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LINEAR SYSTEMS ON TORIC VARIETIES

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Abstract. A version of Fujita's conjectures is proven for $\boldsymbol{Q}$-Gorenstein toric varieties.

Introduction. In 1985, Fujita posed the following conjectures:

(0.1) CONJECTURES [Fuj]. *Let (X, L) be a projective (possibly mildly singular) polarized variety of dimension n. Then*
(i) *K_X+tL is base point free for $t>n$;*
(ii) *K_X+tL is very ample for $t>n+1$.*

As a first step towards proving this, Fujita proved:

(0.2) THEOREM. *Let (X, L) be a projective non-singular polarized variety over $\boldsymbol{C}$ of dimension n. Then*
(i) *K_X+tL is nef for $t>n$;*
(ii) *K_X+tL is ample for $t>n+1$.*

The proof of (0.2) depends on Mori theory (in fact, only on Kawamata vanishing—cf. [De, 8.3]), which is why X has to be complex.

Evidently, Fujita's conjectures hold true for projective space. In this paper, I amplify on this obvious observation, finding a slightly larger class of varieties for which (0.1) holds:

(0.3) THEOREM. (0.1) *is true for $\boldsymbol{Q}$-Gorenstein projective toric varieties (see* (3.2) *for precise statements).*

The proof of (0.3) faithfully follows Fujita's original approach. First, (0.2) is proved for toric varieties, using toric Mori theory as developed by Reid [Re]. Then (0.1) is seen to follow from (0.2), owing to the combinatorics of line bundles on toric varieties.

Note that since toric Mori theory is properly combinatorial (and in particular can do without Kodaira-like vanishing statements), we do not need to restrict ourselves to the complex case here: (0.3) holds in arbitrary characteristic.

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1. Preliminaries. In this first section I group together notation, definitions and standard results to be used in the sequel. Any reader familiar with the canonical texts on toric varieties [Da], [O] and [Fu] can conveniently skip this section.

(1.1) Toric notation. Let M, N be two mutually dual free abelian groups of rank n, and let $M_{\boldsymbol{R}}$, $N_{\boldsymbol{R}}$ be their real scalar extensions.

For a fan $\Delta\subset N_{\boldsymbol{R}}$, denote

$$\Delta(i):=\{\mu\in\Delta\mid\dim\mu=i\}\,.$$

A fan $\Delta\subset N_{\boldsymbol{R}}$ determines an n-dimensional toric variety $X(\Delta)$ by pasting together the $U_\sigma:=\operatorname{Spec}k[\sigma^\vee\cap M]$, $\sigma\in\Delta(n)$, where k is an arbitrary base field.

$X(\Delta)$ is complete if and only if Δ covers $N_{\boldsymbol{R}}$. $X(\Delta)$ is $\boldsymbol{Q}$-factorial if and only if it is simplicial, i.e. every $\sigma\in\Delta(n)$ has just n rays.

(1.2) Cycles. Let $X=X(\Delta)$ be an arbitrary toric variety. A cone $\mu\in\Delta(i)$ determines an $(n-i)$-dimensional closed, torus-stable subvariety $V(\mu)$ of X, defined as the closure of the orbit $O_\mu:=\operatorname{Spec}k[\mu^\perp\cap M]$. $V(\mu)$ is also a toric variety, corresponding to the fan $p(\mathrm{Star}(\mu))$, where $\mathrm{Star}(\mu):=\{\sigma\mid\mu\prec\sigma\}$, and $p\colon N_{\boldsymbol{R}}\to N_{\boldsymbol{R}}/(\mu-\mu)$ the natural projection.

Let $A_*(X)$ denote the Chow group of cycles modulo rational equivalence. Then one proves (cf. [Da, 10.3]) that for any toric variety X, $A_i(X)$ is generated by classes $[V(\mu)]$, where $\mu\in\Delta(n-i)$.

In particular, I will denote by

$$D_i:=V(e_i)\,,\qquad e_i\in\Delta(1)$$

the torus-stable Weil divisors that generate $A_{n-1}(X)$.

(1.3) The Picard group. A Cartier divisor D on a toric variety $X(\Delta)$ is given by a collection $\{u(\sigma)\in M\}_{\sigma\in\Delta(n)}$. This collection corresponds to what is called a *piecewise linear function* φ_D on the support $|\Delta|$:

$$\varphi_D(v):=\langle u(\sigma),v\rangle\qquad\text{if}\quad v\in\sigma\,.$$

A nef Cartier divisor $D=\sum_{i=1}^r a_iD_i$ also determines a convex lattice polyhedron P_D in $M_{\boldsymbol{R}}$, defined as

$$P_D:=\{u\in M_{\boldsymbol{R}}\mid\langle u,e_i\rangle\geq-a_i,\ i=1,\dots,r\}\,.$$

Two nef Cartier divisors are rationally equivalent if and only if their polyhedra differ by an element $m\in M$.

