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THE COMMUTATOR OF THE BOCHNER-RIESZ OPERATOR

GUOEN HU AND SHANZHEN LU

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Abstract. L^p mapping properties are considered for the commutator of the Bochner-Riesz operator.

1. Introduction and the statement of results. As well known, commutators generated by some classical operators and BMO functions are useful in the study of partial differential equations (see [3], [10]). Thus it is of great interest to consider the L^p boundedness of these commutators. In 1978, Coifman and Meyer [4] observed that for the classical Calderón-Zygmund singular integral operators, the L^p boundedness for the corresponding first order commutators can be obtained by appropriate weighted norm inequalities with A_p weights for the singular integral operators, where A_p denotes the weight function class of Muckenhoupt (see [7] for the definition and properties of A_p). Recently, Alvarez, Bagby, Kurtz and Pérez [1] developed the idea of Coifman and Meyer and proved the following result.

THEOREM A. *Let $1<p,q<\infty$. Suppose that the linear operator T satisfies the weighted norm estimate*

$$\| Tf \|_{p,w} \le \bar{C} \| f \|_{p,w}$$

for all $w \in A_q$, where the constant $\bar{C}$ depends only on n,p and the A_q constant of w, but not on the weight w. Then for any positive integer k and $b_1, b_2, \dots, b_k \in \mathrm{BMO}$, the commutator defined by

$$T_{b_1,b_2,\dots,b_k}f(x)=T\left(\prod_{j=1}^{k}(b_j(x)-b_j(\cdot))f(\cdot)\right)(x)$$

is bounded on $L^p(R^n)$ with norm $C(p,n,k)\prod_{j=1}^{k}\| b_j \|_{\mathrm{BMO}}$.

The purpose of this paper is to study the L^p boundedness for the commutator of the Bochner-Riesz operator. The Bochner-Riesz operator is defined in terms of Fourier transform by

$$(T^{\alpha}f)^{\wedge}(\xi)=(1-|\xi|^2)_+^{\alpha}\hat{f}(\xi)\,, \tag{1}$$

where $\alpha \in R$, $\hat{f}$ denotes the Fourier transform of f. For k a positive integer and $b_1(x)$,

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$b_2(x), \ldots, b_k(x) \in \mathrm{BMO}(R^n)$, define the commutator of T^α by

$$T^\alpha_{b_1,b_2,\ldots,b_k} f(x) = T^\alpha\left(\prod_{j=1}^{k}(b_j(x)-b_j(\cdot))f(\cdot)\right)(x)\,. \tag{2}$$

If $\alpha \geq (n-1)/2$, a result of Shi and Sun [11] states that T^α is bounded on $L^p_w(R^n)$ provided $1<p<\infty$ and $w \in A_p$. In view of Theorem A we thus have:

THEOREM B. *If* $\alpha \geq (n-1)/2$, *then for any positive integer* k *and* $b_1, b_2, \ldots, b_k \in$ BMO, *the commutator* $T^\alpha_{b_1,b_2,\ldots,b_k}$ *is bounded on* $L^p(R^n)$ *for all* $1<p<\infty$ *with norm* $C(n,p,k)\prod_{j=1}^{k}\|b_j\|_{\mathrm{BMO}}$.

In the case of $0<\alpha<(n-1)/2$, Herz [8] proved that if T^α is bounded on $L^p(R^n)$, then $2n/(n+1+\alpha)<p<2n/(n-1-2\alpha)$. Thus by the result of Coifman and Rochberg [5], a standard duality argument shows that in this case T^α does not satisfy the assumption of Theorem A for any $1<p, q<\infty$ (see also [9, Corollary 3]). In this paper, we will prove that the commutator of T^α enjoys some L^p mapping properties which are parallel to that of the operator T^α. Our main results can be stated as follows:

THEOREM 1. *Let* $0<\alpha<1/2$ *and* $b_1, b_2, \ldots, b_k \in \mathrm{BMO}(R^2)$. *If* $4/(3+2\alpha)<p<4/(1-2\alpha)$, *then* $T^\alpha_{b_1,b_2,\ldots,b_k}$ *is bounded on* $L^p(R^2)$ *with norm* $C(p,k)\prod_{j=1}^{k}\|b_j\|_{\mathrm{BMO}}$.

