^n via F is a Lagrangian H -umbilical submanifold of C^n satisfying (1.1) with $\lambda = \mu$. A Lagrangian H -umbilical submanifold with $\lambda = \mu$ is simply called a Lagrangian-umbilical submanifold.

EXAMPLE 2.4. Let $a \in C$ and θ be a real number such that $ae^{-i\theta} \notin R$. Then the complex extensor of the unit hypersphere via $F(s) = a + e^{i\theta}s$ is a Lagrangian H -umbilical

submanifold satisfying (1.1) with $\lambda=0$.

Using Theorem 2.3 and Example 2.4 we may obtain the following existence result.

COROLLARY 2.4. *Given a function $\tilde{\lambda}=\tilde{\lambda}(s)$ and an integer $n\geq 2$, there exists a Lagrangian H -umbilical submanifold of C^n which satisfies (1.1) with $\lambda=\tilde{\lambda}$.*

PROOF. If $\tilde{\lambda}=0$, this result follows from Example 2.4. If $\tilde{\lambda}$ is a nonzero function, we choose an anti-derivative f of $\tilde{\lambda}$ and an antiderivative F of e^{if} . Then F is a unit speed curve in C . From the proof of Theorem 2.3, we know that the complex extensor of the unit hypersphere of E^n via F is a Lagrangian H -umbilical submanifold of C^n satisfying (1.1) with $\lambda=\tilde{\lambda}$. □

3. Geometric characterization of Lagrangian pseudo-spheres. In this section we prove the following geometric characterization of Lagrangian pseudo-spheres.

THEOREM 3.1. *Let $L: M \rightarrow C^n$ be a Lagrangian isometric immersion. Then, up to rigid motions of C^n , L is a Lagrangian pseudo-sphere if and only if L is a Lagrangian H -umbilical immersion satisfying*

$$(3.1) \quad \begin{aligned} h(e_1, e_1) &= 2bJe_1, & h(e_2, e_2) &= \dots = h(e_n, e_n) = bJe_1, \\ h(e_1, e_j) &= bJe_j, & h(e_j, e_k) &= 0, \quad j \neq k, \quad j, k = 2, \dots, n, \end{aligned}$$

for some nontrivial function b with respect to some suitable orthonormal local frame field. Moreover, in this case, b is a nonzero constant.

PROOF. Let $L: M \rightarrow C^n$ be a Lagrangian H -umbilical isometric immersion satisfying (3.1). Then the covariant derivative of the second fundamental form of L satisfies

$$(3.2) \quad \begin{aligned} (\bar{\nabla}_{e_1} h)(e_j, e_1) &= D_{e_1} h(e_j, e_1) - h(\nabla_{e_1} e_j, e_1) - h(e_j, \nabla_{e_1} e_1) = (e_1 b)Je_j, \\ (\bar{\nabla}_{e_j} h)(e_1, e_1) &= 2(e_j b)Je_1, \quad j = 2, \dots, n. \end{aligned}$$

(3.2) and Codazzi's equation imply that b is a nonzero constant. Therefore, by the equation of Gauss, M is a real-space-form of constant sectional curvature b^2 . Hence M is locally isometric to an open portion of the warped product $I \times_{\cos(bs)/b} S^{n-1}$ with $I = (-\pi/2b, \pi/2b)$ whose metric is given by

$$(3.3) \quad g = ds^2 + \frac{1}{b^2} \cos^2(bs)g_0,$$

where g_0 is the standard metric on the unit $(n-1)$ -sphere S^{n-1} . With respect to a spherical coordinate chart $\{u_2, \dots, u_n\}$ on S^{n-1} , we have

$$(3.4) \quad g_0 = du_2^2 + \cos^2 u_2 du_3^2 + \dots + \cos^2 u_2 \dots \cos^2 u_{n-1} du_n^2.$$

From (3.3) and (3.4) we obtain

$$\begin{aligned} \nabla_{\partial/\partial s} \frac{\partial}{\partial s} &= 0, \quad \nabla_{\partial/\partial s} \frac{\partial}{\partial u_k} = -b \tan(bs) \frac{\partial}{\partial u_k}, \quad \nabla_{\partial/\partial u_2} \frac{\partial}{\partial u_2} = \frac{\sin(2bs)}{2b} \frac{\partial}{\partial s}; \\ (3.5) \quad \nabla_{\partial/\partial u_i} \frac{\partial}{\partial u_j} &= -\tan u_i \frac{\partial}{\partial u_j}, \quad 2 \leq i < j; \\ \nabla_{\partial/\partial u_j} \frac{\partial}{\partial u_j} &= \frac{\sin(2bs)}{2b} \prod_{l=2}^{j-1} \cos^2 u_l \frac{\partial}{\partial s} + \sum_{k=2}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{l=k+1}^{j-1} \cos^2 u_l \right) \frac{\partial}{\partial u_k}, \quad j \geq 3. \end{aligned}$$

By (3.1), (3.5) and Gauss' formula, we have

$$(3.6) \quad L_{ss} = 2biL_s, \quad L_s = \frac{\partial L}{\partial s}, \quad L_{ss} = \frac{\partial^2 L}{\partial s^2},$$

$$(3.7) \quad YL_s = (ib - b \tan(bs))Y,$$

$$(3.8) \quad YZL = ib \langle Y, Z \rangle L_s + L_*(\nabla_Y Z),$$

where Y, Z are vector fields tangent to the second component S^{n-1} of the warped product and ∇ is the Levi-Civita connection of S^{n-1} .