(1.4) Intersecting curves and divisors. Let $X=X(\Delta)$ be a complete toric variety. Suppose $D\in\operatorname{Pic}(X)$ is given by $\{u(\sigma)\}_{\sigma\in\Delta(n)}$ and $C\in A_1(X)$ is a curve $V(\mu)$ with $\mu\in\Delta(n-1)$.

Writing $\mu=\sigma_1\cap\sigma_2$, with $\sigma_1,\sigma_2\in\Delta(n)$, we find that C is a smooth rational curve, given by the fan $\{\bar\sigma_1,\bar\mu,\bar\sigma_2\}$ in the 1-dimensional vector space $N_{\boldsymbol{R}}/(\mu-\mu)$. It follows that the intersection number (C,D) is

$$(C,D)=\deg(D|_C)=\lambda_v\langle u(\sigma_2)-u(\sigma_1),v\rangle\,,$$

where v is any element of $(\sigma_1 \setminus \mu) \cap N$, and λ_v some positive constant depending on the choice of v.

(1.5) Proposition. *Let $X = X(\Delta)$ be a complete toric variety and $D \in \mathrm{Pic}(X)$.*
(i) *D is base point free if and only if it is nef;*
(ii) *D is ample if and only if $(C, D) > 0$ for every effective curve C on X.*

Proof. This follows from (1.4), analogous to [O, 2.18] where (ii) is proven in the simplicial case. □

(1.6) Intersecting curves and divisors—reprise. On a *simplicial* complete toric variety, we can actually be more explicit about intersection numbers than in (1.4). By (1.2), the intersection pairing $\mathrm{Pic}(X) \times A_1(X) \to \boldsymbol{Z}$ is fully determined by the values

$$(D_i, V(\mu)), \qquad i = 1, \dots, r, \quad \mu \in \Delta(n-1).$$

These values can be expressed in terms of the fan Δ as follows. Any $\mu \in \Delta(n-1)$ is the intersection of two uniquely determined n-dimensional cones, say σ_1 and σ_2. We number the $e_i \in \Delta(1)$ in such a way that

$$\mu = \langle e_1, \dots, e_{n-1} \rangle, \quad \sigma_1 = \langle e_1, \dots, e_{n-1}, e_n \rangle, \quad \sigma_2 = \langle e_1, \dots, e_{n-1}, e_{n+1} \rangle.$$

After eventually renumbering σ_1 and σ_2, we find a unique vector $(a_1, \dots, a_{n+1}) \in \boldsymbol{Q}^{n+1}$ with $a_n \le a_{n+1} = 1$ satisfying

$$a_1 e_1 + \cdots + a_{n+1} e_{n+1} = \bar{0};$$

this vector will be denoted $\mathrm{Rel}(\mu)$.

The formula in (1.4) now implies that

$$(V(\mu), D_i) = \begin{cases} c_\mu a_i, & \text{if} \quad i = 1, \dots, n+1; \\ 0, & \text{otherwise}, \end{cases}$$

where c_μ is some constant with $0 < c_\mu \le 1$ (actually, $c_\mu = \mathrm{mult}(\mu)/\mathrm{mult}(\sigma_2)$).

(1.7) Dualizing sheaf. Let $X = X(\Delta)$ be a complete toric variety; then its dualizing sheaf is given by

$$K_X = -\sum_{i=1}^{r} D_i.$$

This implies: X is $\boldsymbol{Q}$-Gorenstein if and only if for every $\sigma \in \Delta(n)$, the minimal elements on the rays of σ lie in a hyperplane of $N_{\boldsymbol{R}}$.

(1.8) Cone of curves. Let Y be an arbitrary $\boldsymbol{Q}$-Gorenstein projective variety. Defining

$$N^1(Y) := \mathrm{Pic}(Y) \otimes \boldsymbol{Q}/\equiv;$$

$$N_1(Y) := A_1(Y) \otimes \boldsymbol{Q}/\equiv$$

(where $\equiv$ denotes numerical equivalence), one gets two dual $\boldsymbol{Q}$-vector spaces of dimension $\rho(Y)$, the Picard number of Y. (In fact, on a *toric* variety X, this definition boils down to $N^1(X)=\mathrm{Pic}(X)\otimes\boldsymbol{Q}$, $N_1(X)=A_1(X)\otimes\boldsymbol{Q}$, since for curves on X rational and numerical equivalence coincide (Brion—cf. [FS]).)