THEOREM 2. *Let* $n\geq 3$ *and* $(n-1)/(2n+2)<\alpha<(n-1)/2$, $b_1, b_2, \ldots, b_k \in \mathrm{BMO}(R^n)$. If $2n/(n+1+2\alpha)<p<2n/(n-1-2\alpha)$, *then* $T^\alpha_{b_1,b_2,\ldots,b_k}$ *is bounded on* $L^p(R^n)$ *with norm* $C(n,p,k)\prod_{j=1}^{k}\|b_j\|_{\mathrm{BMO}}$.

2. Proof of the theorems.

PROOF OF THEOREM 1. To simplify the exposition, we only deal with the case $k=2$. Write

$$T^\alpha_{b_1,b_2} f(x) = \int_{R^2}\prod_{j=1}^{2}(b_j(x)-b_j(y))B^\alpha(x-y)f(y)dy\,,$$

where

$$B^\alpha(x) = C_\alpha\frac{J_{1+\alpha}(|x|)}{|x|^{1+\alpha}}\,,$$

and $J_\beta(t)$ denotes the Bessel function of order β. Note that

$$J_\beta(t) = Ct^{-1/2}\cos\left(t-\frac{\pi\beta}{2}-\frac{\pi}{4}\right)+r(t)\,, \qquad t\to\infty\,,$$

$$r(t) = O(t^{-3/2})\,, \qquad t\to\infty\,,$$

hence we have

$$\begin{aligned}
T^{\alpha}_{b_1,b_2}f(x)&=C\int_{|x-y|\geq 1}\cos\left(|x-y|-\frac{\pi(3+2\alpha)}{4}\right)\prod_{j=1}^{2}(b_j(x)-b_j(y))\frac{f(y)}{|x-y|^{3/2+\alpha}}dy\\
&\quad+C\int_{|x-y|\geq 1}r(|x-y|)\prod_{j=1}^{2}(b_j(x)-b_j(y))\frac{f(y)}{|x-y|^{1+\alpha}}dy\\
&\quad+C\int_{|x-y|\leq 1}\frac{J_{1+\alpha}(|x-y|)}{|x-y|^{1+\alpha}}\prod_{j=1}^{2}(b_j(x)-b_j(y))f(y)dy\\
&=Pf(x)+Qf(x)+Rf(x)\,.
\end{aligned}$$

Since

$$|J_\beta(t)|\leq C_\beta|t|^\beta\,,\qquad t\to 0\,,$$

it follows that

$$\left|\int_{|x-y|\leq 1}\frac{J_{1+\alpha}(|x-y|)}{|x-y|^{1+\alpha}}f(y)dy\right|\leq CMf(x)\,,$$

where Mf denotes the Hardy-Littlewood maximal function of f. Thus by the weighted estimate for M (see [7]) and Theorem A, we get

$$\|Rf\|_p\leq C\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\|f\|_p\,,\qquad 1<p<\infty.$$

Recall that $|r(t)|\leq C|t|^{-3/2}$ if $t\to\infty$ and $\alpha>0$, so

$$\left|\int_{|x-y|\geq 1}r(|x-y|)\frac{f(y)}{|x-y|^{1+\alpha}}dy\right|\leq C\prod_{k=1}^{\infty}\int_{2^{k-1}\leq|x-y|<2^k}\frac{|f(y)|}{|x-y|^{5/2+\alpha}}dy\leq CMf(x)\,,$$

which implies that

$$\|Qf\|_p\leq C\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\|f\|_p\,,\qquad 1<p<\infty\,.$$

Obviously, the L^p norm of P can be controlled by that of the operator $\tilde{P}$ defined by

$$\tilde{P}f(x)=\int_{|x-y|\geq 1}e^{i|x-y|}\prod_{j=1}^{2}(b_j(x)-b_j(y))\frac{f(y)}{|x-y|^{3/2+\alpha}}dy\,.$$

Furthermore, since $\alpha<1/2$ and

$$\left|\int_{|x-y|<1}e^{i|x-y|}\frac{f(y)}{|x-y|^{3/2+\alpha}}dy\right|\leq CMf(x)\,,$$