Let $\{u_2, \dots, u_n\}$ be a spherical coordinate chart on S^{n-1} . By solving (3.6) we obtain

$$(3.9) \quad L(s, u_2, \dots, u_n) = A(u_2, \dots, u_n)e^{2bsi} + B(u_2, \dots, u_n),$$

for some C^n -valued functions A, B . (3.7), (3.9) and a direct computation imply

$$(3.10) \quad A_{u_j} = B_{u_j}, \quad j = 2, \dots, n,$$

where A_{u_j} denotes the partial derivative of A with respect to u_j . Condition (3.10) implies $B = A + b_0$ where b_0 is a constant vector in C^n . By applying a translation if necessary, we may assume $b_0 = 0$. Therefore, $B = A$. Thus

$$(3.11) \quad L(s, u_2, \dots, u_n) = A(u_2, \dots, u_n)(e^{2bsi} + 1),$$

which implies

$$(3.12) \quad L_s = 2ibAe^{2bsi}, \quad L_{u_2u_2} = A_{u_2u_2}(e^{2bsi} + 1).$$

On the other hand, by using (3.3), (3.4), (3.5), (3.8), (3.11) and (3.12), we find

$$(3.13) \quad L_{u_2u_2} = -A(e^{2bsi} + 1).$$

(3.12) and (3.13) give

$$(3.14) \quad A_{u_2u_2} = -A.$$

Therefore

$$(3.15) \quad A = b_1 \sin u_2 + b_2 \cos u_2$$

for some \mathbb{C}^n -valued functions b_1, b_2 of u_3, \dots, u_n .

If $n=2$, then b_1, b_2 are constant vectors in \mathbb{C}^2 . Thus, (3.11) and (3.15) yield

$$(3.16) \quad L(s, u_2) = (e^{2bsi} + 1)(b_1 \sin u_2 + b_2 \cos u_2).$$

Because M is Lagrangian in \mathbb{C}^2 , (3.3) and (3.4) imply that we may choose the following initial conditions:

$$L_s(0, 0) = (1, 0), \quad L_{u_2} = \left(0, \frac{1}{2bi}\right).$$

(3.16) together with the initial conditions yield

$$(3.17) \quad L = \frac{e^{2bsi} + 1}{2bi} (\cos u_2, \sin u_2).$$

Thus L is a 2-dimensional Lagrangian pseudo-sphere (cf. Example 2.2).

If $n > 2$, then, by putting $Y = \partial/\partial u_2, Z = \partial/\partial u_3$ into (3.8) and also putting $Y = Z = \partial/\partial u_3$ into (3.8) and applying (3.11) and (3.15), we obtain as before that

$$(3.18) \quad b_1 = b_1(u_4, \dots, u_n), \quad b_2 = b_3(u_4, \dots, u_n) \sin u_3 + b_4(u_4, \dots, u_n) \cos u_3.$$

Continuing such procedure $(n-1)$ -times, we may obtain

$$(3.19) \quad L = (e^{2bsi} + 1) \left\{ c_1 \sin u_2 + c_2 \sin u_3 \cos u_2 + \dots + c_{n-1} \sin u_n \prod_{j=2}^{n-1} \cos u_j + c_n \prod_{j=2}^n \cos u_j \right\}$$

for some constant vectors $c_1, \dots, c_n \in \mathbb{C}^n$.

Because M is a Lagrangian submanifold in \mathbb{C}^n , (3.3) and (3.4) imply that we may choose the following initial conditions:

$$(3.20) \quad L_s(0, \dots, 0) = (1, 0, \dots, 0), \\ L_{u_2}(0, \dots, 0) = \left(0, \frac{1}{2bi}, \dots, 0\right), \dots, L_{u_n}(0, \dots, 0) = \left(0, \dots, 0, \frac{1}{2bi}\right).$$

By using (3.17) and (3.18) we obtain

$$(3.21) \quad L = \frac{e^{2bsi} + 1}{2bi} \left(\prod_{j=2}^n \cos u_j, \sin u_2, \sin u_3 \cos u_2, \dots, \sin u_n \prod_{j=2}^{n-1} \cos u_j \right),$$

which implies that, up to rigid motions of \mathbb{C}^n , L is a Lagrangian pseudo-sphere.

Conversely, if L is a Lagrangian pseudo-sphere given in Example 2.2, then Theorem 2.3 together with its proof imply that L is a Lagrangian H -umbilical submanifold in \mathbb{C}^n satisfying (2.18) and (2.19) with

$$(3.22) \quad F(s) = \frac{e^{2bsi} + 1}{2bi}, \quad f(s) = 2bs,$$

which implies (3.1). □

4. Classification of Lagrangian H -umbilical submanifolds in C^n . The main result of this section is to classify Lagrangian H -umbilical submanifolds of complex Euclidean spaces.