$N_1(Y)$ contains the cone of effective 1-cycles, defined as

$$\mathrm{NE}(Y):=\{Z\equiv\sum a_iC_i \mid a_i\geq 0\}\ .$$

(Note that for a toric variety X, it follows from (1.2) that $\mathrm{NE}(X)$ is polyhedral, generated by torus-stable curves $V(\mu)$, $\mu\in\Delta(n-1)$.)

A 1-dimensional subspace $R\subset\mathrm{NE}(Y)$ on which K_Y is negative is an *extremal ray* if $C_1, C_2\in\mathrm{NE}(Y)$, $C_1+C_2\in R$ implies $C_1, C_2\in R$. An element of $A_1(Y)$ whose class is non-trivial and lies in an extremal ray is called an *extremal curve*.

The *length* $l(R)$ of an extremal ray $R\in\mathrm{NE}(Y)$ is defined as

$$l(R):=\min\{-(K_Y, C) \mid R\ni C \text{ extremal curve}\}\ .$$

An extremal curve $C\in R$ for which $-(K_Y, C)=l(R)$ is called *minimal*.

2. On the length of extremal rays.

In this section, I will prove:

(2.1) PROPOSITION. *Let X be a $\boldsymbol{Q}$-Gorenstein projective toric variety of dimension n. Then one has for any extremal ray R of* $\mathrm{NE}(X)$:

$$l(R)\leq n+1\ ,$$

with equality only if $X=\boldsymbol{P}^n$.

Then by (1.5) we may conclude the following, which is logically stronger than (0.2):

(2.2) COROLLARY. *Let X be a $\boldsymbol{Q}$-Gorenstein complete toric variety of dimension n, and let $D\in\mathrm{Pic}(X)\otimes\boldsymbol{Q}$.*

(i) *If $(D, C)>n$ for all $C\in\mathrm{NE}(X)$, then K_X+D is nef.*

(ii) *If $(D, C)>n+1$ for all $C\in\mathrm{NE}(X)$, then K_X+D is ample.*

Moreover, if $X\neq\boldsymbol{P}^n$, then the $>$ signs in (i) *and* (ii) *can be changed into $\geq$ signs.*

(2.3) PROOF OF (2.1), FIRST STEP. Reduction to the simplicial case.

If X is not simplicial then $n>2$, and performing what Danilov terms "elementary subdivisions" of Δ, we get a partial crepant resolution $\sigma\colon \tilde{X}\to X$, where $\tilde{X}$ is a simplicial toric variety. By the projection formula,

$$(K_X, \sigma_*\tilde{C})=(\sigma^*K_X, \tilde{C})=(K_{\tilde{X}}, \tilde{C})$$

for all $\tilde{C}\in\mathrm{NE}(\tilde{X})$. Thus we need one easy lemma to reduce the proof of (2.1) to the simplicial case:

(2.4) LEMMA. *Let $\pi\colon \tilde{Y}\to Y$ be a crepant birational morphism of projective $\boldsymbol{Q}$-Gorenstein varieties. If $C_R\in\mathrm{NE}(Y)$ is a minimal extremal curve, then*

$$C_R \equiv r \cdot \pi_* \tilde{C}_{\tilde{R}},$$

where $\tilde{C}_{\tilde{R}}$ *is a minimal extremal curve in* $\mathrm{NE}(\tilde{Y})$, *and* $r \in \boldsymbol{Q}$ *with* $0 < r \leq 1$.

PROOF. The class $C_R \in \mathrm{NE}(Y)$ is the direct image $\pi_* \tilde{C}$ of some $\tilde{C} \in \mathrm{NE}(\tilde{Y})$. Since $(K_{\tilde{Y}}, \tilde{C}) < 0$, we can write

$$\tilde{C} \equiv \sum_{i=1}^{k} r_i \tilde{C}_i,$$

with $r_i \in \boldsymbol{Q}_{\geq 0}$ and $\tilde{C}_i \in \mathrm{NE}(\tilde{Y})$ minimal extremal curves. Now the condition "$C_R = \sum_{i=1}^{k} r_i \pi_* \tilde{C}_i$ extremal" implies $\pi_* \tilde{C}_i \in R$ for $i = 1, \dots, j$, and $\pi_* \tilde{C}_i = 0 \in \mathrm{NE}(Y)$ for $i = j+1, \dots, k$.