Theorem A tells us that for all $1<p<\infty$,

$$\left\| \int_{|x-y|<1} e^{i|x-y|} \prod_{j=1}^{2} (b_j(x)-b_j(y)) \frac{f(y)}{|x-y|^{3/2+\alpha}} dy \right\|_p \leq C \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} \| f \|_p .$$

Thus we may view the operator P as

$$Pf(x)=\int_{R^2} e^{i|x-y|} \prod_{j=1}^{2} (b_j(x)-b_j(y)) \frac{f(y)}{|x-y|^{3/2+\alpha}} dy . \tag{3}$$

By Stein's interpolation theorem (see [12]) and Theorem B, to prove Theorem 1, it is enough to show that for any $0<\alpha<1/2$, the operator P defined by (3) is bounded on $L^4(R^2)$. Denote $I=[0,1]$, $I^2=I\times I$, and $F(I^2)=[-1.5, 2.5]^2\setminus[-0.5, 1.5]^2$. For fixed $\lambda>0$, define

$$P^\lambda f(x)=\int_{I^2} e^{i\lambda|x-y|} \frac{f(y)}{|x-y|^{3/2+\alpha}} dy$$

and the corresponding commutator

$$P^\lambda_{b_1,b_2} f(x)=\int_{I^2} e^{i\lambda|x-y|} \prod_{j=1}^{2} (b_j(x)-b_j(y)) \frac{f(y)}{|x-y|^{3/2+\alpha}} dy .$$

Set $S^\lambda f(x)=\lambda^{1/2-\alpha} P^\lambda f(x)$ and $S^\lambda_{b_1,b_2} f(x)=\lambda^{1/2-\alpha} P^\lambda_{b_1,b_2} f(x)$. Note that if $b(x)\in \mathrm{BMO}(R^n)$, then $b(tx)\in \mathrm{BMO}(R^n)$ and $\| b(t\cdot) \|_{\mathrm{BMO}}=\| b \|_{\mathrm{BMO}}$ for any $t>0$. By the same argument as in [2], we see that the proof of Theorem 1 can be reduced to the following:

LEMMA. *There exists a positive constant* $\delta>0$, *such that*

$$\| S^\lambda_{b_1,b_2} f \|_{L^4(F(I^2))} \leq C\lambda^{-\delta} \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} \| f \|_{L^4(I^2)} .$$

Now we prove this Lemma. Let s be a small positive constant which will be chosen later. Set $0<r<1/2$ and $\sigma>0$ such that

$$\frac{1}{4+\sigma}=\frac{1}{4}-\frac{r}{2} .$$

Observe that if $x\in F(I^2)$, then

$$|P^\lambda f(x)| \leq C\int_{I^2} |f(y)|dy \leq C_r \int_{R^2} \frac{1}{|x-y|^{2-r}} |f(y)\chi_{I^2}(y)|dy = C_r I_r(f\chi_{I^2})(x) ,$$

where χ_{I^2} is the characteristic function of I^2, and I_r is the usual fractional integral operator of order r. By the Hardy-Littlewood-Sobolev theorem, it follows that

$$\| S^\lambda f \|_{L^{4+\sigma}(F(I^2))} \leq C\lambda^{1/2-\alpha} \| P^\lambda f \|_{L^{4+\sigma}(R^2)} \leq C\lambda^{1/2-\alpha} \| f \|_{L^4(I^2)} . \tag{4}$$

Similarly, if σ is small enough, we have

(5) $$\| S^{\lambda}f \|_{L^4(F(I^2))} \leq C\lambda^{1/2-\alpha} \| f \|_{L^{4-\sigma}(I^2)} .$$

By the key estimate used in [2], we have

(6) $$\| S^{\lambda}f \|_{L^4(F(I^2))} \leq C\lambda^{-\varepsilon} \| f \|_{L^4(I^2)} ,$$

where $\varepsilon > 0$. Interpolation between the inequalites (4) and (6) yields

(7) $$\| S^{\lambda}f \|_{L^{4+s\sigma}(F(I^2))} \leq C\lambda^{-\varepsilon+(1/2-\alpha+\varepsilon)s} \| f \|_{L^4(I^2)}$$

with $0 < s < 1$. On the other hand, interpolation between the inequalities (5) and (6) gives