THEOREM 4.1. *Let $n \geq 3$ and $L: M \rightarrow C^n$ be a Lagrangian H -umbilical isometric immersion.*

- (i) *If M is of constant sectional curvature, then either M is flat or, up to rigid motions of C^n , L is a Lagrangian pseudo-sphere.*
- (ii) *If M contains no open subset of constant sectional curvature, then, up to rigid motions of C^n , L is a complex extensor of the unit hypersphere of E^n .*

PROOF. Let $n \geq 3$ and $L: M \rightarrow C^n$ be a Lagrangian H -umbilical isometric immersion whose second fundamental form satisfies

$$(4.1) \quad \begin{aligned} h(e_1, e_1) &= \lambda Je_1, & h(e_2, e_2) &= \dots = h(e_n, e_n) = \mu Je_1, \\ h(e_1, e_j) &= \mu Je_j, & h(e_j, e_k) &= 0, \quad j \neq k, \quad j, k = 2, \dots, n \end{aligned}$$

for some functions λ and μ with respect to some suitable orthonormal local frame field.

If M is of constant sectional curvature, then (4.1) implies $\mu(\lambda - 2\mu) = 0$. If $\mu = 0$ identically, then M is flat. If $\mu \neq 0$, then $\lambda = 2\mu \neq 0$ on a nonempty open subset V of M . Thus, according to Theorem 3.1, λ and μ are nonzero constants on V . Hence, by continuity, $V = M$. Put $b = \mu$. Then by applying Theorem 3.1 again, we know that, up to rigid motions of C^n , M is a Lagrangian pseudo-sphere. This proves Statement (i).

For Statement (ii) we assume M contains no open subset of constant sectional curvature. In this case

$$(4.2) \quad U := \{p \in M : \mu(\lambda - 2\mu) \neq 0 \text{ at } p\}$$

is an open dense subset of M .

Let e_1, \dots, e_n be an orthonormal local frame field on M satisfying Condition (4.1) and $\omega^1, \dots, \omega^n$ be the dual 1-forms of e_1, \dots, e_n . Let (ω_B^A) , $A, B = 1, \dots, n, 1^*, \dots, n^*$, be the connection forms on M defined by

$$(4.3) \quad \tilde{\nabla} e_i = \sum_{j=1}^n \omega_i^j e_j + \sum_{j=1}^n \omega_i^{j*} e_{j*}, \quad \tilde{\nabla} e_{i*} = \sum_{j=1}^n \omega_{i*}^j e_j + \sum_{j=1}^n \omega_{i*}^{j*} e_{j*},$$

where $e_{i*} = Je_i$, $\omega_i^j = -\omega_j^i$, $\omega_{i*}^{j*} = -\omega_{j*}^{i*}$, $i, j = 1, \dots, n$.

For a Lagrangian submanifold M in C^n , we have (cf. [8])

$$(4.4) \quad \omega_j^{i*} = \omega_i^{j*}, \quad \omega_i^j = \omega_j^{i*}, \quad \omega_j^{i*} = \sum_{k=1}^n h_{jk}^i \omega^k.$$

From (4.1) and (4.4) we find

$$(4.5) \quad \omega_1^{1*} = \lambda \omega^1, \quad \omega_i^{1*} = \mu \omega^i, \quad \omega_i^{i*} = \mu \omega^i, \quad \omega_j^{i*} = 0, \quad 2 \leq i \neq j \leq n.$$

By (4.1), (4.5) and Codazzi's equation, we obtain

$$(4.6) \quad e_1 \mu = (\lambda - 2\mu) \omega_1^2(e_2) = \dots = (\lambda - 2\mu) \omega_1^n(e_n),$$

$$(4.7) \quad e_j \lambda = (2\mu - \lambda) \omega_j^1(e_1), \quad j > 1,$$

$$(4.8) \quad (\lambda - 2\mu) \omega_1^j(e_k) = 0, \quad 1 < j \neq k \leq n.$$

$$(4.9) \quad e_j \mu = 3\mu \omega_j^1(e_1),$$

$$(4.10) \quad \mu \omega_1^j(e_1) = 0, \quad j > 1.$$

We remark that (4.8) and (4.10) occur only for the case $n \geq 3$.

Since $n \geq 3$, (4.6), (4.9) and (4.10) imply

$$(4.11) \quad \omega_1^j = \left(\frac{e_1 \mu}{\lambda - 2\mu} \right) \omega^j, \quad e_j \lambda = e_j \mu = 0, \quad j = 2, \dots, n.$$

$$(4.12) \quad \omega_1^j(e_k) = 0, \quad 1 < j \neq k \leq n.$$

From (4.11) and Cartan's structure equations, we obtain $d\omega^1 = 0$ and $\nabla_{e_1} e_1 = 0$ which imply that the integral curves of e_1 are geodesics.

For $j, k > 1$, (4.12) yields $\langle [e_j, e_k], e_1 \rangle = \omega_k^1(e_j) - \omega_j^1(e_k) = 0$. Thus the distribution \mathcal{D}^\perp spanned by $\{e_2, \dots, e_n\}$ is integrable. Let \mathcal{D} denote the distribution spanned by e_1 . Then \mathcal{D} is also integrable, since \mathcal{D} is one-dimensional.

Since \mathcal{D} and \mathcal{D}^\perp are both integrable, there exists a local coordinate system $\{x_1, \dots, x_n\}$ such that (a) \mathcal{D} is spanned by $\{\partial/\partial x_1\}$ and \mathcal{D}^\perp is spanned by $\{\partial/\partial x_2, \dots, \partial/\partial x_n\}$ and (b) $e_1 = \partial/\partial x_1$, $\omega^1 = dx_1$.