So one gets that

$$C_R \equiv r \cdot \pi_* \tilde{C}_1,$$

where the constant $r \in \boldsymbol{Q}_{>0}$ is ≤ 1 since C_R is supposed to be minimal. □

(2.5) PROOF OF (2.1), SECOND STEP. For simplicial toric varieties, (2.1) follows from toric Mori theory [Re]. We only need Reid's combinatorial condition for a torus-stable curve to be extremal:

(2.6) PROPOSITION ([Re, 2.10]). *Let* X *be a simplicial projective toric variety of dimension* n, $R \subset \mathrm{NE}(X)$ *an extremal ray. Suppose* $V(\mu) \in R$ *for* $\mu \in \Delta(n-1)$ *and* $\mathrm{Rel}(\mu) = (a_1, \dots, a_{n+1})$. *Then for every* $a_i > 0$,

$$\sigma_i := \langle e_1, \dots, \hat{e}_i, \dots, e_{n+1} \rangle \in \Delta(n),$$

and the curves corresponding to the cones

$$\mu' := \langle e_1, \dots, \hat{e}_i, \dots, \hat{e}_n, e_{n+1} \rangle,$$

$$\mu'' := \langle e_1, \dots, \hat{e}_i, \dots, e_n, \hat{e}_{n+1} \rangle$$

belong to R.

(2.7) END OF PROOF OF (2.1). Suppose $\mu \in \Delta(n-1)$ such that $R \ni V(\mu)$ is an extremal curve and

$$-(K_X, V(\mu)) = c_\mu \sum_{i=1}^{n+1} a_i \geq n+1,$$

where $\mathrm{Rel}(\mu) = (a_1, \dots, a_{n+1})$ with $a_n \leq a_{n+1} = 1$ (cf. (1.6)). Then necessarily $a_k := \max\{a_i\}_{1 \leq i \leq n-1} \geq 1$.

By (2.6),

$$\mu' = \langle e_1, \dots, \hat{e}_k, \dots, e_n, \hat{e}_{n+1} \rangle$$

gives another curve in R, and we can write $\mathrm{Rel}(\mu') = (a'_1, \dots, a'_{n+1})$ with $a'_k = 1$.

Comparing the linear relations

$$\begin{aligned} \sum a_i e_i + a_k e_k + \quad e_{n+1} \quad &= 0 ; \\ \sum a'_i e_i + \quad e_k + a'_{n+1} e_{n+1} &= 0 , \end{aligned}$$

one finds $a'_i = a_i / a_k \leq 1$, $i = 1, \ldots, n+1$, hence

$$-(K_X, V(\mu')) = c_{\mu'} \sum_{i=1}^{n+1} a'_i \leq n+1 .$$

Finally, to prove the last statement of (2.1), suppose $(K_X, V(\mu)) = -(n+1)$ for all $V(\mu) \in R$. Then the above reasoning implies that $\mathrm{Rel}(\mu) = (1, 1, \ldots, 1)$ for all $V(\mu) \in R$.

Define

$$\Delta(n-1)_R := \{\mu \in \Delta(n-1) \mid V(\mu) \in R\} ,$$

and suppose that $\Delta(n-1)_R \neq \Delta(n-1)$. Then since Δ is complete, there exist cones $\mu_1 \in \Delta(n-1)_R$, $\mu_2 \in \Delta(n-1) \setminus \Delta(n-1)_R$, which are faces of the same $\sigma \in \Delta(n)$, say

$$\sigma = \langle e_1, \ldots, e_n \rangle , \quad \mu_1 = \langle e_1, \ldots, e_{n-1} \rangle , \quad \mu_2 = \langle e_2, \ldots, e_n \rangle .$$

But this is absurd, since then by (2.6)

$$\mu_2 = (\mu_1)' \in \Delta(n-1)_R .$$

This proves that in case equality holds in (2.1), $\Delta(n-1)_R$ must equal $\Delta(n-1)$, so the Picard number $\rho(X) = 1$, Since also $\mathrm{mult}(\sigma) = 1$ for all $\sigma \in \Delta(n)$, this implies that $X = \boldsymbol{P}^n$. □

3. Very ampleness. As announced in the introduction, a version of Fujita's conjecture will be proven for $\boldsymbol{Q}$-Gorenstein toric varieties. Since base-point-freeness is already established, we will concentrate on very ampleness of the adjoint linear system.

(3.1) DEFINITION. A $\boldsymbol{Q}$-Cartier divisor D is said to be *$\boldsymbol{Q}$-base point free* (resp. *$\boldsymbol{Q}$-very ample*) if the smallest positive multiple of D which is a Cartier divisor, is base point free (resp. very ample).