(8) $$\| S^{\lambda}f \|_{L^4(F(I^2))} \leq C\lambda^{-\varepsilon+(1/2-\alpha+\varepsilon)s} \| f \|_{L^{4-s\sigma}(I^2)} .$$

We can also get by the inequalities (7) and (8) that

(9) $$\| S^{\lambda}f \|_{L^{4+s^2\sigma}(F(I^2))} \leq C\lambda^{-\varepsilon+(1/2-\alpha+\varepsilon)s} \| f \|_{L^{4-s^2\sigma}(I^2)} .$$

Let $\phi(x) \in C_0^{\infty}(R^2)$ such that $\phi(x)=1$ if $|x| \leq 50$ and $\operatorname{supp}\phi \subset \{x : |x| \leq 100\}$. Denote

$$\tilde{b}_j(y) = [b_j(y) - m_{10I^2}(b_j)]\phi(y) ,$$

where $m_{10I^2}(b_j)$ denotes the mean value of b_j on $10I^2$. Obviously, if $x \in F(I^2)$, then

$$\begin{aligned} S^{\lambda}_{b_1,b_2}f(x) &= \tilde{b}_1(x)\tilde{b}_2(x)S^{\lambda}f(x) + \tilde{b}_1 S^{\lambda}(\tilde{b}_2 f)(x) + \tilde{b}_2(x)S^{\lambda}(\tilde{b}_1 f)(x) + S^{\lambda}(\tilde{b}_1\tilde{b}_2 f)(x) \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} . \end{aligned}$$

For the first term, we have

$$\begin{aligned} \| \mathrm{I} \|_{L^4(F(I^2))} &\leq \| \tilde{b}_1\tilde{b}_2 \|_{L^q(R^2)} \| S^{\lambda}f \|_{L^{4+s\sigma}(F(I^2))} \\ &\leq C(\sigma, s) \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} \lambda^{-\varepsilon+(1/2-\alpha+\varepsilon)s} \| f \|_{L^4(I^2)} , \end{aligned}$$

where $1/q = 1/4 - 1/(4+s\sigma)$, and the second inequality follows from the inequality (7) and the fact that

$$\begin{aligned} \| \tilde{b}_1\tilde{b}_2 \|_{L^q(R^2)} &\leq \left(\int_{|y| \leq 100} |b_1(y) - m_{10I^2}(b_1)|^{2q} dy \right)^{1/2q} \left(\int_{|y| \leq 100} |b_2(y) - m_{10I^2}(b_2)|^{2q} dy \right)^{1/2q} \\ &\leq C(s, \sigma) \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} . \end{aligned}$$

The estimate for the fourth term follows from the inequality (8) by

$$\begin{aligned} \| \mathrm{IV} \|_{L^4(F(I^2))} &\leq C\lambda^{-\varepsilon+(1/2-\alpha+\varepsilon)s} \| \tilde{b}_1\tilde{b}_2 f \|_{L^{4-s\sigma}(I^2)} \\ &\leq C\lambda^{-\varepsilon+(1/2-\alpha+\varepsilon)s} \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} \| f \|_{L^4(I^2)} . \end{aligned}$$

In the same way, using (9), we can obtain

$$\| \mathrm{II}+\mathrm{III} \|_{L^4(F(I^2))} \leq C\lambda^{-\varepsilon+(1/2-\alpha+\varepsilon)s} \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} \| f \|_{L^4(I^2)} .$$

Choose s so small that $\delta=\varepsilon-(1/2-\alpha+\varepsilon)s>0$. Combining the estimates above we get

$$\| S^{\lambda}_{b_1,b_2} f \|_{L^4(F(I^2))} \leq C\lambda^{-\delta} \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} \| f \|_{L^4(I^2)} .$$

This concludes the proof of our Lemma.