From (4.7), (4.9) and (4.10) we know that λ and μ depend only on $s (= x_1)$. Furthermore, by (4.11) and the structure equations, we have

$$(4.13) \quad k' + k^2 = \mu^2 - \lambda \mu, \quad k = \frac{\mu'}{\lambda - 2\mu},$$

where ' denotes the differentiation with respect to s .

From (4.1), Codazzi's equation and a direct computation, we obtain

$$(4.14) \quad \langle \nabla_X Y, e_1 \rangle = \left(\frac{e_1 \mu}{2\mu - \lambda} \right) \langle X, Y \rangle.$$

Therefore \mathcal{D}^\perp is a spherical distribution, i.e., \mathcal{D}^\perp is an integrable distribution whose

leaves are totally umbilical submanifolds with parallel mean curvature vector in M . Moreover, by (4.1), (4.14) and Gauss' equation, we know that each leaf of \mathcal{D}^\perp is of constant sectional curvature $\mu^2 + k^2$. Hence, by applying a result of Hiepko [9], U is a warped product $I \times_{f(s)} S^{n-1}$, where S^{n-1} is the unit $(n-1)$ -sphere and $f(s)$ is the warping function. Moreover, each vector tangent to I is in the distribution \mathcal{D} and each vector tangent to S^{n-1} is in the complementary distribution \mathcal{D}^\perp .

With respect to a spherical coordinate chart $\{u_2, \dots, u_n\}$ on S^{n-1} , the metric on $I \times_f S^{n-1}$ is given by

$$(4.15) \quad g = ds^2 + f^2(s) \{ du_2^2 + \cos^2 u_2 du_3^2 + \dots + \cos^2 u_2 \dots \cos^2 u_{n-1} du_n^2 \}.$$

From (4.15) we obtain

$$(4.16) \quad \begin{aligned} \nabla_{\partial/\partial s} \frac{\partial}{\partial s} &= 0, \quad \nabla_{\partial/\partial s} \frac{\partial}{\partial u_k} = \frac{f'}{f} \frac{\partial}{\partial u_k}, \quad \nabla_{\partial/\partial u_2} \frac{\partial}{\partial u_2} = -ff' \frac{\partial}{\partial s}, \\ \nabla_{\partial/\partial u_i} \frac{\partial}{\partial u_j} &= -\tan u_i \frac{\partial}{\partial u_j}, \quad 2 \leq i < j, \\ \nabla_{\partial/\partial u_j} \frac{\partial}{\partial u_j} &= -ff' \prod_{i=2}^{j-1} \cos^2 u_i \frac{\partial}{\partial s} + \sum_{k=2}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{l=k+1}^{j-1} \cos^2 u_l \right) \frac{\partial}{\partial u_k}, \quad j > 2. \end{aligned}$$

(4.1), (4.16) and Codazzi's equation imply

$$(4.17) \quad \frac{f'}{f} = k, \quad k = \frac{\mu'}{\lambda - 2\mu}.$$

Thus, there is a real number $c \neq 0$ such that

$$(4.18) \quad f = c \exp \left(\int k(x) dx \right).$$

By applying (4.15) and (4.16), we know that the sectional curvature of the plane section spanned by $\partial/\partial u_2, \partial/\partial u_3$ is given by

$$(4.19) \quad K \left(\frac{\partial}{\partial u_2} \wedge \frac{\partial}{\partial u_3} \right) = c^{-2} e^{-2 \int k(s) ds} - k^2.$$

On the other hand, (4.1) and Gauss' equation yield

$$(4.20) \quad K \left(\frac{\partial}{\partial u_2} \wedge \frac{\partial}{\partial u_3} \right) = \mu^2.$$

Therefore U is an open portion of the warped product $I \times_{f(s)} S^{n-1}$, where

$$(4.21) \quad f(s) = c \exp \left(\int k(x) dx \right) = \frac{1}{\sqrt{\mu^2 + k^2}}, \quad k = \frac{\mu'}{\lambda - 2\mu}.$$

By (4.1), (4.15), (4.16), (4.17) and Gauss' formula, we get

$$(4.22) \quad L_{ss} = \lambda iL_s, \quad YL_s = (i\mu + k)Y,$$

$$(4.23) \quad YZL = \mu \langle Y, Z \rangle iL_s + L_* (\nabla_Y Z),$$

where Y, Z are vector fields tangent to the second component S^{n-1} of the warped product.

Solving the first equation of (4.22) yields

$$(4.24) \quad L = A(u_2, \dots, u_n) \int^s e^{i \int^s \lambda(t) dt} ds + B(u_2, \dots, u_n)$$

for some C^n -valued functions A and B , where $\int^s \lambda(t) dt$ denotes an antiderivative of $\lambda(s)$.

By applying the second equation of (4.22) with $Y = \partial/\partial u_j$, we find

$$(4.25) \quad (i\mu + k)B_{u_j} = \left(e^{i \int^s \lambda(t) dt} - (i\mu + k) \int^s e^{-i \int^x \lambda(t) dt} dx \right) A_{u_j}$$

for $j = 2, \dots, n$. Since A and B are independent of the variable s , (4.25) implies $B = \alpha A + C$ for some $\alpha \in C$ and $C \in C^n$. Combining this with (4.24) we conclude that after applying a suitable translation of C^n , we have

$$(4.26) \quad L(s, u_2, \dots, u_n) = \left(\alpha + \int^s e^{i \int^s \lambda(t) dt} ds \right) A(u_2, \dots, u_n).$$

Now by applying the same argument as given in the proof of Theorem 3.1, we may conclude that L is of the following form:

$$(4.27) \quad L = \left(\alpha + \int^s e^{i \int^s \lambda(t) dt} ds \right) \left(c_1 \sin u_2 + c_2 \sin u_3 \cos u_2 + \dots \right. \\ \left. + c_{n-1} \sin u_n \prod_{j=2}^{n-1} \cos u_j + c_n \prod_{j=2}^n \cos u_j \right)$$

for some constant vectors $c_1, \dots, c_n \in C^n$.