With this definition, the precise form of the statement (0.3) in the introduction is:

(3.2) THEOREM. *Let (X, L) be a polarized $\boldsymbol{Q}$-Gorenstein toric variety of dimension n. Then*

(i) *$K_X + tL$ is $\boldsymbol{Q}$-base point free for $t > n$;*

(ii) *$K_X + tL$ is $\boldsymbol{Q}$-very ample for $t > n+1$.*

Moreover, if $(X, L) \neq (\boldsymbol{P}^n, \mathcal{O}(1))$, then $\geq$ signs will do in (i) *and* (ii).

By (1.5), (i) is a consequence of (2.2)(i). The rest of this section will be spent on proving (ii).

(3.3) Very ampleness criterion. A Cartier divisor D on a toric variety is very ample if and only if for any vertex v of $P_D \subset M_{\boldsymbol{R}}$, the lattice points in the cone

$$\sigma_v := \boldsymbol{R}_{\geq 0}(P_D - v)$$

are non-negatively and integrally generated by the lattice points in $P_D - v$.

(3.4) REMARK. On a *smooth* toric variety, every ample Cartier divisor is very ample (Demazure—cf. [O, 2.15]). This immediately proves (3.2)(ii) in the smooth case.

Criterion (3.3) is used in [EW] to prove the following

(3.5) THEOREM (Ewald-Wessels). *Let X be an n-dimensional projective toric variety, and $L \in \operatorname{Pic}(X)$ ample. Then kL is very ample as soon as $k \geq n-1$.*

(3.6) COROLLARY (numerical criterion for very ampleness). *Let X be an n-dimensional projective toric variety, and $D \in \operatorname{Pic}(X)$. If $(D, C) \geq n-1$ for every effective curve C on X, then D is very ample.*

PROOF. Since $D = \{u(\sigma)\}_{\sigma \in \Delta(n)}$ is ample, the polyhedron $P_D \subset M_{\boldsymbol{R}}$ is n-dimensional, and its vertices are exactly the elements $u(\sigma)$, $\sigma \in \Delta(n)$ ([Fu, p. 70]). It is not hard to see that $u(\sigma_i)$ and $u(\sigma_j)$ are connected by a 1-dimensional side L_{ij} of P_D if and only if $\sigma_i \cap \sigma_j =: \mu_{ij} \in \Delta(n-1)$, and that in this case

$$(D, V(\mu_{ij})) = \#(L_{ij} \cap M) - 1.$$

So the assumption on D translates into the fact that the polyhedron P_D has at least n lattice points on every 1-dimensional side. But then for every vertex v of P_D, there exists an n-dimensional lattice polyhedron Q_v generating the cone σ_v and such that $(n-1)Q_v \subset P_D - v$; hence it follows from (3.5) that P_D satisfies the very ampleness criterion (3.3). $\square$

(3.7) PROOF OF (3.2)(ii). Let $t > n+1$, and in case $(X, L) \neq (\boldsymbol{P}^n, \mathcal{O}(1))$, let $t \geq n+1$; then we know that $K_X + tL$ is ample (2.2). Let r be the smallest positive integer such that $r(K_X + tL)$ is a Cartier divisor.

Suppose $tL = \sum_{i=1}^{r} ta_i D_i$. Then its polyhedron is given by

$$P_{tL} = \{u \in M_{\boldsymbol{R}} \mid \langle u, e_i \rangle \geq -ta_i,\ i = 1, \ldots, r\}.$$

Since $K_X + tL$ is an ample $\boldsymbol{Q}$-Cartier divisor, its polyhedron is an n-dimensional rational polyhedron, given by

$$P_{K_X + tL} = \{u \in M_{\boldsymbol{R}} \mid \langle u, e_i \rangle \geq 1 - ta_i,\ i = 1, \ldots, r\};$$

thus $P_{K_X + tL} \subset P_{tL}$ contains all interior lattice points of P_{tL}, and its walls are parallel to those of P_{tL}.

In particular, the rational polyhedron $P_{K_X + tL}$ contains the lattice polyhedron $P_{(t-1)L}$, which satisfies the very ampleness criterion (3.3) according to the result of Ewald and

Wessels (3.5) (note that by assumption, $t \geq n+1$). Then a fortiori, the lattice polyhedron $rP_{K_X+tL} \supset P_{K_X+tL}$ contains $P_{(t-1)L}$, so $r(K_X+tL)$ is very ample for the right t. □

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