PROOF OF THEOREM 2. By duality and interpolation, it is enough to consider the situation where $2n/(n+1+2\alpha)<p<(2n+2)/(n+3)$. We only treat the case that $k=2$. Let $\psi_0(x)$, $\psi(x)\in C_0^\infty(R^n)$ be radial functions such that

$$\operatorname{supp}\psi \subset \{x:\, 1/4\leq |x| \leq 4\} ,$$

and that for any $|x|\neq 0$,

$$\psi_0(x)+\sum_{l=1}^{\infty} \psi(2^{-l}x)=1 .$$

Denote $B^\alpha(x)=((1-|\cdot|)^\alpha_+)^\wedge(x)$. Write $\psi_l(x)=\psi(2^{-l}x)$ for a positive integer l and define T^α_l by

$$T^\alpha_l f(x)=(B^\alpha\psi_l)*f(x) .$$

It follows that

$$T^\alpha f(x)=\sum_{l=0}^{\infty} (B^\alpha\psi_l)*f(x)=\sum_{l\geq 0} T^\alpha_l f(x) .$$

As in the proof of Theorem 1, it is not difficult to see that

$$\| T^\alpha_{0;b_1,b_2} f \|_p \leq C \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} \| f \|_p , \qquad 1<p<\infty .$$

Our goal is to obtain a refined L^p estimate for $T^\alpha_{l;b_1,b_2}$ for $l\geq 1$, i.e., we want to show that there exists a positive constant $\varepsilon=\varepsilon(p)$, such that

$$\| T^\alpha_{l;b_1,b_2} f \|_p \leq C 2^{-\varepsilon l} \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} \| f \|_p . \tag{10}$$

If we can do so, then the summation of the inequality (10) over all $l\geq 1$ concludes the proof of Theorem 2.

We turn our attention to the operator

$$\tilde{T}^\alpha_l f(x)=\int_{R^n} B^\alpha(2^l(x-y))\psi(x-y)f(y)dy$$

$(l\geq 1)$, and the corresponding commutator

$$\tilde{T}^{\alpha}_{l;b_1,b_2}f(x)=\int_{R^n}\prod_{j=1}^{2}(\tilde{b}_j(x)-\tilde{b}_j(y))B^{\alpha}(2^l(x-y))\psi(x-y)f(y)dy\,,$$

with $\tilde{b}_j(y)=b_j(2^l y)$. To prove (10), it is enough to show that

$$\|\tilde{T}^{\alpha}_{l;b_1b_2}f\|_p\le C2^{-(n+\varepsilon)l}\prod_{j=1}^{2}\|b_j\|_{\mathrm{BMO}}\|f\|_p\,. \tag{11}$$

Write $R^n=\bigcup I_i$, where $\{I_i\}$ is a collection of cubes of side length 1 with disjoint interiors. Set $f_i=f\chi_{I_i}$. Since $\operatorname{supp}\psi\subset\{x\colon 1/4\le|x|\le 4\}$, the support of $\tilde{T}^{\alpha}_{l;b_1,b_2}f_i$ is contained in a fixed multiple of I_i, so the supports of various terms $\tilde{T}^{\alpha}_{l;b_1,b_2}f_i$ have bounded overlaps. Thus

$$\|\tilde{T}^{\alpha}_{l;b_1,b_2}f\|_p^p\le C\sum_i\|\tilde{T}^{\alpha}_{l;b_1,b_2}f_i\|_p^p\,.$$

For each fixed i, let $\phi_i\in C_0^{\infty}$ be a function such that $0\le\phi_i\le 1$, that ϕ_i is identically one on $100nI_i$, and that $\operatorname{supp}\phi_i\subset 200nI_i$. Denote $\tilde{I}_i=400nI_i$ and

$$\tilde{b}^i_j(y)=[\tilde{b}_j(y)-m_{\tilde{I}_i}(\tilde{b}_i)]\phi_i(y)\,.$$

Obviously,

$$\tilde{T}^{\alpha}_{l;b_1,b_2}f_i(x)=\int_{R^n}\prod_{j=1}^{2}(\tilde{b}^i_j(x)-\tilde{b}^i_j(y))B^{\alpha}(2^l(x-y))\psi(x-y)f_i(y)dy\,.$$