Because M is a Lagrangian submanifold in C^n , by applying (4.15) we may choose the same initial conditions (3.20). Then, by (4.27), we obtain

$$(4.28) \quad L = \left(\alpha + \int^s e^{i \int^s \lambda(t) dt} ds \right) \left(\prod_{j=2}^n \cos u_j, \sin u_2, \sin u_3 \cos u_2, \dots, \sin u_n \prod_{j=2}^{n-1} \cos u_j \right).$$

Since U is dense in M , (4.28) and continuity imply that, up to rigid motions of C^n , M is the complex extensor of the unit hypersphere in E^n . □

Theorem 4.1 implies the following:

COROLLARY 4.2. *Let M be a Lagrangian submanifold of C^n with $n \geq 3$. Then, up*

to rigid motions of C^n , M is a Lagrangian pseudo-sphere if and only if M is a Lagrangian H -umbilical submanifold with nonzero constant sectional curvature.

PROOF. Follows trivially from Theorem 4.1. □

For Lagrangian H -umbilical surfaces of C^2 we have the following:

THEOREM 4.3. (i) *If M is a minimal Lagrangian surface of C^2 without totally geodesic points, then M is a Lagrangian H -umbilical surface of C^2 .*

(ii) *Let $L: M \rightarrow C^2$ be a Lagrangian H -umbilical surface satisfying*

$$h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_2) = \mu J e_2, \quad h(e_2, e_2) = \mu J e_1$$

such that the integral curves of e_1 are geodesics in M . Then we have:

(ii-1) *If M is of constant sectional curvature, then either M is flat or, up to rigid motions of C^2 , L is a Lagrangian pseudo-sphere.*

(ii-2) *If M contains no open subset of constant sectional curvature, then, up to rigid motions of C^2 , L is a complex extensor of the unit circle of E^2 .*

PROOF. (i) Let M be a Lagrangian minimal surface in C^2 without totally geodesic points. We define a function γ_p by

$$\gamma_p: UM_p \rightarrow \mathbf{R}: v \mapsto \gamma_p(v) = \langle h(v, v), Jv \rangle,$$

where $UM_p = \{v \in T_p M: \langle v, v \rangle = 1\}$. Since UM_p is a compact set, there exists a vector v in UM_p such that γ_p attains an absolute minimum at v . Since p is not totally geodesic, it follows from (1.3) that $\gamma_p \neq 0$. By linearity, we have $\gamma_p(v) < 0$. Because γ_p attains an absolute minimum at v , it follows from (1.3) that $\langle h(v, v), Jw \rangle = 0$ for all w orthogonal to v . So, using (1.3), v is an eigenvector of the symmetric operator A_{Jv} . By choosing an orthonormal basis $\{e_1, e_2\}$ of $T_p M$ with $e_1 = v$, we obtain $h(e_1, e_1) = \lambda J e_1$, $h(e_1, e_2) = -\lambda J e_2$, $h(e_2, e_2) = -\lambda J e_1$ for some λ . This proves Statement (i).

Statement (ii) can be proved in the same way as Theorem 4.1 with minor modification. □

REMARK 4.1. Since minimal Lagrangian surfaces of C^2 are not complex tensors of the unit circle in general, the assumption on the integral curves of e_1 given in Statement (ii) of Theorem 4.3 cannot be omitted.

REMARK 4.2. If M is a Lagrangian H -umbilical submanifold satisfying (1.1) with $\lambda = \mu$, then, by (4.13), we obtain $\lambda = a/(b - s)$ for some real numbers b and $a \neq 0$. By applying a reparametrization of s if necessary, we have $\lambda = -a/s$. Combining this with (4.28) we conclude that a Lagrangian-umbilical submanifold of C^n with $n \geq 3$ is a complex extensor given by Example 2.3.

REMARK 4.3. If M is a Lagrangian H -umbilical submanifold of C^n , $n \geq 3$, satisfying (1.1) with $\lambda = 0$, then, according to (4.28), M is the complex extensor of the unit

hypersphere of E^n via a linear function $F(s) = a + e^{i\theta}s$ for some constant $a \in \mathbb{C}$ and $\theta \in \mathbb{R}$. Since the complex extensor is not totally geodesic, Proposition 2.2 implies $ae^{-i\theta} \notin \mathbb{R}$. Therefore, M is a complex extensor given by Example 2.4.

In this case, (2.19) implies

$$(4.29) \quad \mu = \frac{\langle\langle e^{ib}, ia \rangle\rangle}{\langle\langle a + e^{ib}s, a + e^{ib}s \rangle\rangle}.$$

It is easy to verify that λ and μ given above satisfy the second order differential equation (4.13).