Now we estimate $\|\tilde{T}^{\alpha}_{l;b_1,b_2}f_i\|_p$. By the argument of [6], we know that if $2n/(n+1+2\alpha)<p<(2n+2)/(n+3)$, then

$$\|T^{\alpha}_l h\|_p\le C2^{l(n/p-(n+1+2\alpha)/2)}\|h\|_p\,,$$

which implies that

$$\|\tilde{T}^{\alpha}_l h\|_p\le C2^{-ln}2^{l(n/p-(n+1+2\alpha)/2)}\|h\|_p\le C2^{-l((3n+1+2\alpha)/2-n/p)}\|h\|_p\,. \tag{12}$$

Noting that $|B^{\alpha}(y)|\le C$ for all $|y|\ge 1$, we have, for any $0<r<n$,

$$|\tilde{T}^{\alpha}_l h(x)|\le\int_{1\le|x-y|\le 2}|h(y)|dy\le C_r\int_{R^n}\frac{|h(y)|}{|x-y|^{n-r}}dy=C_rI_rh(x)\,. \tag{13}$$

Let $2n/(n+1+2\alpha)<p<(2n+2)/(n+3)$ and s be small positive number. By the inequalities (12) and (13), as in the proof of Theorem 1, we can find that

$$\|\tilde{T}^{\alpha}_l h\|_{p+s\sigma}\le C2^{-\delta_1 l}\|h\|_p\,,$$

$$\|\tilde{T}^{\alpha}_l h\|_p\le C2^{-\delta_2 l}\|h\|_{p-s\sigma}\,,$$

$$\|\tilde{T}^{\alpha}_l h\|_{p+s\sigma}\le C2^{-\delta_1 l}\|h\|_{p-s_1\sigma}\,,$$

where $0<\delta_1<\delta_2$, and $\delta_1\to\delta_0=(3n+1+2\alpha)/2-n/p>n$ as $s\to 0$. We can choose s, σ

small enough such that $\delta_1 > n$. The same argument as in the proof of Theorem 1 then yields

$$\| \tilde{T}^{\alpha}_{l;b_1,b_2} f_i \|_p \leq C \prod_{j=1}^{2} \| b_j \|_{\mathrm{BMO}} 2^{-(n+\varepsilon)l} \| f_i \|_p ,$$

with $\varepsilon = \varepsilon(p) > 0$. This leads to the estimate (11), and then completes the proof of Theorem 2.

REFERENCES

[1] J. ALVAREZ, R. J. BAGBY, D. S. KURTZ AND C. PÉREZ, Weighted estimates for commutators of linear operators, Studia Math. 104 (1993), 195–209.

[2] L. CARLESON AND P. SJÖLIN, Oscillatory integrals and multiplier problem for the disc, Studia Math. 44 (1972), 287–299.

[3] R. COIFMAN, P. L. LIONS, Y. MEYER AND S. SEMMES, Compensated compactness and Hardy Space, J. Math. Pure Appl. 72 (1993), 247–286.

[4] R. COIFMAN AND Y. MEYER, Au déla des opérateurs pseudo-différentiles, Astérisque 57 (1978), 1–185.

[5] R. COIFMAN AND R. ROCHBERG, Another characterization of BMO, Proc. Amer. Math. Soc. 79 (1980), 249–254.

[6] C. FEFFERMAN, A note on spherical summation multipliers, Israeli J. Math. 15 (1973), 44–52.

[7] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, Weighted Norm Inequalities and Related Topics, North-Holland Math. Stud. 116, North-Holland, Amsterdam, 1985.

[8] C. HERZ, On the mean inversion of Fourier and Hankel transforms, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 996–999.

[9] J. L. RUBIO DE FRANCIA, Factorization theory and A_p weights, Amer. J. Math. 106 (1986), 533–547.

[10] C. SEGOVIA AND J. L. TORREA, Vector-valued commutators and applications, Indiana Univ. Math. J. 38 (1989), 959–971.

[11] X. SHI AND Q. SUN, Weighted norm inequalities for Bochner-Riesz operators and singular integral operators, Proc. Amer. Math. Soc. 116 (1992), 665–673.

[12] E. M. STEIN AND G. WEISS, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, New Jersey, 1971.

DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
BEIJING, 100875
PEOPLE'S REPUBLIC OF CHINA