REMARK 4.4. Similarly, from (4.13) and (4.28), we know that a minimal Lagrangian H -umbilical submanifold of C^n , $n \geq 3$, is a complex extensor of the unit hypersphere given by (4.28), where $\lambda = \lambda(s)$ is a solution of the following second order differential equation:

$$(4.30) \quad \left(\frac{\lambda'}{\lambda}\right)' - \frac{1}{n+1} \left(\frac{\lambda'}{\lambda}\right)^2 + \frac{n(n+1)}{(n-1)^2} \lambda^2 = 0.$$

5. Flat Lagrangian H -umbilical submanifolds in C^n . Let N_1, N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2 , respectively and f a positive function on $N_1 \times N_2$. Then the metric $g = f^2g_1 + g_2$ is called a *twisted product metric* on $N_1 \times N_2$. The manifold $N_1 \times N_2$ together with the twisted product metric $g = f^2g_1 + g_2$ is called a *twisted product manifold*, which is denoted by ${}_fN_1 \times N_2$. The function f is called the *twisting function* of the twisted product manifold.

We recall the following existence and uniqueness theorems of Lagrangian immersions (cf. [3], [4], [5]).

THEOREM A. *Let M be a simply-connected Riemannian n -manifold and σ a TM -valued symmetric bilinear form on M satisfying*

- (1) $\langle \sigma(X, Y), Z \rangle$ is totally symmetric,
- (2) $(\nabla \sigma)(X, Y, Z) = \nabla_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ is totally symmetric,
- (3) $R(X, Y)Z = \sigma(\sigma(Y, Z), X) - \sigma(\sigma(X, Z), Y)$.

Then there exists a Lagrangian isometric immersion $x: M \rightarrow C^n$ whose second fundamental form h is given by $h(X, Y) = J\sigma(X, Y)$.

THEOREM B. *Let $L_1, L_2: M \rightarrow C^n$ be two Lagrangian isometric immersion of a Riemannian n -manifold M with second fundamental forms h^1 and h^2 , respectively. If*

$$\langle h^1(X, Y), JL_{1*}Z \rangle = \langle h^2(X, Y), JL_{2*}Z \rangle$$

for all vector fields X, Y, Z tangent to M , then there exists an isometry ϕ of C^n such that $L_1 = L_2 \circ \phi$.

The purpose of this section is to investigate flat Lagrangian H -umbilical sub-

manifolds in \mathbf{C}^n which occurs in Case (i) of Theorem 4.1. More precisely, we prove the following:

THEOREM 5.1. (i) *Let M be a simply-connected open portion of the twisted product manifold ${}_f\mathbf{R} \times \mathbf{E}^{n-1}$ with twisted product metric*

$$(5.1) \quad g = f^2 dx_1^2 + \sum_{j=2}^n dx_j^2,$$

where the twisted function is of the form:

$$(5.2) \quad f = \beta(x_1) + \sum_{j=2}^n \alpha_j(x_1)x_j$$

for some functions $\beta, \alpha_2, \dots, \alpha_n$ of x_1 . Then, up to rigid motions of \mathbf{C}^n , there is a unique Lagrangian isometric immersion $L_f: M \rightarrow \mathbf{C}^n$ without totally geodesic points whose second fundamental form satisfies

$$(5.3) \quad h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_j) = h(e_j, e_k) = 0, \quad 2 \leq j, k \leq n,$$

where

$$\lambda = f^{-1}, \quad e_1 = \lambda \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2}, \dots, e_n = \frac{\partial}{\partial x_n}.$$

(ii) *If $n \geq 3$ and $L: M \rightarrow \mathbf{C}^n$ is a Lagrangian H -umbilical isometric immersion of a flat manifold into \mathbf{C}^n without totally geodesic points, then M is an open portion of a twisted product manifold ${}_f\mathbf{R} \times \mathbf{E}^{n-1}$ with twisted product metric given by (5.1) and twisted function f given by (5.2) for some functions $\beta, \alpha_2, \dots, \alpha_n$.*

Moreover, up to rigid motions of \mathbf{C}^2 , L is given by the unique Lagrangian immersion L_f given in Statement (i).

(iii) *If $L: M \rightarrow \mathbf{C}^2$ is a Lagrangian H -umbilical isometric immersion of a flat surface into \mathbf{C}^2 without totally geodesic points, then one of the following two cases occurs.*

(iii-1) *M is an open portion of a twisted product surface ${}_f\mathbf{R} \times \mathbf{E}^1$ with twisted product metric given by (5.1) and twisted function f given by (5.2) for some functions β, α_2 . Moreover, up to rigid motions of \mathbf{C}^n , L is given by the unique Lagrangian immersion L_f mentioned in Statement (i).*

(iii-2) *L is the complex extensor $\phi = F \otimes G$ of a circle of radius, say r , in \mathbf{E}^2 via F , where F is the unit speed curve in \mathbf{C} given by*

$$(5.4) \quad F(s) = \frac{e^{(i+b)s}}{\sqrt{1+b^2}}$$

for some $b \in \mathbf{R}$.

PROOF. Assume M is a simply-connected open portion of the twisted product

manifold ${}_fR \times E^{n-1}$ with twisted product metric given by (5.1) and twisted function f given by (5.2) for some functions $\beta, \alpha_2, \dots, \alpha_n$.

We define a symmetric bilinear form σ on M by

$$(5.5) \quad \sigma\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_1}, \quad \sigma\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}\right) = 0, \quad \sigma\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = 0, \\ j, k = 2, \dots, n.$$

Then $\langle \sigma(X, Y), Z \rangle$ is totally symmetric in X, Y and Z .

From (5.1) and (5.2), we find

$$(5.6) \quad \nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_1} = \frac{f_1}{f} \frac{\partial}{\partial x_1} - f \sum_{k=2}^n f_k \frac{\partial}{\partial x_k}, \quad \nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_j} = \frac{f_j}{f} \frac{\partial}{\partial x_1}, \quad \nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_k} = 0,$$

$$(5.7) \quad f_1 = \frac{\partial f}{\partial x_1} = \beta'(x_1) + \sum_{j=2}^n \alpha'(x_1)x_j, \quad f_j = \frac{\partial f}{\partial x_j} = \alpha_j, \quad j, k = 2, \dots, n.$$

From (5.5)–(5.7) we know that $(\nabla\sigma)(X, Y, Z)$ is also totally symmetric in X, Y and Z . Furthermore, (5.5), (5.6) and (5.7) imply that σ and the Riemannian curvature tensor R of M satisfy

$$(5.8) \quad R(X, Y)Z = \sigma(\sigma(Y, Z), X) - \sigma(\sigma(X, Z), Y).$$

Therefore, according to Theorems A and B, up to rigid motions of C^n there exists a unique Lagrangian isometric immersion $L_f: M \rightarrow C^n$ whose second fundamental form is given by $h = J\sigma$. If we put

$$(5.9) \quad e_1 = \lambda \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2}, \dots, e_n = \frac{\partial}{\partial x_n}, \quad \lambda = f^{-1},$$

we obtain (5.3). Since the twisted function f is positive, the immersion L_f has no totally geodesic points. This proves Statement (i).

Now we prove Statement (ii). Assume $L: M \rightarrow C^n$ be a Lagrangian H -umbilical isometric immersion of a flat manifold into C^n without totally geodesic points. Suppose the second fundamental form of L is given by (1.1) for some suitable functions λ and μ with respect to some suitable orthonormal local frame field.

Since $n > 2$ and M is Lagrangian H -umbilical in C^n , the flatness of M and Gauss' equation imply $\mu = 0$. Thus, we have

$$(5.10) \quad h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_j) = h(e_j, e_k) = 0, \quad j, k = 2, \dots, n,$$

where λ is nowhere zero. Without loss of generality, we may assume $\lambda > 0$. From (5.10) and Codazzi's equation, we find

$$(5.11) \quad e_j \ln \lambda = \omega_1^j(e_1), \quad \omega_1^j(e_k) = 0, \quad 2 \leq j, k \leq n.$$

Let \mathcal{D} and \mathcal{D}^\perp be the distributions spanned by $\{e_1\}$ and $\{e_2, \dots, e_n\}$, respectively.

Then (5.9) and (5.10) imply that \mathcal{D}^\perp is integrable and moreover the leaves of \mathcal{D}^\perp are totally geodesic submanifolds of C^n . Because \mathcal{D} and \mathcal{D}^\perp are both integrable and they are perpendicular, there exist local coordinates $\{x_1, x_2, \dots, x_n\}$ such that $\partial/\partial x_1$ spans \mathcal{D} and $\{\partial/\partial x_2, \dots, \partial/\partial x_n\}$ spans \mathcal{D}^\perp . Since \mathcal{D} is one-dimensional, we may choose x_1 such that $\partial/\partial x_1 = (1/\lambda)e_1$. Let N^{n-1} be an integral submanifold of \mathcal{D}^\perp . Then N^{n-1} is a totally geodesic submanifold of C^n . Thus, N^{n-1} is an open portion of a Euclidean $(n-1)$ -space E^{n-1} . Therefore, M is an open portion of the twisted product manifold $fI \times E^{n-1}$, where $f=1/\lambda$ and I is an open interval on which λ is defined. Therefore, M admits the metric $g=f^2 dx_1^2 + g_0$, where g_0 is the standard Euclidean metric of E^{n-1} . In particular, if x_2, \dots, x_n denote the canonical Euclidean coordinates on E^{n-1} , then

$$(5.12) \quad g = f^2 dx_1^2 + dx_2^2 + \dots + dx_n^2.$$

(5.12) implies that the Levi-Civita connection of M satisfies

$$(5.13) \quad \nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_1} = \frac{f_1}{f} \frac{\partial}{\partial x_1} - f \sum_{k=2}^n f_k \frac{\partial}{\partial x_k}, \quad \nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_j} = \frac{f_j}{f} \frac{\partial}{\partial x_1}, \quad \nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_k} = 0$$

for $2 \leq j, k \leq n$. Using (5.13) we obtain

$$(5.14) \quad R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}\right) \frac{\partial}{\partial x_1} = f \sum_{k=2}^n f_{jk} \frac{\partial}{\partial x_k}, \quad j=2, \dots, n.$$

Since M is flat, (5.14) yields $f_{jk}=0, j, k=2, \dots, n$. Therefore, f satisfies (5.2) for some functions $\beta, \alpha_2, \dots, \alpha_n$. Consequently, by Statement (i), up to rigid motions, L is given by immersion L_f as mentioned in Statement (i).

For Statement (iii), we assume $L: M \rightarrow C^2$ is a Lagrangian H -umbilical isometric immersion of a flat surface into C^2 without totally geodesic points which satisfies $h(e_1, e_1) = \lambda J e_1, h(e_1, e_2) = \mu J e_2, h(e_2, e_2) = \mu J e_1$, with respect to some orthonormal frame field e_1, e_2 . Then, by the flatness of M , we have either $\mu=0$ or $\lambda=\mu$ identically on M .

If $\mu=0$, then we obtain (iii-1) by using the same argument as in Statement (ii).

If $\lambda=\mu$ identically on M , then we have

$$(5.15) \quad h(e_1, e_1) = \lambda J e_1, \quad h(e_1, e_2) = \lambda J e_2, \quad h(e_2, e_2) = \lambda J e_1.$$

By (5.15) and Codazzi's equation, we obtain

$$(5.16) \quad e_2 \lambda = \omega_1^2(e_1) = 0, \quad e_1 \lambda = -\lambda \omega_1^2(e_2),$$

which implies

$$(5.17) \quad \left[\frac{1}{\lambda} e_1, \frac{1}{\lambda} e_2 \right] = 0.$$

Therefore, there exists a coordinate chart $\{x, y\}$ such that $\partial/\partial x = \lambda^{-1} e_1, \partial/\partial y = \lambda^{-1} e_2$. With respect to x, y , we have

$$(5.18) \quad g = \lambda^{-2}(dx^2 + dy^2), \quad e_1 = \lambda \frac{\partial}{\partial x}, \quad e_2 = \lambda \frac{\partial}{\partial y}.$$

(5.16) and (5.18) imply that $\lambda = \lambda(x)$ is a function of x . Since M is flat, $\lambda = \lambda(x)$ and (5.18) yield $\lambda\lambda_{xx} - \lambda_x^2 = 0$. By solving this second order ordinary differential equation, we find

$$(5.19) \quad \lambda(x) = \frac{e^{-bx}}{r}$$

for some real numbers $b, r \neq 0$. Using (5.18) and (5.19) we get

$$(5.20) \quad \nabla_{\partial/\partial x} \frac{\partial}{\partial x} = b \frac{\partial}{\partial x}, \quad \nabla_{\partial/\partial x} \frac{\partial}{\partial y} = b \frac{\partial}{\partial y}, \quad \nabla_{\partial/\partial y} \frac{\partial}{\partial y} = -b \frac{\partial}{\partial x}.$$

(5.15), (5.19), (5.20) and Gauss' formulas yield

$$(5.21) \quad L_{xx} = (i+b)L_x, \quad L_{xy} = (i+b)L_y, \quad L_{yy} = (i-b)L_x.$$

Solving the first equation of (5.21), we obtain

$$(5.22) \quad L(x, y) = A(y) + e^{(i+b)x}B(y)$$

for some C^2 -valued functions $A(y), B(y)$. Using (5.22) and the second equation in (5.21), we know that $A(y)$ is a constant vector. Thus, by applying a suitable translation if necessary, we may choose $A = 0$. Solving the last equation in (5.21) and using (5.22) with $A = 0$, we obtain

$$(5.23) \quad L(x, y) = e^{(i+b)x}(c_1 \cos(\sqrt{1+b^2}y) + c_2 \sin(\sqrt{1+b^2}y))$$

for some vectors $c_1, c_2 \in C^2$.

Because L is Lagrangian, we may choose the following initial conditions:

$$(5.24) \quad L_x(0, 0) = \left(\frac{r(i+b)}{\sqrt{1+b^2}}, 0 \right), \quad L_y(0, 0) = (0, r)$$

by virtue of (5.18) and (5.19).

Combining (5.23) and (5.24), we obtain

$$(5.25) \quad L(x, y) = \frac{e^{(i+b)x}}{\sqrt{1+b^2}} (r \cos(\sqrt{1+b^2}y), r \sin(\sqrt{1+b^2}y)),$$

which implies that L is the complex extensor of a circle of radius r in E^2 with F given by (5.4). □

REMARK 5.1. The Lagrangian immersion L_f mentioned in Statement (i) of Theorem 5.1 is not necessary a complex extensor in general. For example, if $M = {}_fI \times E^1$ is the twisted product surface with $g = f^2 dx^2 + dy^2$, $f(x) = \beta(x) + y$ for some function β ,

then up to rigid motions of \mathbb{C}^2 the immersion L_f takes the following form:

(5.26)

$$\begin{aligned} \phi = & \left(y \exp\left(\frac{1+\sqrt{5}}{2} ix\right) + \frac{1+\sqrt{5}}{2} i \int_0^x \beta(x) \exp\left(\frac{1+\sqrt{5}}{2} ix\right) dx \right) \left(\sqrt{\frac{5-\sqrt{5}}{10}}, 0 \right) \\ & + \left(y \exp\left(\frac{1-\sqrt{5}}{2} ix\right) + \frac{1-\sqrt{5}}{2} i \int_0^x \beta(x) \exp\left(\frac{1-\sqrt{5}}{2} ix\right) dx \right) \left(0, \sqrt{\frac{5+\sqrt{5}}{10}} \right), \end{aligned}$$

which is not necessary a complex extensor of any plane curve in general.

REMARK 5.2. For a general study of Lagrangian isometric immersions of real-space-forms of constant sectional curvature c into a complex-space-form of constant holomorphic sectional curvature $4c$, see [6].

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